

Heat transfer augmentation in MHD hybrid nanofluid flow induced by a rotating disk in the presence of heat source

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Abstract

The present investigation explores the impact caused by a applied magnetic field and internal thermal effect in the area of the flow dynamics and temperature behavior from a hybrid nanofluid motion over a rotating disk. The analysis is based on the classical Von Karman rotating flow model, considering alumina (Al_2O_3) and copper (Cu) nanoparticles were mixed within the carrier fluid such as water. The similarity transformations were retained to convert nonlinear partial differential equations into corresponding ordinary differential equations, which were computed numerically by bvp4c algorithm in MATLAB. The impacts of nondimensional quantities such as Eckert number, magnetic interaction parameter, heat source and Prandtl number on the fluid flow velocities and temperature distributions are evaluated. The findings indicate that the magnetic field strongly influences the flow pattern and enhances the heat transport rate. The present work highlights the improved thermal efficiency analysis of hybrid nanofluids, demonstrating their potential in cooling systems, rotating machinery, electronic devices and thermal energy storage applications where efficient heat dissipation is essential.

Keywords and phrases: Hybrid nanofluid, MHD, Heat Source, Rotating disk, Eckert number.
2020 AMS Subject Classification: 76W05, 65Nxx, 76D05, 76Dxx.

Nomenclature

u, v, w	velocity components of radial, tangential, axial
F, G, H	dimensionless velocities of radial, tangential, axial
η	similarity variable
r, θ, z	cylindrical coordinates
Ω	angular velocity
T	temperature
M^2	magnetic field
Ec	Eckert number
Q	heat generation coefficient
ϕ_1	copper nanoparticles
ϕ_2	alumina nanoparticles

Sub scripts

hnf	hybrid nanofluid
nf	nanofluid
f	fluid
∞	far away from the disc

1 Introduction

The rotating disk flow concept finds wide applications in areas including aerospace systems, thermal power plants, rotating equipment, medical devices, gas turbines, electronic cooling and air purification systems. Heat transfer analysis in rotating disk configurations remains a key subject of interest in engineering and applied physics research. Von Karman [11] was first pioneer to examine steady, axisymmetric boundary layer flow of the fluid produced by a rotating disk using similarity transformation technique. His theoretical analysis established the foundation for understanding the interaction between centrifugal and viscous forces and continues to serve as a cornerstone subsequent research. Cochran [7] expanded Von Karman's work by solving his equations, thereby improving the accuracy of velocity and pressure distributions near the disk. These classical studies laid the theoretical groundwork for modern analyses of rotating disk flows.

Moreover, Gregory et al. [10] provided one of the earliest validations of these theories by examining the stability characteristics of rotating disk boundary layers. Their results clarified the onset of flow instabilities and transition mechanisms, effectively linking theory with experiment. Later, Benton [4] advanced the

theoretical understanding of rotating disk flow stability through asymptotic analysis, showing how small disturbances influence transition behavior. Together, these works strengthened the understanding of rotating boundary layers and laying the groundwork for further explorations of unsteady, magnetohydrodynamic, and non-Newtonian effects. Building on these foundations, Bhatnagar and Zago [5] examined the impacts of viscoelastic on velocity study and secondary flow structures between coaxial rotating disks, demonstrating how elasticity affects boundary layer growth and flow bifurcation. Further, Szeri et al. [23] analyzed the fluid motion between two finite rotating disks and identified the mid-radius Batchelor profile that balances radial and rotational effects. Their Laser-Doppler velocity measurements showed excellent agreement with Galerkin B-spline simulations, validating theoretical predictions. Evans et al. [8] studied movement and thermal transfer between parallel disks using similarity transformations and highlighted the influence of inlet conditions and Reynolds number on shear stress and heat flux findings relevant to rotating disk reactors. Oztekin et al. [16] investigated the stability of viscoelastic Oldroyd-B fluids between large rotating disks and determined the critical parameters for spiral vortex formation, which correlated well with Boger-fluid experiments. Collectively, these progressive studies have significantly advanced the understanding of complex fluid behavior in rotating geometries.

The rotating disk system offers several distinct advantages for fluid studies, such as its well-defined geometry, inherent cylindrical symmetry and suitability for obtaining explicit or semi-analytical solutions. This configuration enables detailed examination of boundary-layer characteristics, flow instabilities, turbulence behavior and viscoelastic effects under controlled laboratory conditions, making it an excellent platform for studying advanced fluids like nanofluids. Compared to standard fluids, nanofluids demonstrate superior heat conduction, enhanced thermal transfer rates and improved convective performance, which make them particularly effective in rotating disk applications. The idea of employing nanoparticles to improve fluid thermal performance was initially presented by Choi and Eastman in [6] who demonstrated in their pioneering work that dispersing nanoparticles in base fluids significantly increases thermal conductivity, thereby making nanofluids highly promising for advanced heat transfer applications. Nanofluids are essentially liquids containing extremely fine particles (1–100 nm) suspended in base fluids, which substantially improve their thermal transport capabilities.

