

First and higher-order differential equations with different arguments

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(Received: December 31, 2025 Accepted: June 3, 2026)

Abstract

Exact solutions are obtained to first-order and higher-order linear and non-linear differential equations with different arguments. The equations are subject to initial conditions.

1 Introduction

In [2], Lefebvre considered the delay differential equations (DDEs)

$$y'(t) = c_1 e^{c_2 t} y(c_0 t) + c_3 y(t) \quad (1.1)$$

and

$$y'(t) = c_0 + c_1 y^2(t/2) \quad (1.2)$$

for $t \geq 0$, where c_i is a constant for $i = 0, 1, 2, 3$. These equations were generalizations of DDEs that appeared, in particular, in [1] and [3]. In these references,

Keywords and phrases: Delay differential equation, non-linear differential equation, initial value problem.

2020 AMS Subject Classification: 34K05, 34K06.

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the above equations were considered in the interval $[0, 1]$, but it is clear that the solutions obtained are valid in $[0, \infty)$.

A DDE is often written in the form

$$y'(t) = y(t - \tau), \quad (1.3)$$

where the delay τ could be negative. In the case of Equation (1.1), the term $y(c_0 t)$ (with $c_0 \neq 1$) can be written as follows:

$$y(c_0 t) = y(t + (c_0 - 1)t). \quad (1.4)$$

Therefore, the delay $\tau = (1 - c_0)t$ is a function of t . Similarly for the term $y^2(t/2)$ in Equation (1.2).

In the case of a constant positive delay τ , we need the value of $y(t)$ in the interval $[-\tau, 0]$, whereas we only need the value of $y(t)$ at time $t = 0$ in Equations (1.1) and (1.2).

DDEs are used in a variety of applications, notably in epidemiology and in mathematical biology; see Smith [5] and Rihan [4]. To obtain approximate solutions to DDEs with a positive delay, many algorithms have been proposed. Sometimes, the solution provided by these algorithms is actually the exact solution to a given equation.

Mathematical software packages such as *Mathematica* and *Maple* can currently only provide numerical solutions to DDEs. In this paper, DDEs that appear in [2] will be further generalized and *exact* solutions that satisfy given initial conditions will be presented. New first-order DDEs will be solved in Section 2, and higher-order equations will be considered in Section 3.

2 First-order equations

I. In [2], it was shown that $y(t) = e^{kt^2}$ is an exact solution to the first-order differential equation

$$y'(t) = c_1 t y(c_2 t) e^{c_3 t^2} \quad \text{for } t \geq 0, \quad (2.1)$$

subject to $y(0) = 1$, where $c_1 \neq 0$, $c_2 \neq 1$ and $c_3 \neq 0$, if and only if

$$k = \frac{c_1}{2} \quad \text{and} \quad k = \frac{c_3}{1 - c_2^2}. \quad (2.2)$$

Equation (2.1) can be generalized as follows:

$$y'(t) = c_1 t^{n-1} y(c_2 t) e^{c_3 t^n} \quad \text{for } t \geq 0, \quad (2.3)$$

where $n \in \{2, 3, \dots\}$. A solution that satisfies the initial condition $y(0) = 1$ is $y(t) = e^{kt^n}$ iff

$$k = \frac{c_1}{n} \quad \text{and} \quad k = \frac{c_3}{1 - c_2^n}. \quad (2.4)$$

We could in fact assume that $n \in (0, \infty)$.

Remark. The fact that $y(t) = e^{kt^n}$ is a solution of Equation (2.3), under the conditions in Equation (2.4), can be checked by substituting the proposed solution into the equation. The same holds true for the various solutions proposed below.

II. The non-linear Equation (1.2) has the solution $y(t) = \sin(kt)$ for $t \geq 0$ such that $y(0) = 0$ iff (see [2]) $k = c_0 = -c_1/2$. We can also state that it has the solution $y(t) = \sinh(kt)$ if $k = c_0 = c_1/2$.

III. Similarly, $y(t) = \sinh(kt)$ satisfies Equation (17) in [2], namely

$$[y'(t)]^2 = c_1 + c_2 y(2t) \quad \text{for } t \geq 0, \quad (2.5)$$

together with the initial condition $y(0) = 0$, iff $k = \sqrt{2c_2}$ and $c_1 = -c_2 < 0$.

Next, we will consider two new DDEs.

IV. Let

$$y'(t) = c_0 + 4y'(t/2) + c_1 y^4(t/4) \quad \text{for } t \geq 0, \quad (2.6)$$

subject to $y(0) = 0$. Using the trigonometric identity

$$8 \sin^4(x) = 3 - 4 \cos(2x) + \cos(4x), \quad (2.7)$$

we find that $y(t) = \sin(kt)$ is a solution to the above initial value problem (IVP) iff $k = -c_0/3 = c_1/8$.

