

Global existence and decay of solutions for a p -Biharmonic equations with nonlinear damping term

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(Received: April 8, 2026 Accepted: June 30, 2026)

Abstract

The aim of this work, we consider the global existence and decay of solutions for a p -Biharmonic equations with nonlinear damping and source terms in a bounded domain. Firstly, we investigate the global existence of solutions. Later, we prove the decay of solutions by the Nakao's inequality.

1 Introduction

Let $\Omega \subset R^n$ ($n \geq 1$) be a bounded domain, $\partial\Omega$ is a smooth boundary. We investigate the following p -biharmonic equation with initial-boundary value problem

$$\begin{cases} u_{tt} + \Delta_p^2 u + m^2 u + u_t |u_t|^{q-1} = u |u|^{r-1}, & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) = \frac{\partial}{\partial \nu} u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where ν is the unit outward normal vector on $\partial\Omega$, the exponents $p, q, r \geq 1$ and $m > 0$. The p -biharmonic Δ_p^2 is the nonlinear differential operator defined by

$$\Delta_p^2 u = \Delta(|\Delta u|^{p-1} \Delta u).$$

Keywords and phrases: Decay, p -biharmonic equation, nonlinear damping term.

2020 AMS Subject Classification: 35A01, 35B40.

When $p = 1$ and $m = 0$, Eq. (1.1) becomes the following Petrovsky equation

$$u_{tt} + \Delta^2 u + |u_t|^{q-1} u_t = |u|^{r-1} u. \quad (1.2)$$

Messaoudi [5] proved the local existence and nonexistence of solutions of the problem (1.2). Later, Wu and Tsai [18] proved the decay and blow up of solutions of the problem (1.2). Also, Pişkin and Polat [14] proved the decay of solutions of the problem (1.2) with strong damping term $(-\Delta u_t)$.

Messaoudi [6] studied the p -Laplacian equation as follows

$$u_{tt} - \operatorname{div}(|\nabla u|^p \nabla u) - \Delta u_t + u_t |u_t|^{q-1} = u |u|^{r-1}. \quad (1.3)$$

He proved the decay of solution of the equation (1.3). Later, the eq. (1.3) was considered by Wu and Xue [19]. They studied the decay of the solutions, by using the multiplier method. Also, the problem (1.3) was studied by Pişkin [8]. He proved the decay and nonexistence of the solutions. Some other important works related to global existence, nonexistence and decay are [2, 4, 9–12, 16, 17, 21, 22].

In this paper, we study the global existence and decay of solutions for the p -biharmonic equation (1.1), motivated by above works.

This paper consists of two parts in addition to the introduction. In part 2, we give needed materials. Then, in part 3, we prove the global existence and decay of solutions.

2 Preliminaries

In this part, we will give some notations and lemmas which shall be used throughout this work. Let $\|\cdot\|_2 = \|\cdot\|$ denotes the usual $L^2(\Omega)$ norm and $\|\cdot\|_p$ denotes the usual $L^p(\Omega)$ norm, for $p \in (1, \infty)$.

Lemma 2.1. (Sobolev-Poincaré inequality) [1, 15]. Let q be a number with

$$\begin{cases} 2 \leq q < +\infty, & \text{if } n \leq 4, \\ 2 \leq q \leq \frac{2n}{n-4}, & \text{if } n > 4. \end{cases}$$

Then there is constant C depending on Ω and s such that

$$\|u\|_q \leq C \|\Delta u\|, \quad \forall u \in H_0^2(\Omega).$$

Lemma 2.2. (Nakao's Inequality) [7] Suppose that $\phi(t)$ be nonnegative and non-

increasing function defined on $[0, T]$, $T > 1$, satisfying

$$\phi^{1+\alpha}(t) \leq w_0 (\phi(t) - \phi(t+1)), \quad t \in [0, T]$$

for α a nonnegative constant and w_0 a positive constant. Then we get, for each $t \in [0, T]$,

$$\begin{cases} \phi(t) \leq \phi(0) e^{-w_1[t-1]^+}, & \alpha = 0, \\ \phi(t) \leq (\phi(0)^{-\alpha} + w_0^{-1} \alpha [t-1]^+)^{-\frac{1}{\alpha}}, & \alpha > 0 \end{cases}$$

here $[t-1]^+ = \max\{t-1, 0\}$ and $w_1 = \ln\left(\frac{w_0}{w_0-1}\right)$.

Now, we present the local existence theorem which can be considered by combining the arguments of [3, 13, 20].