Following this development, Wang et al. [27] experimentally analyzed the heat conduction characteristics containing (Al_2O_3) and (CuO) nanofluids formulated with water, engine oil and ethylene glycol as different base fluids. Their results revealed substantial enhancements over traditional fluids, although conventional

theoretical models could not completely explain the observed behavior. Maiga et al. [14] performed a quantitative analysis of laminar forced convection of nanofluids in heated tubes and between parallel disks reporting a marked increase in heat transfer with elevated nanoparticle concentrations. Similarly, Wang et.al. [26] presented a detailed assessment of convective heat transfer using nanofluids which are suspensions of nanoparticles within base fluids and concluded that nanoparticle addition greatly improves both thermal and flow characteristics of conventional fluids. Their analysis covered forced and free convection systems highlighting the enhanced heat transfer performance of nanofluids and their potential applications in advanced cooling techniques. Bachok et al. [3] analyzed the flow of nanofluid and associated transfer of heat across a porous rotating disk computed through the Keller–Box method, focusing on the impacts of suction/injection and nanoparticle volume fraction. Their results indicated that the Patel conductivity model predicts greater surface heat transfer enhancement than the Maxwell–Garnett model.

Rashidi et al. [18] analyzed production of entropy in steady state magnetohydrodynamic (MHD) nanofluid movement over a porous spinning disk under magnetic field. By applying Von Karman transformations, they numerically studied how magnetic interaction, suction and nanoparticle fraction influence flow, temperature and entropy generation. Their results were consistent with existing literature and identified optimal conditions that minimize entropy valuable for MHD and thermal management applications. Building upon these studies, Turkiilmazoglu [25] extended the analysis fluids with dispersed nanoparticles in water containing various metallic and oxide nanoparticles (Cu , Ag , Al_2O_3 , CuO , TiO_2), using the spectral collocation method. The study revealed that copper nanoparticles provided the highest thermal enhancement among all tested materials. Shah et al. [20] investigated thin film flow of pasty nanofluids over a horizontally rotating disk under nonlinear heat radiation and a constant magnetic field. Applying the homotopy analysis method and ND-Solve, they found that improving the magnetic parameter reduced film thickness, whereas stronger thermal radiation elevated nanofluid temperature results that showed excellent agreement between analytical and numerical solutions.

Khan et al. [12] examined the flow in transient state and thermal transmission of a chemically reactive Maxwell nanofluid on a rotating and vertical motion of disk affected due to magnetic fields, heat radiation and chemical reactions. Through similarity transformations and numerical techniques, they found that Lorentz forces suppressed velocity while increasing temperature and thermophoretic effects thickened both thermal and concentration boundary layers. Ramzan et al. [17] investigated the flow of MHD of an (Al_2O_3) nanoparticles dispersed

in water on top of a rotating disk by incorporating nonlinear radiative thermal energy, viscous dissipation, the influence of nanoparticle size and spacing. Through similarity transformations and the homotopy analysis method, they observed that increasing suction, Eckert number and nanoparticle concentration improved both velocity and heat transfer performance for various nanoparticle sizes. Recently, Yasmin et al. [30] examined electrically conducting nanofluid flow of titanium and aluminum alloy-based nanofluids above a rotating disk under nonlinear radiative thermal energy, chemical reaction and magnetically induced effects. Applying similarity transformations and MATLAB bvp4c solver, they demonstrated that stretching/shrinking disk and magnetic parameters significantly affect velocity, temperature and skin friction, with titanium nanofluids showing stronger thermal responses.

As a result of these advancements, the current research has shifted toward hybrid nanofluids combining two types of nanoparticles to overcome the limitations of mono-nanofluids such as particle agglomeration, instability and saturation of thermal conductivity at high concentrations. Hybrid nanofluids provide superior thermal transport characteristics and improved sustainability, enabling their suitability for high-performance cooling implications. Turcu et al. [24] synthesized hybrid composites of polypyrrole (PPY) and multi-walled carbon nanotubes (MWCNTs), exhibiting enhanced electrical conductivity and mechanical stability. The synergistic effect of conducting polymers and carbon nanostructures resulted in composites with multifunctional properties verified by SEM, TEM and XRD analyses, indicating potential for energy and thermal applications. Significant progress was reported by Suresh et al. [21] who prepared hybrid nanofluids containing a mixture of alumina (Al_2O_3) and copper (Cu) nanoparticles suspended in water using a two-step preparation technique and examined their thermophysical characteristics. These results demonstrated higher thermal conductivity and stability compared to single-particle systems, with (Al_2O_3) enhancing dispersion and Cu contributing superior heat conduction. Subsequent research study by Suresh et al. [22] investigated the characteristics of convective heat transfer also flow resistance for water-based hybrid nanofluids containing (Al_2O_3) and (Cu) nanoparticles in heated tubes, observing up to a 13.5 % rise in Nusselt number at moderate Reynolds numbers. These studies evidenced that combining nanoparticles can synergistically improve heat transfer with manageable increases in viscosity.

