

Convergence of C –class enriched Hardy-Rogers contractions in some Banach spaces

Arsalan Hojjat Ansari^{1,2,3} and Olaoluwa Jeremiah Omidire⁴

¹ Department of Mathematics and Applied Mathematics
Sefako Makgatho Health Sciences University
Ga-Rankuwa, Pretoria, Medunsa-0204, South Africa

²Department of Mathematics, Karaj Branch
Islamic Azad University, Karaj, Iran

³Department of Mathematics
Payame Noor University, Tehran, Iran

⁴ Department of Mathematical Sciences
Osun State University, Osogbo, Nigeria

Email: olaoluwa.omidire@uniosun.edu.ng, omidireolaoluwa@gmail.com

(Received: May 20, 2026 Accepted: June 31, 2026)

Abstract

The applicability of classical Banach contraction mapping principle in solving diverse problems caught the attention of several researchers in various fields of science and engineering. Since its introduction, many extensions and generalizations of this principle continue to emerge in different directions. Some are seen in form of common fixed point using commuting mappings, A-contractions, rational-type contractions, enriched contractions amongst others. In this paper, C -class function is used to generalizing enriched Hargy-Rogers contractions. Convergence of unique invariant point and common invariant points of these general class of operators were established using some related iterative schemes. Results obtained are generalizations of some recently announced related results in literature.

Keywords and phrases: Enriched Jungck-type Hardy-Rogers C –.class contraction, enriched Hardy-Rogers C –class contraction.

2020 AMS Subject Classification: Primary 47H10; Secondary 54H25.

1 Background

Diverse real life problems in Mathematics, Engineering and several other fields may be formulated as Mathematical equation, such as systems of non-linear equations, optimization theory, differential equation amongst others. These problems may be solved using fixed point approach.

If $\mathbf{Z}$ is a nonempty set and $P : \mathbf{Z} \rightarrow \mathbf{Z}$. Any element $u \in \mathbf{Z}$ is an invariant point of P , if $P(u) = u$, $F_P = \{u \in \mathbf{Z} : P(u) = u\}$ denote the set of all invariant points of P .

Given a complete metric space $\mathbf{Z}$ and a continuous self-map P on $\mathbf{Z}$ such that

$$d(Pu, Pv) \leq a.d(u, v), \quad \forall u, v \in \mathbf{Z} \quad (1.1)$$

with $a \in [0, 1)$ fixed; Banach [5], established that P has a unique invariant point in $\mathbf{Z}$. In generalizing this, Rakotch [9] proved that the conclusion of the above claim is valid, if given non-increasing function $a : (0, \infty) \rightarrow [0, 1)$ satisfying, for each $u, v \in \mathbf{Z}, u \neq v$,

$$d(Pu, Pv) \leq a(d(u, v)). \quad (1.2)$$

Also, Kannan [7] proved that P need not be continuous to have a unique invariant point using the inequality: $\exists a \in [0, 1/2)$ satisfying

$$d(Pu, Pv) \leq a[d(u, Pu) + d(v, Pv)], \quad \forall u, v \in \mathbf{Z}. \quad (1.3)$$

Several generalizations and extensions of (1.1) are available in literature. Meanwhile, in 1976, Jungck [6] prove the notion of common invariant point for pair of commuting self-operators. He established that, the pair (S, P) on a complete metric space $(\mathbf{Z}, d)$ satisfying

$$d(Pu, Pv) \leq c.d(Su, Sv), \quad \forall u, v \in \mathbf{Z}, c \in [0, 1), \quad (1.4)$$

have a unique common invariant point in $\mathbf{Z}$.

Definition 1.1. [3] If $\mathbf{Z} \neq \emptyset$. Two mappings $S, P : \mathbf{Z} \rightarrow \mathbf{Z}$ are said to commute $\iff SP = PS$.

2 Method

In [1], Ansari defined what he termed C -class functions as follows:

Definition 2.1. [1] A continuous function $G : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ was referred to as C -class function if the following axioms are satisfied:

$$(1) \ G(u, v) \leq u;$$

$$(1) \ G(u, v) = u \implies \text{either } u = 0 \text{ or } v = 0; \text{ for all } u, v \in \mathbb{R}_+.$$

Note for some G we have that $G(0, 0) = 0$.

We denote the set of C -class functions by $\mathcal{C}$. See [1] for examples.

Definition 2.2. [2] $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ was referred to as altering distance function if:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(u) = 0$ if and only if $u = 0$.

We denote the set of altering distance functions by Ψ .

Definition 2.3. [1] Let Φ_u denote the class of the functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following conditions:

(i) φ is continuous;

(ii) $\varphi(u) > 0, u > 0$ and $\varphi(0) \geq 0$.

Definition 2.4. [1] A tripled $(\psi, \varphi, G) \in (\Psi, \Phi, \mathcal{C})$ was said to be monotone, provided $\forall u, v \in \mathbb{R}_+$

$$u \leq v \implies G(\psi(u), \varphi(u)) \leq G(\psi(v), \varphi(v)).$$

Example 2.1. [1] Let $G(u, v) = u - v, \phi(u) = \sqrt{u}$

$$\psi(u) = \begin{cases} \sqrt{u} & \text{if } 0 \leq u \leq 1, \\ u^2, & \text{if } u > 1 \end{cases},$$

then (ψ, ϕ, G) is monotone.

Example 2.2. [1] Let $G(u, v) = u - v, \phi(u) = u^2$

$$\psi(u) = \begin{cases} \sqrt{u} & \text{if } 0 \leq u \leq 1, \\ u^2, & \text{if } u > 1 \end{cases},$$

then (ψ, ϕ, G) is not monotone.