Theorem 2.1. (Local existence) Assume that $p+1 < r+1 < \frac{n(p+1)}{n-(p+1)}$, $p+1 < n$ and further $u_0(x) \in W_0^{2,p+1}(\Omega)$ and $u_1(x) \in L^2(\Omega)$ such that the problem (1.1) has a unique local solution,

$$u \in C([0, T]; W_0^{2,p+2}(\Omega)) \text{ and } u_t \in C([0, T]; L^2(\Omega)) \cap L^{q+1}(\Omega \times [0, T]).$$

Furthermore, at least one of the following statements is true:

- (i) $T = \infty$,
- (ii) $\|u_t\|^2 + \|\Delta u\|_{p+1}^{p+1} + m^2 \|u\|^2 \rightarrow \text{as } t \rightarrow T$.

3 Global existence and decay of solutions

In this part, we prove the global existence and energy decay of the solutions for the problem (1.1).

Firstly, we define the following functionals

$$J(t) = \frac{1}{p+1} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2}{2} \|u\|^2 - \frac{1}{r+1} \|u\|_{r+1}^{r+1} \quad (3.1)$$

and

$$I(t) = \|\Delta u\|_{p+1}^{p+1} + m^2 \|u\|^2 - \|u\|_{r+1}^{r+1}. \quad (3.2)$$

Now, we define the energy functionals

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{p+1} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2}{2} \|u\|^2 - \frac{1}{r+1} \|u\|_{r+1}^{r+1}, \quad (3.3)$$

we also define

$$\mathcal{W} = \left\{ u : u \in W_0^{2,p+1}(\Omega), I(t) > 0 \right\} \cup \{0\}. \quad (3.4)$$

Lemma 3.1. *The energy functional $E(t)$ is a non-increasing function for $t \geq 0$ and*

$$E'(t) = -\|u_t\|_{q+1}^{q+1} \leq 0. \quad (3.5)$$

Proof. Multiplying the first equation in (1.1) by u_t and integrating over Ω . Using integration by parts, we obtain

$$E(t) - E(0) = - \int_0^t \|u_\tau\|_{q+1}^{q+1} d\tau \quad \text{for } t \geq 0 \quad (3.6)$$

□

Lemma 3.2. *If $(u_0(x), u_1(x)) \in \mathcal{W} \times L^2(\Omega)$, $r > p$ and*

$$\beta = C_*(C_*E(0))^{\frac{r-p}{p+1}} < 1, \quad (3.7)$$

then we have $u \in \mathcal{W}$ for each $t \geq 0$.

Proof. Since $I(0) > 0$, then there exists (by the continuity of $u(x, t)$) such that

$$I(t) > 0$$

for some interval near $t = 0$. Let $T_p > 0$ be a maximal time, when (3.2) holds on $[0, T_p]$.

By (3.1) and (3.2), we obtain

$$\begin{aligned} J(t) &= \frac{1}{r+1} I(t) + \frac{r-p}{(r+1)(p+1)} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2(r-1)}{2(r+1)} \|u\|^2 \\ &\geq \frac{r-p}{(r+1)(p+1)} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2(r-1)}{2(r+1)} \|u\|^2. \end{aligned} \quad (3.8)$$

Therefore, by (3.3) and $E(t)$ being non-increasing by (3.5), we obtain

$$\begin{aligned} m^2 \|u\|^2 + \|\Delta u\|_{p+1}^{p+1} &\leq C^* J(t) \\ &\leq C^* E(t) \\ &\leq C^* E(0), \end{aligned} \quad (3.9)$$

where $C^* = \max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\}$. Thanks to Sobolev-Poincaré inequality, (3.7) and (3.9), we get

$$\begin{aligned} \|u\|_{r+1}^{r+1} &\leq C_* \|\Delta u\|^{r+1} \\ &\leq C_* \|\Delta u\|_{p+1}^{r+1} \\ &= C_* \|\Delta u\|_{p+1}^{r-p} \|\Delta u\|_{p+1}^{p+1} \\ &\leq C_* (C_* E(0))^{\frac{r-p}{p+1}} \|\Delta u\|_{p+1}^{p+1} \\ &= \beta \|\Delta u\|_{p+1}^{p+1} \\ &< \|\Delta u\|_{p+1}^{p+1} \text{ on } t \in [0, T_p]. \end{aligned} \quad (3.10)$$

Consequently, by using (3.2), we conclude that $I(t) > 0$ for all $t \in [0, T_p]$. By repeating this procedure, T_p is extended to T . Therefore, the proof is completed. $\square$

Lemma 3.3. *Let the conditions of the above Lemma be satisfied. Then there exists $\eta_1 = 1 - \beta$ such that*

$$\|u\|_{r+1}^{r+1} \leq (1 - \eta_1) \|\Delta u\|_{p+1}^{p+1}$$

Proof. From (3.10), we have

$$\|u\|_{r+1}^{r+1} \leq \beta \|\Delta u\|_{p+1}^{p+1}.$$

□

Let $\eta_1 = 1 - \beta$, then we obtain the following result.