Fallah et al. [9] examined $SiC - TiO_2/DO$ of hybrid nanofluid flow across rotating disk under magnetic field action and suction impacts, demonstrating the influence of MHD forces on thermal and velocity distributions. Khan et al. [13] investigated influence of Hall current in copper (Cu) – alumina (Al_2O_3)/water based

hybrid nanofluid move over a time-dependent rotating and radially extending disk which considering Lorentz force, heat radiation and viscous dissipation. Through the use of similarity transformations and MATLAB's `bvp4c`, they detected that Hall currents enhance radial velocity, while magnetic fields reduce velocity and increase temperature, with hybrid nanofluids exhibiting higher heat transfer rates than single nanofluids. Zubair et al. [31] executed a semi analytical study of couple pressure of hybrid nanofluid flow in a spinning Darcy–Forchheimer porous disk influenced by magnetic fields and thermal radiation. By applying similarity transformations and the HAM, they determined that momentum increases with inertial, porosity and couple stress parameters while heat transfer is enhanced by radiation but reduced by thermophoresis, Brownian motion and thermal stratification.

Algehyne et al. [1] inspected MHD mixed convective hybrid nanofluid flow material containing graphene and silver nanoparticles dispersed in a 50:50 mixture of ethylene glycol and water in a rotating disk. Implementing homotopy analysis method, they observed an improvement in heat transfer with higher Biot number and nanoparticle concentration while velocity decreased with stronger magnetic fields. Yaseen et al. [28] investigated the unsteady buoyancy and stagnation point. Their results suggested an increase in velocities in assisting flow which enhanced heat transfer with rising Reynolds number, relevant to many rotating disk applications. Sakkaravarthi et al. [19] evaluated the entropy generation mechanisms of Casson tetra hybrid nanofluid transport in a rotating disk using a Levenberg Marquardt based neural network approach. Extending their approach, the present study applies the Runge–Kutta shooting method with LMA-based optimization to enhance prediction accuracy of thermal and flow behaviors for applications in energy, biomedical, and industrial systems. Mishra et al. [15] explored transient radiative hybrid nanofluid motion over a rotating porous disk considering slip boundary conditions. Expanded upon this concept the magnetic field regulation, radiation, as well as slip effects on flow rate and thermal dispersion characteristics applying similarity transformations for enhanced thermal performance.

The present study focuses on behavior of hybrid nanofluid flow in a rotating disk subjected to magnetic flux influence as well as internal thermal generation. The analysis focuses on radial, tangential and axial velocities together with thermal distribution considering parameters such as magnetic field, heat source, Eckert number and Prandtl number which are additionally expressed through the dimensionless heat-transfer parameter and the surface shear-force coefficient. The thermophysical characteristics of the hybrid nanofluid containing alumina and copper nanoparticles uniformly mixed within the base fluid are modeled to enhance non-dimensional ODEs were numerically solved through the use of the MATLAB

bvp4c procedure. The investigation provides insights into how governing parameters influence velocities and temperature with applications in turbomachinery, cooling of rotating machinery, disk drives, power generation systems, thermal management in aerospace and electronic devices.

2 Mathematical formulation

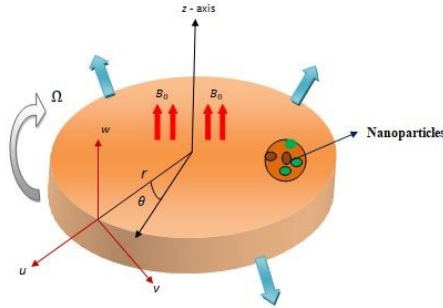


Figure 1: Schematic representation of the system model

The proposed model is formulated using a cylindrical spatial coordinate system (r, θ, z) . Assume a steady, laminar, and electrically conductive hybrid nanofluid flowing above a rotating disk positioned in the xy -plane. The disk rotates about the z -axis at a constant angular speed Ω , inducing motion in the adjacent fluid layers. A constant B_0 is applied along surface normal direction to disk surface which is acting in the z -direction. The disk surface is maintained at a fixed temperature T_w , which exceeds T_∞ . In this study a hybrid nanofluid containing Al_2O_3 and Cu nanoparticles are dispersed within base fluid as water also energy equation include Ohmic heating, viscous dissipation and heat generation which enhances heat transfer rate. The governing equations for mass, momentum and energy conservation were stated in the reference of Bachok et al. [3] and Turkyilmazoglu [25]

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad (2.1)$$

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho_{hnf}} \frac{\partial p}{\partial r} + \nu_{hnf} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} \right) - \frac{\sigma_{hnf}}{\rho_{hnf}} B_0^2 u \quad (2.2)$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \nu_{hnf} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2} \right) - \frac{\sigma_{hnf}}{\rho_{hnf}} B_0^2 v \quad (2.3)$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_{hnf}} \frac{\partial p}{\partial z} + \nu_{hnf} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right) \quad (2.4)$$

$$\begin{aligned} u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \alpha_{hnf} \left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{\sigma_{hnf}}{(\rho C_p)_{hnf}} B_0^2 (u^2 + v^2) \\ + \frac{\mu_{hnf}}{(\rho C_p)_{hnf}} \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right] + \frac{Q_0(T - T_\infty)}{(\rho C_p)_{hnf}} \end{aligned} \quad (2.5)$$