Definition 2.5. [3, 4] Let S and f be two self mappings of a nonempty subset M of a normed linear space N . Then (S, f) is a Banach operator pair, if any one of the following conditions is satisfied:

- (i) $S(Ff) \subseteq Ff$ (i.e) Ff is S -invariant.
- (ii) $fSx = Sx$ for each $x \in Ff$.
- (iii) $fSx = Sfx$ for each $x \in Ff$.
- (iv) $\|Sfx - fx\| \leq k\|fx - x\|$ for some $k \geq 0$.

Definition 2.6. [3] Let $(N, \|\cdot\|)$ be a normed linear space. A mapping $f : N \rightarrow N$ is said to be an enriched Hardy-Rogers contraction if for any $\delta \in [0, \infty)$ and non-negative $c_i \in \mathbb{R}_+$, ($i = 1, 2, \dots, 5$) with $\sum_{i=1}^5 c_i < 1$, such that $\forall u, v \in N$ we have

$$\begin{aligned} \|\delta(u - v) + fu - fv\| &\leq c_1\|u - v\| + c_2\|u - fu\| + c_3\|u - fv\| \\ &+ c_4\|v - fu\| + c_5\|v - fv\|. \end{aligned} \quad (2.1)$$

Definition 2.7. [3] Let $(N, \|\cdot\|)$ be a normed linear space. A mapping $f : N \rightarrow N$ is said to be an enriched Jungck-type Hardy-Rogers contraction if for any $\delta \in [0, \infty)$ and non-negative $c_i \in \mathbb{R}_+$, ($i = 1, 2, \dots, 5$) with $\sum_{i=1}^5 c_i < 1$, there is a mapping $S : N \rightarrow N$ which commutes with operator f , such that $\forall u, v \in N$ we have

$$\begin{aligned} \|\delta(Su - Sv) + fu - fv\| &\leq c_1\|Su - Sv\| + c_2\|Su - fu\| + c_3\|Su - fv\| \\ &+ c_4\|Sv - fu\| + c_5\|Sv - fv\|. \end{aligned} \quad (2.2)$$

Definition 2.8. [12, 13] A mapping $S : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be convex if for any $x, y \in \mathbb{R}^n$ and $\aleph \in [0, 1]$, we have

$$S(x + \aleph(y - x)) \leq \aleph S(x) + (1 - \aleph)S(y).$$

Theorem 2.1. [3] Let $(B, \|\cdot\|)$ be a real Banach space and $f : B \rightarrow B$ a convex mapping satisfying Definition 2.6. Then, f has a unique fixed point u in B .

Theorem 2.2. [3] Let $(B, \|\cdot\|)$ be a real Banach space and $(f, S) : B \rightarrow B$ be commuting convex Banach operator pair satisfying Definition 2.7. If $f(B) \subseteq S(B)$, then f and S have a unique common fixed point u in B .

Lemma 2.1. [10, 11] Given K a convex subset of a normed linear space N and $f : K \rightarrow K$. Then, $f_c : K \rightarrow K$ given by

$$f_c(u) = (1 - c)u + cfu \quad \forall u \in N, \quad c \in [0, 1]$$

is called averaged mapping and it has the following property: $Fix(f_c) = Fix(f)$.

The C -class function will be used in generalizing enriched Hardy-Rogers contraction and enriched Jungck-type Hardy-Rogers contraction which is a generalization and extension of Banach, Jungck, Kannan, Hardy-Rogers contractive definitions and many more in literature. Also, some iterative procedure together with some basic analysis tools such as continuity, convexity, triangle inequality and many more were employed in establishing our results.

3 Preliminary results

We introduce the following definitions which are generalization of Definitions 2.6 and 2.7 and many more in literature.

Definition 3.1. Let $(\mathbf{Z}, ||\cdot||)$ be a normed linear space. A mapping $P : \mathbf{Z} \rightarrow \mathbf{Z}$ is said to be an enriched Hardy-Rogers C -class contraction if for any $\delta \in [0, \infty)$ and non-negative $c_i \in \mathbb{R}_+$, $(i = 1, 2, \dots, 5)$ with $\sum_{i=1}^5 c_i = 1$, such that $\forall u, v \in \mathbf{Z}$ we have

$$\begin{aligned} \psi(||\delta(u - v) + Pu - Pv||) &\leq G\left(\psi(c_1||u - v|| + c_2||u - Pu|| + c_3||u - Pv|| \right. \\ &\quad + c_4||v - Pu|| + c_5||v - Pv||), \\ &\quad \varphi(c_1||u - v|| + c_2||u - Pu|| + c_3||u - Pv|| \\ &\quad \left. + c_4||v - Pu|| + c_5||v - Pv||)\right), \end{aligned} \quad (3.1)$$

where $(\psi, \varphi, G) \in \Psi \times \Phi_u \times C$

Definition 3.2. Let $(\mathbf{Z}, ||\cdot||)$ be a normed linear space. A mapping $P : \mathbf{Z} \rightarrow \mathbf{Z}$ is said to be an enriched Jungck-type Hardy-Rogers C -class contraction if for any $\delta \in [0, \infty)$ and non-negative $c_i \in \mathbb{R}_+$, $(i = 1, 2, \dots, 5)$ with $\sum_{i=1}^5 c_i = 1$, there is a mapping $S : \mathbf{Z} \rightarrow \mathbf{Z}$ which commutes with operator P , such that $\forall u, v \in \mathbf{Z}$ we

have

$$\begin{aligned}
\psi(\|\delta(Su - Sv) + Pu - Pv\|) &\leq G\left(\psi(c_1\|Su - Sv\| + c_2\|Su - Pu\| + c_3\|Su - Pv\| \right. \\
&\quad \left. + c_4\|Sv - Pu\| + c_5\|Sv - Pv\|), \right. \\
&\quad \left. \varphi(c_1\|Su - Sv\| + c_2\|Su - Pu\| + c_3\|Su - Pv\| \right. \\
&\quad \left. + c_4\|Sv - Pu\| + c_5\|Sv - Pv\|)\right), \tag{3.2}
\end{aligned}$$

where $(\psi, \varphi, G) \in \Psi \times \Phi_u \times C$.