Remark 3.1. *By the above lemma, we get*

$$\|u\|_{r+1}^{r+1} \leq \frac{1}{\eta_1} I(t). \quad (3.11)$$

3.1 Global existence

In this subsection, we will prove that solutions to the problem (1.1) exist for all time.

Theorem 3.1. *(Global existence) Assume that $p+1 < r+1 < \frac{n(p+1)}{n-(p+1)}$ and $p+1 < n$ holds. If $u_0(x) \in \mathcal{W}$ satisfying (3.7). Then the solution of system (1.1) is bounded and global.*

Proof. It suffices to show that

$$\|u_t\|^2 + \|\Delta u\|_{p+1}^{p+1} + m^2 \|u\|^2$$

is bounded independently of t . By using (3.2) and (3.3), we get

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{p+1} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2}{2} \|u\|^2 - \frac{1}{r+1} \|u\|_{r+1}^{r+1} \\ &= \frac{1}{2} \|u_t\|^2 + \frac{r-p}{(r+1)(p+1)} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2(r-1)}{2(r+1)} \|u\|^2 + \frac{1}{r+1} I(t) \\ &\geq \frac{1}{2} \|u_t\|^2 + \frac{r-p}{(r+1)(p+1)} \|\Delta u\|_{p+1}^{p+1} + \frac{m^2(r-1)}{2(r+1)} \|u\|^2 \end{aligned}$$

since $I(t) \geq 0$, which implies that

$$\|u_t\|^2 + \|\Delta u\|_{p+1}^{p+1} + m^2 \|u\|^2 \leq CE(0),$$

$$\text{here } C = \max \left\{ 2, \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\}.$$

□

3.2 Decay

In this subsection, we will prove that the solutions to the problem (1.1) converges to zero as time approaches infinity.

Theorem 3.2. (Decay) Assume that $p + 1 < r + 1 < \frac{n(p+1)}{n-(p+1)}$, $p + 1 < n$, and (3.7) hold and further $u_0(x) \in \mathcal{W}$. Thus, we have the following estimates

$$E(t) \leq \begin{cases} E(0) e^{-\omega_1[t-1]^+}, & \text{if } q = 1, p = 0, \\ (E(0)^{-\alpha} + C_7^{-1} \alpha [t-1]^+)^{-\frac{1}{\alpha}}, & \text{if } q > 1 \end{cases}$$

here w_1, α and C_7 are positive constants that will be specified later.

Proof. Integrating (3.5) over $[t, t+1]$, for $t > 0$, we obtain

$$E(t) - E(t+1) = D^{q+1}(t),$$

where

$$D^{q+1}(t) = \int_t^{t+1} \|u_\tau\|_{q+1}^{q+1} d\tau. \quad (3.12)$$

By virtue of (3.12) and Hölder's inequality, we get

$$\begin{aligned} \int_t^{t+1} \int_\Omega |u_t|^2 dx dt &\leq \int_t^{t+1} |\Omega|^{\frac{q-1}{q+1}} \|u(t)\|_{q+1}^2 dt \\ &= |\Omega|^{\frac{q-1}{q+1}} D^2(t) \\ &= CD^2(t). \end{aligned} \quad (3.13)$$

Thus, from (3.13), there exist $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t + 1]$ such that

$$\|u_t(t_i)\|_2 \leq CD(t), \quad i = 1, 2.$$

Multiplying Eq. (1.1) by u and integrating it over $\Omega \times [t_1, t_2]$, we obtain

$$\begin{aligned} \int_{t_1}^{t_2} I(t) &= \int_{t_1}^{t_2} \int_\Omega uu_{tt} dx dt + \int_{t_1}^{t_2} \int_\Omega u \Delta(|\Delta u|^{p-1} \Delta u) dx dt \\ &\quad + \int_{t_1}^{t_2} \int_\Omega um^2 u dx dt + \int_{t_1}^{t_2} \int_\Omega u |u_t|^{q-1} u_t dx dt. \end{aligned} \quad (3.14)$$