The boundary conditions are

$$\begin{aligned} u = 0, v = v_w = r\Omega, w = 0, T = T_w, \text{ at } z = 0 \\ u \rightarrow 0, v \rightarrow 0, T \rightarrow T_\infty, \text{ as } z \rightarrow \infty. \end{aligned} \quad (2.6)$$

The components of motion radial, tangential and axial directions are represented by u, v and w , where T represents nanofluid temperature, B_0 represents the magnetic field strength where $\alpha_{hnf}, \rho_{hnf}, k_{hnf}, \mu_{hnf}$ correspond to the thermal diffusivity, density, thermal conductivity and viscosity of the hybrid nanofluid respectively. Taking into account the Von Karman relations [11]

$$\begin{aligned} \eta &= \sqrt{\frac{\Omega}{\nu_f}} z \\ [u, v, w] &= [r\Omega F(\eta), r\Omega G(\eta), \sqrt{\nu_f \Omega} H(\eta)] \\ [p, T] &= [p_\infty + 2\Omega\mu_f P(\eta), T_\infty + (T_w - T_\infty)\theta(\eta)] \end{aligned} \quad (2.7)$$

Table 1: Thermal parameters of nanoparticles and base fluid [Yashkun et al. [29]]

Property	Water(H ₂ O)	Copper(Cu)	Alumina(Al ₂ O ₃)
Specific heat [$C_p(J/KgK)$]	4179	385	765
Density of the fluid [$\rho(Kg/m^3)$]	997.1	8933	3970
Electrical conductivity $\sigma(S/m)$	5.5×10^{-6}	5.96×10^7	3.69×10^7
Thermal conductivity [$K(W/mk)$]	0.613	400	40

The following ordinary differential equations can be reduced for equations (2.1) through (2.6) by using the similarity transformations given in (2.7)

$$H' + 2F = 0 \quad (2.8)$$

$$\frac{\nu_{hnf}}{\nu_f} F''' - H F' - F^2 + G^2 - \frac{\sigma_{hnf}/\sigma_f}{\rho_{hnf}/\rho_f} M^2 F = 0 \quad (2.9)$$

$$\frac{\nu_{hnf}}{\nu_f} G'' - H G' - 2FG - \frac{\sigma_{hnf}/\sigma_f}{\rho_{hnf}/\rho_f} M^2 G = 0 \quad (2.10)$$

$$\begin{aligned} \frac{\alpha_{hnf}}{\alpha_f} \theta'' - Pr H \theta' + \frac{(\rho C_p)_f}{(\rho C_p)_{hnf}} Pr \left[\frac{\sigma_{hnf}}{\sigma_f} Ec M^2 (F^2 + G^2) \right. \\ \left. + \frac{\mu_{hnf}}{\mu_f} Ec (F'^2 + G'^2) + Q \theta \right] = 0 \end{aligned} \quad (2.11)$$

The boundary conditions (2.6) were converted into

$$\begin{aligned} F(0) = 0, G(0) = 1, H(0) = 0, \theta(0) = 1 \text{ at } \eta = 0 \\ F(\infty) = 0, G(\infty) = 0, \theta(\infty) = 0 \text{ as } \eta \rightarrow \infty \end{aligned} \quad (2.12)$$

in which $M^2 = \frac{\sigma_f B_0^2}{\rho_f \Omega}$ represents magnetic interaction parameter, $Pr = \frac{\nu_f}{\alpha_f}$ denotes Prandtl number, $Q = \frac{Q_0}{\Omega(\rho C_p)_f}$ represents the heat generation parameter, $Ec = \frac{r^2 \Omega^2 \mu_f}{(\rho C_p)_f \nu_f (T_w - T_\infty)}$ denotes Eckert number.

Table 2: Thermophysical characteristics of hybrid nanofluid [Turkyilmazoglu [25], Alkuhayli et al. [2] and Yasmin et al. [30]]