Remark 3.1. *If in 3.1*

$RHS = c_1\|u - v\| + c_2\|u - Pu\| + c_3\|u - Pv\| + c_4\|v - Pu\| + c_5\|v - Pv\|$
then, it reduces to Definition 2.6 (i.e G, ψ and φ are identity operator).

If in 3.2, $RHS = c_1\|Su - Sv\| + c_2\|Su - Pu\| + c_3\|Su - Pv\| + c_4\|Sv - Pu\| + c_5\|Sv - Pv\|$ then, it reduces to Definition 2.7 (i.e G, ψ and φ are identity operator).

4 Main results

Theorem 4.1. *Let $(B, \|\cdot\|)$ be a real Banach space and $f : B \rightarrow B$ a convex mapping satisfying Definition 3.1. Then, f has a unique fixed point u^* in B .*

Proof. For a generic point u_1 in X , we define the sequence $\{u_n\}_{n \in \mathbb{N}}$, by Schaefer iteration

$$u_n = (1 - c)u_{n-1} + cfu_{n-1}, \forall n \geq 1.$$

From (3.1), let $v = fu$, we have

$$\begin{aligned}
&\psi(\|\delta(u - fu) + fu - f^2u\|) \\
&\leq G(\psi(c_1\|u - fu\| + c_2\|u - fu\| + c_3\|u - f^2u\| \\
&\quad + c_4\|fu - fu\| + c_5\|fu - f^2u\|), \\
&\quad \varphi(c_1\|u - fu\| + c_2\|u - fu\| + c_3\|u - f^2u\| \\
&\quad + c_4\|fu - fu\| + c_5\|fu - f^2u\|)),
\end{aligned}$$

then, we have

$$\begin{aligned} & \|\delta(u - fu) + fu - f^2u\| \\ & \leq c_1\|u - fu\| + c_2\|u - fu\| + c_3\|u - f^2u\| + c_5\|fu - f^2u\|. \end{aligned} \quad (4.1)$$

Case 1 When $\delta > 0$, and for any non-negative c . Let $c = \frac{1}{1+\delta} < 1 \implies \delta = \frac{1-c}{c}$ then inequality (4.1) becomes:

$$\begin{aligned} & \|\frac{1-c}{c}(u - fu) + fu - f^2u\| \\ & \leq c_1\|u - fu\| + c_2\|u - fu\| + c_3\|u - f^2u\| + c_5\|fu - f^2u\| \end{aligned} \quad (4.2)$$

($f_c u = (1-c)u + cfu$, $f_c^2 u = (1-c)fu + cf^2u$, since f is a convex mapping.)

$$\begin{aligned} \|u - f_c u\| &= \|u - (1-c)u - cfu\| = \|u - u + cu - cfu\| = \|cu - cfu\| \\ &= c\|u - fu\|. \end{aligned}$$

Hence, inequality (4.2) could be written as:

$$\|f_c u - f_c^2 u\| \leq c_1\|u - f_c u\| + c_2\|u - f_c u\| + c_3\|u - f_c^2 u\| + c_5\|f_c u - f_c^2 u\|. \quad (4.3)$$

Therefore, from inequality (4.3), we have that

$$\begin{aligned} (1 - c_5)\|f_c u - f_c^2 u\| &\leq (c_1 + c_2)\|u - f_c u\| + c_3\|u - f_c^2 u\| \\ &\leq (c_1 + c_2)\|u - f_c u\| + c_3(\|u - f_c u\| + \|f_c u - f_c^2 u\|) \end{aligned}$$

so,

$$(1 - c_5 - c_3)\|f_c u - f_c^2 u\| \leq (c_1 + c_2 + c_3)\|u - f_c u\|, \quad (4.4)$$

since c_i are arbitrary positive real numbers (each) less than 1, without loss of generality we can choose our $c_3 = c_4$. Therefore, we have from the inequality above that

$$\|f_c u - f_c^2 u\| \leq \|u - f_c u\| \quad (4.5)$$

then, we have

$$\begin{aligned}
 \|f_c^2 u - f_c^3 u\| &\leq \|f_c u - f_c^2 u\| \leq \|u - f_c u\| \\
 &\vdots \\
 \|f_c^n u - f_c^{n+1} u\| &\leq \dots \leq \|f_c^2 u - f_c^3 u\| \leq \|f_c u - f_c^2 u\| \leq \|u - f_c u\|,
 \end{aligned}$$

so, u_n is a monotone decreasing, then there is an $r \in \mathbb{R}$ such that

$$\|f_c^n u - f_c^{n+1} u\| \rightarrow r \quad \text{as } n \rightarrow \infty.$$

From (3.1) and the above,

$$\begin{aligned}
 \psi(\|f_c^n u - f_c^{n+1} u\|) &\leq G\left[\psi(c_1\|f_c^{n-1} u - f_c^n u\| + c_2\|f_c^{n-1} u - f_c^n u\| + c_3\|f_c^{n-1} u - f_c^{n+1} u\| + c_5\|f_c^n u - f_c^{n+1} u\|), \right. \\
 &\quad \left. \varphi(c_1\|f_c^{n-1} u - f_c^n u\| + c_2\|f_c^{n-1} u - f_c^n u\| + c_3\|f_c^{n-1} u - f_c^{n+1} u\| + c_5\|f_c^n u - f_c^{n+1} u\|)\right],
 \end{aligned}$$

so we have

$$\psi(r) \leq G(\psi(r), \varphi(r)) \leq \psi(r),$$

thus, from Definitions 2.1 and 2.4,

$\psi(r) = 0$, or $\varphi(r) = 0$, that is $r = 0$, hence,

$$\|f_c^n u - f_c^{n+1} u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.6)$$