By using eq. (1.1) and integrating by parts and applying the Cauchy-Schwarz's inequality in the first term and the Hölder's inequality in the second term of the

right-hand side of (3.14), we get

$$\begin{aligned} \int_{t_1}^{t_2} I(t) dt &\leq \|u(t_2)\| \|u_t(t_2)\| + \|u(t_1)\| \|u_t(t_1)\| \\ &\quad + \int_{t_1}^{t_2} \|u_t\|^2 dt - \int_{t_1}^{t_2} \int_{\Omega} u |u_t|^q dx dt. \end{aligned} \quad (3.15)$$

Now, we aim to estimate the last term on the right side of inequality (3.15). Thanks to Hölder's inequality, we have

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\Omega} u |u_t|^q dx dt &\leq \int_{t_1}^{t_2} \left(\int_{\Omega} |u_t|^{q+1} \right)^{\frac{q}{q+1}} \left(\int_{\Omega} |u|^{q+1} \right)^{\frac{1}{q+1}} \\ &= \int_{t_1}^{t_2} \|u_t\|_{q+1}^q \|u\|_{q+1} dt. \end{aligned} \quad (3.16)$$

Thanks to the inequality of Sobolev-Poincaré and (3.9), we get

$$\begin{aligned} &\int_{t_1}^{t_2} \|u_t\|_{q+1}^q \|u\|_{q+1} dt \\ &\leq C^* \int_{t_1}^{t_2} \|u_t\|_{q+1}^q \|\Delta u\| dt \\ &\leq C^* \int_{t_1}^{t_2} \|u_t\|_{q+1}^q \|\Delta u\|_{p+1} dt \\ &\leq C^* \int_{t_1}^{t_2} \|u_t\|_{q+1}^q \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} E(s) \right)^{\frac{1}{p+1}} dt \\ &\leq C^* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \int_{t_1}^{t_2} \|u_t\|_{q+1}^q E^{\frac{1}{p+1}}(s) dt \\ &\leq C^* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) \int_{t_1}^{t_2} \|u_t\|_{q+1}^q dt \\ &\leq C^* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) D^q(t), \end{aligned}$$

and

$$\begin{aligned}
 \int_{t_1}^{t_2} \|u_t(t_i)\|_2 \|u(t_i)\|_2 dt &\leq \int_{t_1}^{t_2} C \|u_t(t_i)\| \|\Delta u\|_{p+1} dt \quad (3.17) \\
 &= CC^* \|u_t(t_i)\| \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \int_{t_1}^{t_2} E^{\frac{1}{p+1}}(s) dt \\
 &= C_1 \|u_t(t_i)\| \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s).
 \end{aligned}$$

By the (3.9), (3.13) and the Lemma 1, we obtain

$$\|u_t(t_i)\| \|u(t_i)\| \leq C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s),$$

where

$$C_1 = CC^* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}}.$$

Then by (3.15), we obtain

$$\begin{aligned}
 &\int_{t_1}^{t_2} I(t) dt \\
 &= C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) + D^2(t) \\
 &+ C_* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) D^q(t) \\
 &= C_3 \left\{ \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) D(t) + D^2(t) \right. \\
 &\left. + C_* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) D^q(t) \right\} \quad (3.18)
 \end{aligned}$$

On the other hand, by using (3.2), (3.3) and Remark 7, we get

$$E(t) \leq \frac{1}{2} \|u_t\|^2 + C_4 I(t) dt, \quad (3.19)$$

where $C_4 = \max \{C_1, C, 1\}$.

By integrating (3.19) over $[t_1, t_2]$, we get

$$\int_{t_1}^{t_2} E(t) dt \leq \frac{1}{2} \int_{t_1}^{t_2} \|u_t\|^2 dt + C_4 \int_{t_1}^{t_2} I(t) dt. \quad (3.20)$$

Then by (3.13), (3.18) and (3.20), we have

$$\begin{aligned} & \int_{t_1}^{t_2} E(t) dt \\ & \leq \frac{1}{2} \int_{t_1}^{t_2} \|u_t\|^2 dt + C_4 \int_{t_1}^{t_2} I(t) dt \\ & \leq \frac{1}{2} C D^2(t) + C_4 C_3 \left[D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) + D^2(t) \right] \\ & + C_* \left(\max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\} \right)^{\frac{1}{p+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{p+1}}(s) D^q(t). \end{aligned} \quad (3.21)$$