Thermophysical properties	Hybrid nanofluid values
Thermal diffusivity of the hybrid nanofluid	$\alpha_{hnf} = \frac{k_{hnf}}{(\rho C_p)_{hnf}}$
Dynamic viscosity of the hybrid nanofluid	$\mu_{hnf} = \frac{\mu_f}{(1 - \phi_1 - \phi_2)^{2.5}}$
Density of the hybrid nanofluid	$\rho_{hnf} = \phi_1 \rho_1 + \phi_2 \rho_2 + (1 - \phi_1 - \phi_2) \rho_f$
Specific heat capacity of the hybrid nanofluid	$(\rho C_p)_{hnf} = \phi_1 (\rho C_p)_1 + \phi_2 (\rho C_p)_2 + (1 - \phi_1 - \phi_2) (\rho C_p)_f$
Thermal conductivity of the hybrid nanofluid	$\frac{k_{hnf}}{k_f} = \frac{\left(\frac{\phi_1 k_1 + \phi_2 k_2}{\phi_1 + \phi_2} \right) + 2k_f + 2(\phi_1 k_1 + \phi_2 k_2) - 2(\phi_1 + \phi_2)k_f}{\left(\frac{\phi_1 k_1 + \phi_2 k_2}{\phi_1 + \phi_2} \right) + 2k_f - (\phi_1 k_1 + \phi_2 k_2) + (\phi_1 + \phi_2)k_f}$
Electrical conductivity of the hybrid nanofluid	$\frac{\sigma_{hnf}}{\sigma_f} = \frac{\left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\phi_1 + \phi_2} \right) + 2\sigma_f + 2(\phi_1 \sigma_1 + \phi_2 \sigma_2) - 2(\phi_1 + \phi_2)\sigma_f}{\left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\phi_1 + \phi_2} \right) + 2\sigma_f - (\phi_1 \sigma_1 + \phi_2 \sigma_2) + (\phi_1 + \phi_2)\sigma_f}$

Now define skin friction coefficient at disk surface as follows

$$c_f = \frac{\sqrt{\tau_{wr}^2 + \tau_{w\theta}^2}}{\rho_f (r\Omega)^2} \quad (2.13)$$

in which $\tau_{wr} = \mu_{hnf} \left[\frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right]_{z=0}$ the radial stress and $\tau_{w\theta} = \mu_{hnf} \left[\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right]_{z=0}$ represents tangential stress. Using Eqn. (2.7) in Eqn. (2.13) to get

$$\sqrt{Re} c_f = \frac{1}{(1 - \phi_1 - \phi_2)^{2.5}} \sqrt{F'^2(0) + G'^2(0)}. \quad (2.14)$$

where $Re = \frac{r^2 \Omega}{\nu_f}$ indicates the Reynolds number.

The reduced Nusselt number is defined as

$$Nu = \frac{rq_w}{k_f(T_w - T_\infty)}, \quad (2.15)$$

where $q_w = -k_{hnf} \left(\frac{\partial T}{\partial z} \right)_{z=0}$ the surface heat flux from the disk. Using equation (2.7) in equation (2.15) takes following form

$$\frac{Nu}{\sqrt{Re}} = -\frac{k_{hnf}}{k_f} \theta'(0) \quad (2.16)$$

3 Numerical approach

In this study the system of nonlinear equations (2.8) – (2.11) are worked out under the boundary conditions given in (2.12) which is extremely difficult to solve analytically. Hence, a numerical approach to obtain accurate and reliable solutions using MATLAB's bvp4c solver. The numerical scheme effectively handles boundary value problems involving nonlinear differential equations and provides smooth approximations for the desired functions. The obtained results, including the variations related to the velocity components along with the temperature distribution, can be presented graphically to illustrate the influence of key governing parameters. Furthermore, Table 1 and 2 shown the thermal parameters of nanoparticles and base fluid and as well as the thermophysical characteristics of hybrid nanofluid. The current results are compared with the previous studies found by Bachok et al. [3] and Turkyilmazoglu [25] in the absence of magnetic field, Eckert number, thermal source parameter and copper and alumina nanoparticles with the fixed value of $Pr = 6.2$, it is clear that the present values are highly accurate with all previous results as shown in Table 3. The calculated skin friction values, reduced Nusselt number under various magnetic field, Eckert number and thermal source parameter are shown in Table 4.

Table 3: Evaluation of the numerical results of $F'(0)$, $-G'(0)$, $-H(\infty)$, $-\theta(0)$ obtained for $M^2 = 0$, $Ec = 0$, $Q = 0$, $\phi_1 = 0$, $\phi_2 = 0$ and $Pr = 6.2$

	Bachok et al. [3]	Turkyilmazoglu [25]	Present outcomes
$F'(0)$	0.5102	0.51023262	0.5101
$-G'(0)$	0.6159	0.61592201	0.6159
$-H(\infty)$	–	0.88447411	0.8837

4 Results and discussion

The nonlinear ODEs obtained through similarity transformation were solved numerically by applying MATLAB's bvp4c solver. The calculated values demonstrate the impacts of various governing physical parameters including the magnetic field (M^2), nanoparticle volume fractions $\phi_1(Cu)$ and $\phi_2(Al_2O_3)$, heat source parameter (Q) and Eckert number (Ec) affect the velocities and temperature variation of hybrid nanofluid flow in a rotating disk. The subsequent findings are analysed and discussed in detail through graphical representations displayed in Figures 2–13.

Figure 2 depicts impact of varying M^2 on $F(\eta)$. It is noted that radial velocity progressively suppresses as the magnetic field strength increases. This reduction happens because of Lorentz force induced by magnetic field interacts with motion of the electrically conducting hybrid nanofluid, which opposes direction of the fluid motion. However, as M^2 increases to 0, 1, 2 and 4 the radial velocity near the disk surface gradually decreases and shifts closer to the wall. This indicates that stronger magnetic field suppress the radial flow.