And, we write

$$\psi(\|f_c^n u - f_c^{n+1} u\|) \leq G(\psi(\|f_c^{n-1} u - f_c^n u\|), \varphi(\|f_c^{n-1} u - f_c^n u\|)) \leq \psi(\|f_c^{n-1} u - f_c^n u\|),$$

i.e.,

$$\psi(\|u_n - u_{n+1}\|) \leq G(\psi(\|u_{n-1} - u_n\|), \varphi(\|u_{n-1} - u_n\|)) \leq \psi(\|u_{n-1} - u_n\|). \quad (4.7)$$

Next we show that $\{u_n\}$ is a Cauchy sequence. Suppose not, then there exists $\epsilon > 0$ such that for some sub-sequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$, where $n_k > m_k > k$, we have

$$\|u_{n_k} - u_{m_k}\| \geq \epsilon \quad (4.8)$$

so, we can choose n_k depending on m_k in a way that the choosing n_k is the smallest integer with $n_k > m_k$ and this in line with (4.8) gives

$$\|u_{n_{k-1}} - u_{m_k}\| < \epsilon. \quad (4.9)$$

Then, we can say

$$\begin{aligned} 0 < \epsilon \leq \|u_{n_k} - u_{m_k}\| &\leq \|u_{n_k} - u_{n_{k-1}}\| + \|u_{n_{k-1}} - u_{m_k}\| \\ &< \|u_{n_k} - u_{n_{k-1}}\| + \epsilon. \end{aligned} \quad (4.10)$$

Taking limit as $k \rightarrow \infty$ together with (4.6) and Lemma 1.10, we have

$$\lim_{k \rightarrow \infty} \|u_{n_k} - u_{m_k}\| = \epsilon, \quad (4.11)$$

and

$$\begin{aligned} \|u_{n_k} - u_{m_k}\| &\leq \|u_{n_k} - u_{n_{k-1}}\| + \|u_{n_{k-1}} - u_{m_{k-1}}\| + \|u_{m_{k-1}} - u_{m_k}\| \\ &\leq \|u_{n_k} - u_{n_{k-1}}\| + \|u_{n_{k-1}} - u_{n_k}\| + \|u_{n_k} - u_{m_{k-1}}\| \\ &\quad + \|u_{m_{k-1}} - u_{m_k}\| \\ &\leq \|u_{n_k} - u_{n_{k-1}}\| + \|u_{n_{k-1}} - u_{n_k}\| + \|u_{n_k} - u_{m_k}\| \\ &\quad + \|u_{m_k} - u_{m_{k-1}}\| + \|u_{m_{k-1}} - u_{m_k}\|, \end{aligned} \quad (4.12)$$

using (4.6) and (4.11) as $k \rightarrow \infty$, we have

$$\epsilon = \lim_{k \rightarrow \infty} \|u_{n_{k-1}} - u_{m_{k-1}}\|. \quad (4.13)$$

Also, using (4.5), we can write

$$\|u_n - u_{n+1}\| \leq \|u_{n-1} - u_n\|. \quad (4.14)$$

Therefore, using (4.14) in (3.1) with $u = u_{n_k}$ and $v = u_{m_k}$, together with (4.7),

gives

$$\begin{aligned}
 \psi(\|u_{n_k} - u_{m_k}\|) &\leq G\left(\psi(\|u_{n_k} - u_{m_k}\|), \varphi(\|u_{n_k} - u_{m_k}\|)\right) \\
 &\leq G\left(\psi(\|u_{n_{k-1}} - u_{m_{k-1}}\|), \varphi(\|u_{n_{k-1}} - u_{m_{k-1}}\|)\right) \\
 &\leq \psi(\|u_{n_{k-1}} - u_{m_{k-1}}\|)
 \end{aligned} \tag{4.15}$$

taking limit as $k \rightarrow \infty$, and using Equations (4.11) and (4.13), we have

$$\psi(\epsilon) \leq G(\psi(\epsilon), \varphi(\epsilon)) \leq \psi(\epsilon) \tag{4.16}$$

thus, $\psi(\epsilon) = 0$, or $\varphi(\epsilon) = 0$, since ψ is altering distance function, we get a contradiction.

Hence, sequence $\{u_n\}$ is Cauchy and so converges to certain $u^* \in B$.

We claim that $u^* = f_c u^*$. Then using triangle inequality, Definition 3.1 together with (4.1) and (4.3), we have

$$\begin{aligned}
 \psi(\|u^* - f_c u^*\|) &\leq \psi(\|u^* - f_c^{n+1} u\| + \|f_c^{n+1} u - f_c u^*\|) \\
 &= \psi(\|u^* - f_c^{n+1} u\|) + \psi(\|f_c^{n+1} u - f_c u^*\|) \\
 &\leq \psi(\|u^* - f_c^{n+1} u\|) + G\left[\psi\left(c_1\|f_c^n u - u^*\| + c_2\|f_c^n u - f_c^{n+1} u\|\right.\right. \\
 &\quad \left.\left.+ c_3\|f_c^n u - f_c u^*\| + c_4\|u^* - f_c^{n+1} u\| + c_5\|u^* - f_c u^*\|\right), \\
 &\quad \varphi\left(c_1\|f_c^n u - u^*\| + c_2\|f_c^n u - f_c^{n+1} u\| + c_3\|f_c^n u - f_c u^*\|\right. \\
 &\quad \left.\left.+ c_4\|u^* - f_c^{n+1} u\| + c_5\|u^* - f_c u^*\|\right)\right]
 \end{aligned}$$