By integrating (3.5) over $[t, t_2]$, we have

$$E(t) = E(t_2) + \int_t^{t_2} \|u_\tau\|_{q+1}^{q+1} d\tau. \quad (3.22)$$

Hence, given that $t_2 - t_1 \geq \frac{1}{2}$, we conclude that

$$\begin{aligned} \int_{t_1}^{t_2} E(t) dt & \geq (t_2 - t_1) E(t_2) \\ & \geq \frac{1}{2} E(t_2), \end{aligned}$$

which implies that

$$E(t_2) \leq 2 \int_{t_1}^{t_2} E(t) dt. \quad (3.23)$$

Consequently, exploiting (3.13), (3.21), (3.23) and since $t_1, t_2 \in [t, t+1]$, we have

$$\begin{aligned}
E(t) &\leq 2 \int_{t_1}^{t_2} E(t) dt + \int_t^{t_2} \|u_\tau\|_{q+1}^{q+1} d\tau \\
&= 2 \int_{t_1}^{t_2} E(t) dt + D^{q+1}(t).
\end{aligned} \tag{3.24}$$

Then, by (3.21), we get

$$E(t) \leq (C + 2C_4C_3) D^2(t) + D^{q+1}(t) + C_5 E^{\frac{1}{p+1}}(t) (D(t) + D^q(t)),$$

here $C_5 = 2C_4C_3 \max\left(1, C_*, \max\left\{\frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1}\right\}\right)$.

Hence, we get

$$E(t) \leq C_6 \left[D^2(t) + D^{q+1}(t) + D^{\frac{p+2}{p+1}}(t) + D^{\frac{p+2}{p+1}q}(t) \right], \tag{3.25}$$

where $C_6 = \max\{2C + 4C_4C_3 + C_5^2, 2, C_5^2\}$. Note that, since $E(t)$ is non-increasing and $E(t) \geq 0$ on $[0, \infty)$,

$$\begin{aligned}
D^{q+1}(t) &= E(t) - E(t+1) \\
&\leq E(0).
\end{aligned}$$

Thus, we get

$$D(t) \leq E^{\frac{1}{q+1}}(0). \tag{3.26}$$

By the (3.25) and (3.26), we get

$$\begin{aligned}
E(t) &\leq C_6 D^2(t) \left[1 + D^{q-1}(t) + D^{2(q-1)}(t) \right] \\
&\leq C_6 \left[1 + E^{\frac{q-1}{q+1}}(0) + E^{\frac{2(q-1)}{q+1}}(0) \right] D^2(t) \\
&\leq C_7 D^2(t),
\end{aligned}$$

where $C_7 = C_6 \left[1 + E^{\frac{q-1}{q+1}}(0) + E^{\frac{2(q-1)}{q+1}}(0) \right]$. Thus, we get

$$E(t)^{m+1} \leq (C_7 D^2(t))^{m+1},$$

where $m = \max \left\{ \frac{(r+1)(p+1)}{r-p}, \frac{2(r+1)}{r-1} \right\}$. Thus

$$\begin{aligned}
 E(t)^{m+1} &\leq C_7^{m+1} D^{2m+2}(t) \\
 &= C_7^{m+1} D^{q+1}(t) D^{2m-q+1}(t) \\
 &\leq C_7^{m+1} D^{q+1}(t) E^{\frac{2m-q+1}{q+1}}(0) \\
 &= C_8 D^{q+1}(t) \\
 &= C_8 (E(t) - E(t+1)), \tag{3.27}
 \end{aligned}$$

where $C_8 = C_7^{m+1} E^{\frac{2m-q+1}{q+1}}(0)$.

Case 1: When $q = 1$ and $m = 0$ from (3.27), we have

$$E(t) \leq C_8 (E(t) - E(t+1)).$$

Thanks to the Nakao's Inequality, we have

$$E(t) \leq E(0) e^{-w_1[t-1]^+},$$

here $w_1 = \ln \frac{C_7}{C_7-1}$.

Case 2: When $(m+1)q > 1$, applying Nakao's Inequality to (3.27) yield

$$E(t) \leq (E(0)^{-\alpha} + C_7^{-1} \alpha [t-1]^+)^{-\frac{1}{\alpha}},$$

where $\alpha = \frac{(m+1)q-1}{m+2}$. The proof of this theorem is completed. $\square$

Declarations

Competing interests: The authors declare that they have no competing interests.

Authors' contributions: The authors contributed equally in this manuscript.

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