Figure 3 presents how the tangential component of velocity $G(\eta)$ responds differently for different magnetic field values M^2 . It is evident that tangential velocity decreases gradually with M^2 . This reduction arises because M^2 generates a Lorentz force that opposes the rotating movement of the electrically conducting hybrid nanofluid. However, with increasing magnetic field ($M^2 = 0, 1, 2, 4$), the tangential velocity near the disk surface gradually decreases and the velocity profile shifts downward that suppresses the rotational motion.

Figure 4 illustrates the axial velocity $H(\eta)$ is significantly affected by changes in the magnetic field M^2 . It is observed that the axial velocity magnitude shows a progressive damping with increasing magnetic field M^2 . The fluid motion is directed downward towards rotating disk surface when $H(\eta)$ is negative, representing a suction type flow behavior. Increasing the magnetic field, the Lorentz force counteracts fluid motion of the electrically conducting hybrid nanofluid. This electromagnetic braking effect resists the downward flow, thereby reducing the fluid's penetration towards the disk.

Figure 5 shows the effect of the M^2 on the temperature distribution $\theta(\eta)$. It is clear that the temperature rises as M^2 increases. Physically, this occurs because of Lorentz force produced by the applied magnetic field, which resists fluid motion as

well as it generates heat through Joule (Ohmic) heating. The magnetic resistance not only suppresses convective motion but also increases viscous and electromagnetic dissipation, leading to higher fluid temperature. Consequently, thickness of thermal boundary layer grows as M^2 increases.

Figure 6 displays influence of copper nanoparticles (ϕ_1) on the radial direction. The results shows that a slight enhancement in the radial velocity $F(\eta)$ with an increasing copper nanoparticle (ϕ_1). This happens due to the addition of more copper nanoparticles (ϕ_1) improve the fluid's ability at conducting heat, which strengthens the momentum transfer near the disk surface. When ϕ_1 increases from 0.01 to 0.04, the radial velocity shows a small rise near the wall and reaches a slightly higher peak value before gradually decaying to zero away from the disk.

As depicted in Figure 7, increasing copper nanoparticles ϕ_1 from 0.01 to 0.04 suppresses the axial velocity $H(\eta)$ throughout the flow field. This pattern indicates that the axial velocity boundary layer becomes thinner with distance from the surface. Increasing ϕ_1 from 0.01 to 0.04 results in a reduction of axial velocity throughout the region. This behavior suggests that adding more nanoparticles increases the fluid's effective viscosity, which opposes the flow and reduces the flow speed of the hybrid nanofluid.

Figure 8 illustrates the variation of angular direction velocity distribution $G(\eta)$ for varying values of the copper nanoparticles (ϕ_1). The graph reveals that the tangential velocity $G(\eta)$ decreases sharply near the surface and gradually approaches zero as η increases. This behavior indicates that velocity boundary layer reduces showing a smooth decay in the free-stream region.

Figure 9 presents the analysis of radial velocity component $F(\eta)$ for different alumina nanoparticles ϕ_2 which confirms that $F(\eta)$ decreases from wall and approaches zero as η increases. This trend indicates the formation of a boundary layer where the velocity effects are confined close to the surface and diminish further away. The results show that the impact of ϕ_2 on radially directed flow velocity $F(\eta)$ is very small. For all considered values of $\phi_2 = 0.01, 0.02, 0.03, 0.04$ the profiles nearly overlap, which suggests that the alumina nanoparticles have a weak influence on the radial motion of the hybrid nanofluid.

As illustrated in Figure 10 shows $G(\eta)$ for distinct values of alumina nanoparticles ϕ_2 by decreasing the tangential velocity sharply near the wall and gradually

approaches zero as η increases. This indicates that the motion of the fluid is mainly confined to the region near the disk surface also the velocity near-wall layer becomes thinner with distance. The figure also reveals that changes in ϕ_2 have a very small impact on the tangential velocity profile.

Figure 11 depicts variation of $H(\eta)$ for different values of the alumina nanoparticles ϕ_2 when other parameters are kept constant. The results shows that the axial velocity increases with increasing ϕ_2 from 0.01 to 0.04. This behavior is associated with the altered momentum transport caused by the presence of Al_2O_3 nanoparticles. The increased nanoparticle volume fraction strengthens the axial fluid motion near the rotating disk.

Figure 12 illustrates influence of Ec on dimensionless temperature distribution $\theta(\eta)$. It is observed that $\theta(\eta)$ rises progressively with increasing η , indicating that the fluid cools as it progressively departs from the rotating disk surface. However, temperature increases with greater values of Ec (0.001 to 0.004). This occurs because an elevation of the fluid temperature near to the wall and reduction in thickness of thermal boundary layer.

Figure 13 presents the influence of Q on $\theta(\eta)$, while keeping other parameters constants. It is clear that an increment in Q gives a higher fluid temperature across the entire boundary layer region. This is because a larger value of Q represents stronger internal heat generation within the fluid, which enhances the thermal energy and increases the temperature.