taking limit as $n \rightarrow \infty$

$$\begin{aligned}
 &= \psi(||u^* - u^*||) + G \left[\psi \left(c_1 ||u^* - u^*|| + c_2 ||u^* - u^*|| \right. \right. \\
 &+ \left. \left. c_3 ||u^* - f_c u^*|| + c_4 ||u^* - u^*|| + c_5 ||u^* - f_c u^*|| \right), \right. \\
 &\quad \left. \varphi \left(c_1 ||u^* - u^*|| + c_2 ||u^* - u^*|| + c_3 ||u^* - f_c u^*|| \right. \right. \\
 &+ \left. \left. c_4 ||u^* - u^*|| + c_5 ||u^* - f_c u^*|| \right) \right] \\
 &= G \left[\psi \left((c_3 + c_5) ||u^* - f_c u^*|| \right), \varphi \left((c_3 + c_5) ||u^* - f_c u^*|| \right) \right] \\
 &\leq \psi \left((c_3 + c_5) ||u^* - f_c u^*|| \right),
 \end{aligned}$$

that is,

$$\psi(||u^* - f_c u^*||) \leq \psi \left((c_3 + c_5) ||u^* - f_c u^*|| \right),$$

since ψ is a altering distance function and clearly $c_3 + c_5 < 1$, then we conclude that

$$u^* = f_c u^*.$$

And by Lemma 2.1, we have

$$u^* = f_c u^* = f u^*.$$

For the uniqueness part (we prove by contradiction). Suppose not, then

$$Fix_{f_c} = \{u^*, v^*\} \text{ i.e } u^* \neq v^* \text{ and } f_c u^* = u^*; \quad f_c v^* = v^*.$$

By Definition 3.1, we have

$$\begin{aligned}
0 &< \psi(||u^* - v^*||) = \psi(||f_c u^* - f_c v^*||) \\
&\leq G \left[\left(\psi(c_1 ||u^* - v^*|| + c_2 ||u^* - f_c u^*|| + c_3 ||u^* - f_c v^*|| + c_4 ||v^* - f_c u^*|| \right. \right. \\
&\quad \left. \left. + c_5 ||v^* - f_c v^*|| \right) \right), \\
&\quad \left(\varphi(c_1 ||u^* - v^*|| + c_2 ||u^* - f_c u^*|| + c_3 ||u^* - f_c v^*|| + c_4 ||v^* - f_c u^*|| \right. \\
&\quad \left. + c_5 ||v^* - f_c v^*|| \right) \right] \\
&= G \left[\left(\psi(c_1 ||u^* - v^*|| + c_2 ||u^* - u^*|| + c_3 ||u^* - v^*|| + c_4 ||v^* - u^*|| \right. \right. \\
&\quad \left. \left. + c_5 ||v^* - v^*|| \right) \right), \\
&\quad \left(\varphi(c_1 ||u^* - v^*|| + c_2 ||u^* - u^*|| + c_3 ||u^* - v^*|| + c_4 ||v^* - u^*|| \right. \\
&\quad \left. + c_5 ||v^* - v^*|| \right) \right] \\
&= G \left[\psi \left((c_1 + c_3 + c_4) ||u^* - v^*|| \right), \varphi \left((c_1 + c_3 + c_4) ||u^* - v^*|| \right) \right] \\
&\leq \psi \left((c_1 + c_3 + c_4) ||u^* - v^*|| \right),
\end{aligned}$$

but $c_1 + c_3 + c_4 < 1$. Therefore, we conclude that $u^* = v^*$.

Case 2: When $\delta = 0$, here $f_c = f$ and (4.3) becomes

$$||fu - f^2u|| \leq c_1 ||u - fu|| + c_2 ||u - fu|| + c_3 ||u - f^2u|| + c_5 ||fu - f^2u||, \quad (4.17)$$

while inequalities (4.6) and (4.7) respectively becoming

$$||f^n u - f^{n+1} u|| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.18)$$

$$\psi(\|u_n - u_{n+1}\|) \leq G\left(\psi(\|u_{n-1} - u_n\|), \varphi(\|u_{n-1} - u_n\|)\right) \leq \psi(\|u_{n-1} - u_n\|) \quad (4.19)$$

the rest of the prove is similar to that of Case 1. Hence, we have $fu^* = u^*$, this together with the conclusion of Case 1 gives $f_c u^* = fu^* = u^*$. $\square$

Theorem 4.2. *Let $(B, \|\cdot\|)$ be a real Banach space and $S, f : B \rightarrow B$ be commuting convex mappings satisfying Definition 3.2, with $f(B) \subset S(B)$. Then, S and f have a unique common fixed point (say u^*) in B .*

Proof. For a generic point u_0 in B , we define the sequence $\{Su_n\}_{n \in \mathbb{N}}$ $\left(\{fu_n\} = \{Su_{n+1}\}\right)$ by Jungck-Schaefer iteration

$$Su_{n+1} = (1 - c)Su_n + cfu_n \quad \forall n \geq 0.$$

And let $fv = Su$, for any $v > u$. Using Definition 3.2, we have

$$\begin{aligned} & \psi(\|\delta(Su - Sv) + fu - fv\|) \\ \leq & G\left(\psi(c_1\|Su - Sv\| + c_2\|Su - fu\| + c_3\|Su - fv\| + c_4\|Sv - fu\| + c_5\|Sv - fv\|), \right. \\ & \left. \varphi(\psi(c_1\|Su - Sv\| + c_2\|Su - fu\| + c_3\|Su - fv\| + c_4\|Sv - fu\| + c_5\|Sv - fv\|))\right) \\ = & G\left(\psi(c_1\|Su - Sv\| + c_2\|fv - fu\| + c_3\|fv - fv\| + c_4\|Sv - fu\| + c_5\|Sv - Su\|), \right. \\ & \left. \varphi(\psi(c_1\|Su - Sv\| + c_2\|fv - fu\| + c_3\|fv - fv\| + c_4\|Sv - fu\| + c_5\|Sv - Su\|))\right) \\ = & G\left(\psi((c_1 + c_5)\|Su - Sv\| + c_4\|Sv - fu\| + c_2\|fu - fv\|), \right. \\ & \left. \varphi((c_1 + c_5)\|Su - Sv\| + c_4\|Sv - fu\| + c_2\|fu - fv\|)\right) \end{aligned}$$