Remark. To obtain the above identity, we can use the formulae $\sin^2(x) = 1 - \cos^2(x)$ and $\cos(2x) = 2 \cos^2(x) - 1$.

V. Consider finally the DDE

$$\cos^2(kt) y'(t) = c_0 + c_1 \frac{\sin(kt)}{1 - \cos(kt)} y(t/2) \quad \text{for } 0 < t < \pi/2, \quad (2.8)$$

such that $y(0) = 0$. It follows from the identity

$$\tan(\theta/2) = \frac{1 - \cos(\theta)}{\sin(\theta)} = \frac{\sin(\theta)}{1 + \cos(\theta)} \quad (2.9)$$

that $y(t) = \tan(kt)$ iff $c_0 + c_1 = k$, where $k > 0$. Moreover, we can replace Equation (2.8) by

$$\cos^2(kt)y'(t) = c_0 + c_1 \frac{1 + \cos(kt)}{\sin(kt)} y(t/2) \quad \text{for } 0 < t < \pi/2. \quad (2.10)$$

3 Higher-order equations

I. Suppose first that Equation (1.2) is replaced by the second-order non-linear DDE

$$y''(t) = c_0 + c_1 y^2(t/2) \quad \text{for } t \geq 0, \quad (3.1)$$

with $y(0) = 1$ and $y'(0) = 0$. A solution is $y(t) = \cos(kt)$ iff $c_0 = k^2$ and $c_1 = -2k^2$.

We can consider the general case

$$y^{(n)}(t) = c_0 + c_1 y^2(t/2) \quad \text{for } t \geq 0, \quad (3.2)$$

subject to the initial conditions $y(0) = 0, y'(0) = 1, \dots$ if n is odd, and $y(0) = 1, y'(0) = 0, \dots$ if n is even.

II. We now turn to Equation (1.1). This equation can be further generalized to

$$y'(t) = c_1 e^{c_2 t} y(c_0 t) + c_3 y(t) + \sum_{j=2}^m d_j y^{(j)}(t), \quad (3.3)$$

where $m \in \{2, 3, \dots\}$, $c_0 \neq 1$ and all the constants are different from zero. The solution that satisfies the initial conditions $y^{(j)}(0) = k^j$, for $j = 0, \dots, m-1$, is $y(t) = e^{kt}$, provided that

$$k = \frac{c_2}{1 - c_0} \quad \text{and} \quad k = c_1 + c_3 + \sum_{j=2}^m d_j k^j. \quad (3.4)$$

When $k = 1$, we must have

$$c_0 + c_2 = 1 \quad \text{and} \quad c_1 + c_3 + \sum_{j=2}^m d_j = 1. \quad (3.5)$$

III. Finally, we can also generalize Equation (1.2):

$$y''(t) = c_0 + c_1 y^2(t/2) + c_2 y^{(4)}(t). \quad (3.6)$$

Assume that we have the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -k^2$ and $y'''(0) = 0$. The solution to this non-linear, fourth-order IVP is $y(t) = \cos(kt)$, iff

$$c_0 + \frac{c_1}{2} = 0 \quad \text{and} \quad -k^2 = \frac{c_1}{2} + c_2 k^4. \quad (3.7)$$

If we suppose that $c_0 = -1$, $c_1 = 2$ and $c_2 = -3/4$, then we find that $k = \pm\sqrt{2}$.

The function $y(t) = \cos(kt)$ is also a solution to

$$y''(t) = c_0 + c_1 y^2(t/2) + c_2 y^{(4)}(t) + c_3 y^{(6)}(t) + \dots + c_m y^{(2m)}(t) \quad (3.8)$$

for $m \in \{2, 3, \dots\}$, subject to certain initial conditions and under appropriate constraints on the coefficients c_j .

Remark. Equation (2.5) can also be generalized to a higher-order equation.

4 Conclusion

In this note, differential equations for the function $y(t)$ in which some terms were evaluated at values different from t , for example $y(t/2)$, were studied. These equations can be considered as DDEs with a negative delay that depends on t .

In Section 2, the equations treated were first-order differential equations, while equations of order 2 or higher were solved in Section 3. In both sections, linear and non-linear equations were considered. We obtained exact analytical solutions to all the equations studied, subject to given initial conditions. As mentioned above, mathematical software packages are not yet capable of solving such problems.

Finally, as in [2], the solutions obtained in this note were expressed in terms of elementary functions such as $\sinh(kt)$ or $\cos(kt)$. We would also like to find solutions involving special functions, for example Bessel functions, to particular DDEs.

Acknowledgements

This research was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). The authors would also like to thank the anonymous reviewers for their constructive comments.

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