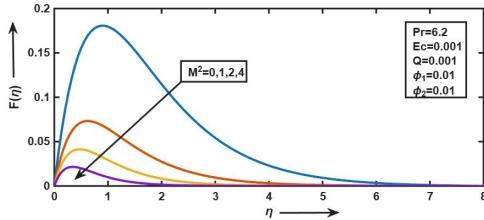


Figure 2: $F(\eta)$ for distinct values of magnetic field

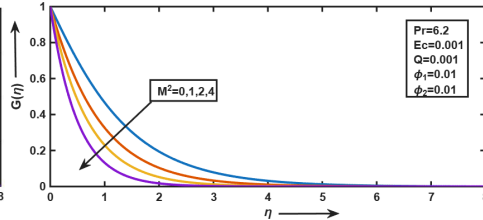


Figure 3: $G(\eta)$ for distinct values of magnetic field

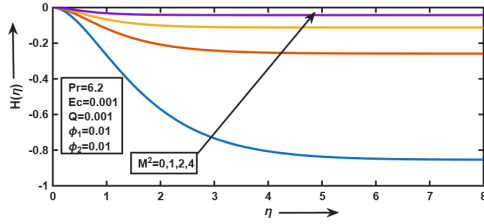


Figure 4: $H(\eta)$ for several values of M^2

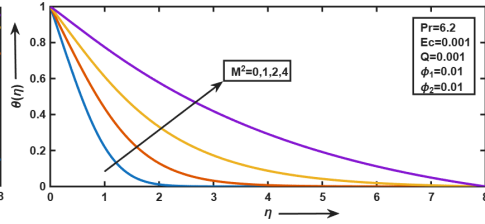


Figure 5: $\theta(\eta)$ corresponding to M^2

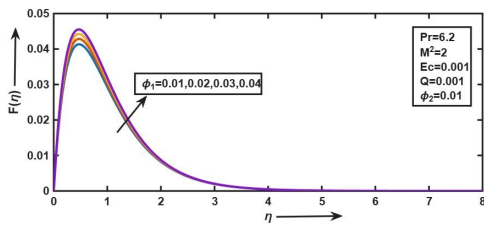


Figure 6: Variation of $F(\eta)$ for ϕ_1

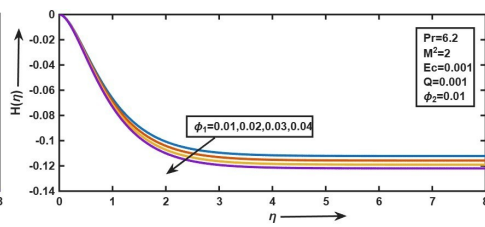


Figure 7: Variation of ϕ_1 on $H(\eta)$

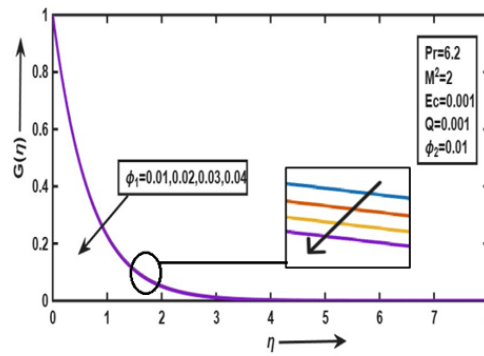


Figure 8: Presence of ϕ_1 on $G(\eta)$

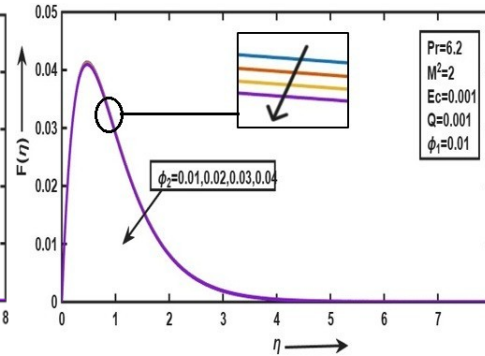


Figure 9: Influence of ϕ_2 on $F(\eta)$

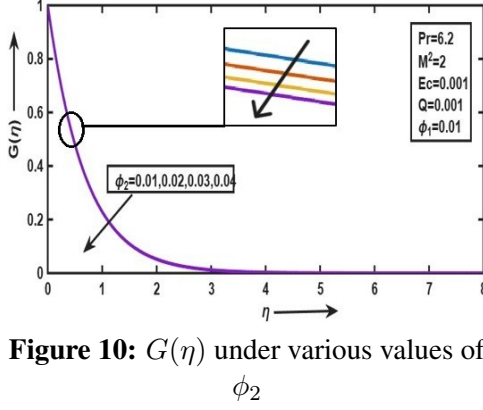


Figure 10: $G(\eta)$ under various values of ϕ_2

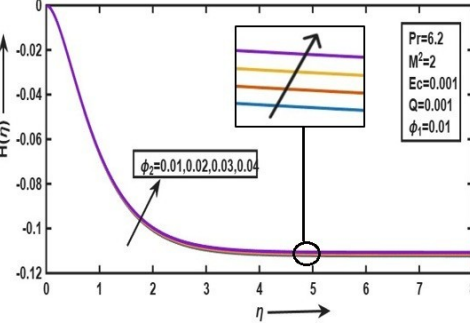


Figure 11: Variations of ϕ_2 in $H(\eta)$

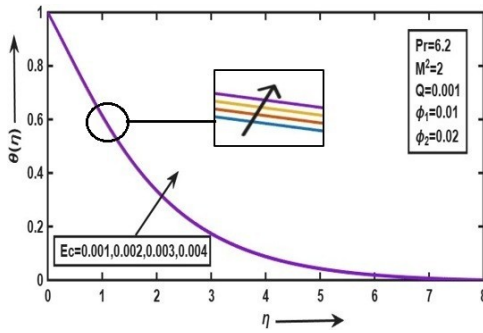


Figure 12: : Different values of Ec in $\theta(\eta)$

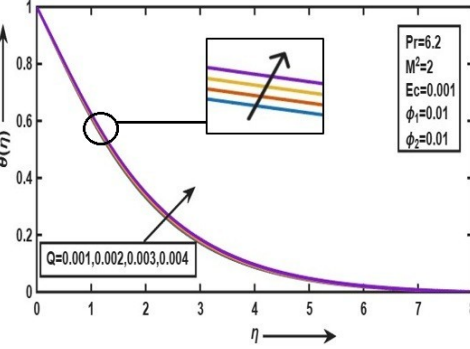


Figure 13: $\theta(\eta)$ corresponding to various values of Q

Table 4 demonstrates the changes in skin friction and reduced Nusselt number obtained for distinct outcomes of magnetic interaction parameter, Eckert number (Ec) and heat production. When M^2 increases as skin friction coefficient also increases and reduced Nusselt number exhibits an opposite variation trend. The Nusselt number decreased with rising Eckert number and the heat generation parameter. Increasing the Ec and Q mainly influences temperature field, with minimal impact on the momentum field. Since the surface velocity gradient remains nearly constant, the skin friction coefficient shows negligible variation. Physically, this indicates that the rise in internal energy from viscous dissipation and heat generation enhances the temperature within the boundary layer but does not alter the shear stress at the wall. Hence, the skin friction values remain stagnant showing negligible.

Table 4: Calculated outcomes of coefficient of skin friction and Nusselt number corresponding to different physical parameters such as $M^2 = 2$, $Ec = 0.001$, $Q = 0.001$, $\phi_1 = 0.01$, $\phi_2 = 0.01$, $Pr = 6.2$

M^2	Ec	Q	$\frac{1}{(1 - \phi_1 - \phi_2)^{2.5}} \sqrt{F'^2 + G'^2}$	$-\frac{k_{hnf}}{k_f} \theta'$
0	0.001	0.001	0.8641	0.9644
1			1.1841	0.6444