$$\begin{aligned}
&\leq G\left(\psi((c_1 + c_5)||Su - Sv|| + c_4(||Su - Sv|| + ||Su - fu||) \right. \\
&\quad \left. + c_2||fu - fv||), \right. \\
&\quad \left. \varphi((c_1 + c_5)||Su - Sv|| + c_4(||Su - Sv|| + ||Su - fu||) \right. \\
&\quad \left. + c_2||fu - fv||)\right) \\
&= G\left(\psi((c_1 + c_4 + c_5)||Su - Sv|| + c_4||fv - fu|| + c_2||fu - fv||), \right. \\
&\quad \left. \varphi((c_1 + c_4 + c_5)||Su - Sv|| + c_4||fv - fu|| + c_2||fu - fv||)\right) \\
&= G\left(\psi((c_1 + c_4 + c_5)||Su - Sv|| + (c_2 + c_4)||fu - fv||), \right. \\
&\quad \left. \varphi((c_1 + c_4 + c_5)||Su - Sv|| + (c_2 + c_4)||fu - fv||)\right)
\end{aligned}$$

$$||\delta(Su - Sv) + fu - fv|| \leq (c_1 + c_4 + c_5)||Su - Sv|| + (c_2 + c_4)||fu - fv||. \quad (4.20)$$

When $\delta = 0$, then inequality (4.20) becomes:

$$||fu - fv|| \leq (c_1 + c_4 + c_5)||Su - Sv|| + (c_2 + c_4)||fu - fv||. \quad (4.21)$$

Therefore, from inequality (4.21), we have

$$(1 - c_2 - c_4)||fu - fv|| \leq (c_1 + c_4 + c_5)||Su - Sv||,$$

since c_i are arbitrary positive real numbers (each) less than 1, without loss of generality we can choose our $c_3 = c_4$. Therefore, from the inequality above we have

$$\begin{aligned}
||fu - fv|| &\leq ||Su - Sv|| \\
||f^2u - f^2v|| &\leq ||fu - fv|| \leq ||Su - Sv|| \\
&\vdots \\
||f^n u - f^{n+1} u|| = ||f^n u - f^n v|| &\leq \dots \leq ||f^3 u - f^3 v|| \leq ||f^2 u - f^2 v|| \\
&\leq ||fu - fv|| \leq ||Su - Sv||, \quad (4.22)
\end{aligned}$$

so

$$||f^n u - f^{n+1} u|| \rightarrow r \quad \text{as } n \rightarrow \infty.$$

Using Definition 3.2 and inequality (4.22), with $\delta = 0$, we arrive at

$$\psi(r) \leq G(\psi(r), \varphi(r)) \leq \psi(r),$$

thus, $\psi(r) = 0$, or $\varphi(r) = 0$, that is $r = 0$, this implies that

$$\|f^n u - f^{n+1} u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Next we show that $\{f^n u_0\}$ is a Cauchy sequence. This follows the same process as we have in the prove of Theorem 4.1.

Thus, $\{f^n u_0\}$ is a Cauchy sequence and so converges to some $u^* \in B$.

When $\delta > 0$, for any non-negative c . Let $c = \frac{1}{1+\delta} < 1 \implies \delta = \frac{1-c}{c}$ then, using Jungck-Schaefer iteration we have,

$$\begin{aligned} \left\| \frac{1-c}{c}(Su - Sv) + fu - fv \right\| &\leq (c_1 + c_4 + c_5)\|Su - Sv\| + (c_2 + c_4)\|fu - fv\| \\ \|(1-c)(Su - Sv) + cfu - cfv\| &\leq c(c_1 + c_4 + c_5)\|Su - Sv\| + c(c_2 + c_4)\|fu - fv\| \\ \|(1-c)Su - (1-c)Sv + cfu - cfv\| &\leq c(c_1 + c_4 + c_5)\|Su - Sv\| + c(c_2 + c_4)\|fu - fv\| \\ \|(1-c)Su + cfu - (1-c)Sv + cfv\| &\leq c(c_1 + c_4 + c_5)\|Su - Sv\| + c(c_2 + c_4)\|fu - fv\| \\ \|f_c u - f_c v\| &\leq c(c_1 + c_4 + c_5)\|Su - Sv\| + (c_2 + c_4)\|f_c u - f_c v\|. \end{aligned}$$

Therefore

$$(1 - c_2 - c_4)\|f_c u - f_c v\| \leq c(c_1 + c_4 + c_5)\|Su - Sv\|$$

since c , and c_i are arbitrary positive real numbers less than 1, without loss of generality, we can choose our $c_3 = c_4$. Therefore, we have from the inequality above that

$$\|f_c u - f_c v\| \leq \|Su - Sv\| \quad (4.23)$$

so (4.23) implies,

$$\begin{aligned}
 \|f_c^2 u - f_c^2 v\| &\leq \|f_c u - f_c v\| \leq \|Su - Sv\| \\
 &\vdots \\
 &\implies \\
 \|f_c^n u - f_c^{n+1} u\| = \|f_c^n u - f_c^n v\| &\leq \dots \leq \|f_c^2 u - f_c^2 v\| \\
 &\leq \|f_c u - f_c v\| \leq \|Su - Sv\|
 \end{aligned}$$

that is

$$\|f_c^n u - f_c^{n+1} u\| \leq \|Su - Sv\| = \|f_c v - f_c^2 v\| \quad (4.24)$$

so

$$\|f_c^n u - f_c^{n+1} u\| = \|f_c^n u - f_c^n v\| \rightarrow r \quad \text{as } n \rightarrow \infty.$$

Using Definition 3.2, in the same way, we also get

$$\psi(r) \leq G(\psi(r), \varphi(r)) \leq \psi(r)$$

thus, $\psi(r) = 0$, or $\varphi(r) = 0$, that is $r = 0$, this implies that

$$\|f_c^n u - f_c^{n+1} u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The subsequent proof lines is similar to that of Theorem 4.1.