2			1.5464	0.4236
4			2.1329	0.1890
2	0.001	0.001	1.5464	0.4236
	0.002		1.5464	0.4154
	0.003		1.5464	0.4071
	0.004		1.5464	0.3988
2	0.001	0.001	1.5464	0.4236
		0.002	1.5464	0.4152
		0.003	1.5464	0.4064
		0.004	1.5464	0.3974

5 Conclusion

This study explores flow of heat transfer behavior in hybrid nanofluid composed of water, copper and alumina nanoparticles above a rotating disk, considering impact of both M^2 (magnetic field) and Q (internal heat generation). By applying similarity transformations, the governing nonlinear PDEs were converted into ODEs and following this, solved numerically in MATLAB through the bvp4c routine to get accurate and convergent results. The obtained results were validated by comparison with previous studies, showing excellent agreement and confirming the reliability of the present model.

From the analysis the following key findings and engineering implications can be drawn

- An improvement in the magnetic field strength (M^2) leads to a decline in

$F(\eta)$ and $G(\eta)$ components, while the rise in magnetic field strength increases the axial velocity and temperature distribution of the fluid flow. In addition, increasing M^2 enhances the skin-friction coefficient but lowers the Nusselt number, signifying greater wall shear and reduced thermal transport.

- The thermal energy rises with increasing Eckert number and heat generation parameter within hybrid nanofluid and decline in the Nusselt number which signifies a reduction in heat transfer efficiency. Moreover, the skin-friction coefficient remains nearly unchanged as the Eckert number and heat-generation parameter increase, showing that these parameters have little effect on the wall shear stress.
- Increasing the copper nanoparticles (ϕ_1) enhances the radial velocity but slightly reduces the tangential and axial velocities indicating a redistribution of momentum transport within the rotating disk boundary layer.
- The addition of alumina nanoparticles (ϕ_2) suppresses the $F(\eta)$ and $G(\eta)$ velocity profiles whereas it enhances the axial velocity profile.

The results of this study have valuable applications in engineering and industrial systems. Rotating disk employing hybrid nanofluids can be effectively utilized in Heat dissipation, energy generation and cooling technologies.

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