Now, for the prove of common fixed point of f and S , by this, we claim that, $Su^* = f_c u^*$. And by our previous arguments together with triangle inequality, we

have:

$$\begin{aligned}
 & \psi \|Su^* - f_c u^*\| \\
 \leq & \psi \left(\|Su^* - f_c^2 u^*\| + \|f_c^2 u^* - f_c u^*\| \right) \\
 = & \psi \left(\|Su^* - f_c^2 u^*\| + \|f_c u^* - f_c(f_c u^*)\| \right) \\
 \leq & \psi \|Su^* - f_c^2 u^*\| + G \left(\psi(c_1 \|Su^* - S(f_c u^*)\| + c_2 \|Su^* - f_c u^*\| \right. \\
 & \left. + c_3 \|Su^* - f_c(f_c u^*)\| + c_4 \|S(f_c u^*) - f_c u^*\| + c_5 \|S(f_c u^*) - f_c(f_c u^*)\|), \right. \\
 & \left. \varphi(\psi(c_1 \|Su^* - S(f_c u^*)\| + c_2 \|Su^* - f_c u^*\| + c_3 \|Su^* - f_c(f_c u^*)\| \right. \\
 & \left. + c_4 \|S(f_c u^*) - f_c u^*\| + c_5 \|S(f_c u^*) - f_c(f_c u^*)\|) \right) \\
 = & \psi \|Su^* - f_c^2 u^*\| + G \left(\psi(c_1 \|Su^* - S(f_c u^*)\| + c_2 \|Su^* - Su^*\| \right. \\
 & \left. + c_3 \|Su^* - f_c(Su^*)\| + c_4 \|S(f_c u^*) - f_c u^*\| + c_5 \|S(f_c u^*) - S(f_c u^*)\|), \right. \\
 & \left. \varphi(\psi(c_1 \|Su^* - S(f_c u^*)\| + c_2 \|Su^* - Su^*\| + c_3 \|Su^* - f_c(Su^*)\| \right. \\
 & \left. + c_4 \|S(f_c u^*) - f_c u^*\| + c_5 \|S(f_c u^*) - S(f_c u^*)\|) \right) \\
 = & \psi \|Su^* - f_c^2 u^*\| + G \left(\psi(c_1 \|Su^* - f_c^2 u^*\| + c_3 \|Su^* - f_c^2 u^*\| \right. \\
 & \left. + c_4 \|f_c^2 u^* - Su^*\|), \right. \\
 & \left. \varphi(\psi(c_1 \|Su^* - f_c^2 u^*\| + c_3 \|Su^* - f_c^2 u^*\| + c_4 \|f_c^2 u^* - Su^*\|) \right) \\
 \leq & \psi \|Su^* - f_c^2 u^*\| + \psi(c_1 \|Su^* - f_c^2 u^*\| + c_3 \|Su^* - f_c^2 u^*\| \\
 & + c_4 \|f_c^2 u^* - Su^*\|),
 \end{aligned}$$

then, we arrive at:

$$\begin{aligned}
 \|Su^* - f_c u^*\| & \leq (1 + c_1 + c_3 + c_4) \|Su^* - f_c^2 u^*\| = (1 + c_1 + c_3 \\
 & \quad + c_4) \|f_c u^* - f_c^2 u^*\| \\
 & \leq (1 + c_1 + c_3 + c_4) \|Su^* - Sv^*\| \text{ using (4.24).}
 \end{aligned}$$

Since $u^* = v^*$, and by our assumption, we have

$$\begin{aligned} \|Su^* - f_c u^*\| &= \|Su^* - Su^*\| \\ &\leq (1 + c_1 + c_3 + c_4) \|Su^* - Sv^*\| \\ &= (1 + c_1 + c_3 + c_4) \|Su^* - Su^*\|. \end{aligned}$$

This holds only for equality part with $c_1 = c_3 = c_4 = 0$ then, we have, $Su^* = f_c u^*$. So, we have from the foregoing that,

$$f_c u^* = Su^* = u^*.$$

□

5 Numerical examples and applications

Example 5.1. Let $B = \mathbb{R}$ endowed with the usual norm $\|\cdot\| = |\cdot|$. Define the mapping $f : B \rightarrow B$ by

$$f(x) = \frac{x}{4}, \quad x \in B.$$

Clearly, f is linear, continuous and convex. Let

$$\psi(t) = t, \quad \phi(t) = \frac{t}{2}, \quad G(s, t) = s - t.$$

Then $(\psi, \phi, G) \in \Psi \times \Phi_u \times C$, where G is a C -class function. Choose

$$\delta = 0,$$

and constants

$$c_1 = 0.2, \quad c_2 = 0.1, \quad c_3 = 0.2, \quad c_4 = 0.2, \quad c_5 = 0.3,$$

satisfying

$$\sum_{i=1}^5 c_i = 1.$$

We verify Definition 3.1. For arbitrary $u, v \in B$,

$$|f(u) - f(v)| = \left| \frac{u}{4} - \frac{v}{4} \right| = \frac{1}{4}|u - v|.$$

Moreover,

$$\begin{aligned} M(u, v) &= c_1|u - v| + c_2|u - f(u)| + c_3|u - f(v)| \\ &\quad + c_4|v - f(u)| + c_5|v - f(v)|. \end{aligned}$$

Since

$$|u - f(u)| = \frac{3}{4}|u|, \quad |v - f(v)| = \frac{3}{4}|v|,$$

it follows that

$$M(u, v) \geq c_1|u - v| = 0.2|u - v|.$$

Hence,

$$\psi(|f(u) - f(v)|) = \frac{1}{4}|u - v| \leq M(u, v) - \frac{1}{2}M(u, v),$$

that is,

$$\psi(\|f(u) - f(v)\|) \leq G(\psi(M(u, v)), \phi(M(u, v))).$$

Therefore, f is an enriched Hardy-Rogers C -class contraction.

Next, consider the Schaefer iteration associated with Theorem 4.1:

$$u_{n+1} = (1 - c)u_n + cf(u_n), \quad n \geq 0,$$

where $c = \frac{1}{2}$. Then

$$u_{n+1} = \frac{1}{2}u_n + \frac{1}{2}\left(\frac{u_n}{4}\right) = \frac{5}{8}u_n.$$

Starting with $u_0 = 8$, we obtain

$$\begin{aligned} u_1 &= 5, \\ u_2 &= 3.125, \\ u_3 &= 1.953125, \\ u_4 &= 1.220703125. \end{aligned}$$

Consequently,

$$u_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since the fixed point equation

$$f(x) = x$$

gives

$$\frac{x}{4} = x \implies x = 0,$$

the mapping f possesses the unique fixed point

$$u^* = 0.$$

Thus, the conclusion of Theorem 4.1 is numerically validated.

Example 5.2. Let $B = \mathbb{R}$ equipped with the usual norm. Define the mappings $S, f : B \rightarrow B$ by

$$S(x) = 2x, \quad f(x) = x, \quad x \in B.$$

Clearly,

$$f(B) \subseteq S(B),$$

and the pair (S, f) is commuting since

$$Sf(x) = 2x = fS(x), \quad \forall x \in B.$$

Let

$$\psi(t) = t, \quad \phi(t) = \frac{t}{3}, \quad G(s, t) = s - t.$$

Choose the constants

$$c_1 = 0.2, \quad c_2 = 0.1, \quad c_3 = 0.3, \quad c_4 = 0.1, \quad c_5 = 0.3,$$

with

$$\sum_{i=1}^5 c_i = 1.$$

For arbitrary $u, v \in B$, we have

$$|f(u) - f(v)| = |u - v|,$$

while

$$|Su - Sv| = 2|u - v|.$$

Also,

$$\begin{aligned} M(u, v) &= c_1|Su - Sv| + c_2|Su - fu| \\ &\quad + c_3|Su - fv| + c_4|Sv - fu| + c_5|Sv - fv|. \end{aligned}$$

Since each term in $M(u, v)$ is nonnegative, it follows immediately that

$$M(u, v) \geq 0.4|u - v|.$$

Therefore,

$$\psi(\|fu - fv\|) = |u - v| \leq G(\psi(M(u, v)), \phi(M(u, v))).$$

Hence, f satisfies Definition 3.2.

Now consider the Jungck-Schaefer iteration

$$Su_{n+1} = (1 - c)Su_n + cfu_n, \quad n \geq 0,$$

where $c = \frac{1}{2}$. Then

$$2u_{n+1} = u_n + \frac{1}{2}u_n = \frac{3}{2}u_n,$$

which yields

$$u_{n+1} = \frac{3}{4}u_n.$$

Taking $u_0 = 8$, we obtain

$$u_1 = 6,$$

$$u_2 = 4.5,$$

$$u_3 = 3.375,$$

$$u_4 = 2.53125.$$

Hence,

$$u_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, solving the common fixed point equations

$$f(x) = x, \quad S(x) = x,$$

gives

$$x = 2x,$$

and consequently,

$$x = 0.$$

Thus, S and f possess the unique common fixed point

$$u^* = 0,$$

which confirms Theorem 4.2.

Remark 5.1. Examples 5.1 and 5.2 illustrate that the proposed enriched Hardy-Rogers C -class framework properly extends several classical contraction models, including Banach, Kannan, Hardy-Rogers and enriched contractions. In particular, the incorporation of the triplet (ψ, ϕ, G) enlarges the admissible class of operators while preserving strong convergence of the associated iterative schemes.

6 Discussion

The results obtained in Theorems 4.1 and 4.2 generalized and extended some of the results contained in [1], [5, 7, 8] and some many others.

7 Conclusion

The C -class function was used in generalizing enriched Hardy-Rogers contraction and enriched Jungck-type Hardy-Rogers contraction and related fixed points and common fixed points results were established which are generalization and extension of many recently introduced related results in literature. Finally, Our results unify, generalize and extend many known results in the literature.

References

- [1] A. H. Ansari, *Note on $\varphi - \psi$ -contractive type mappings and related fixed point*, The 2nd Regional Conference on Mathematics And Applications, PNU, (2014), 377-380.
- [2] M. S. Khan, M. Swaleh, and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., **30**(1)(1984), 1-9.
- [3] O. J. Omidire and M. O. Olatinwo, *Fixed Point and Common Fixed Point of Enriched Hardy-Rogers Contractions in Some Banach Spaces*, <https://doi.org/10.21203/rs.3.rs-4084674/v1>, (2024), 1-13.
- [4] J. Chen and Z. Li, *Common fixed points for Banach operator pairs in best approximation*, J. Math. Anal. Appl., **336** (2007), 1466-1475.
- [5] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133-181.
- [6] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83** (1976), 261-263.
- [7] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **10** (1968), 71-76.
- [8] M. O. Olatinwo and O. J. Omidire, *Fixed point theorems of Akram-Banach type*, Nonlinear Anal. Forum, **21**(2) (2016), 55-64.
- [9] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc., **13** (1962), 459-465.
- [10] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics (2017).
- [11] Anjali, R. Chugh, and C. Batra, *Fixed Point Theorems of Enriched Ciric's Type and Enriched Hardy-Rogers Contractions*, Numer. Algebra Control Optim., (2023). doi:10.3934/naco.2023022.
- [12] C. R. Bector, S. Chandra, and J. Dutta, *Principles of Optimization Theory*, Narosa Publishing House Pvt. Ltd., New Delhi, India (2005).
- [13] O. J. Omidire, *Common Fixed Point of Some Certain Generalized Contractive Conditions in Convex Metric Space Settings*, Int. J. Math. Sci. Optim.: Theory Appl., **10**(3) (2024), 1-9. <https://doi.org/10.5281/zenodo.13152753>.