The Aligarh Bulletin of Mathematics
Volume 44, Number 2 (2025), 99 - 135
ISSN: 0304-9787

Dynamic geometric solution for real roots of polynomial equations of any degree

Narinder Kumar Wadhawan
Civil Servant, Indian Administrative Service (Retired)
House No. 563, Sector 2, Panchkula–134109, Haryana, India
Email: narinderkw@gmail.com

(Received: September 9, 2025, Accepted: December 27, 2025)

Abstract

This paper presents a dynamic geometric method for solving polynomial equations of any degree, focusing on real roots. By constructing a right-angled triangle and drawing alternating perpendiculars to the base and hypotenuse, we form a geometric equation. Varying the angle yields real roots, with the perpendicular lengths corresponding to the roots. Transformations allow extraction of all real roots.

1 Introduction

Mathematicians have always sought ways to solve polynomial equations of any degree. The significance of this pursuit can be gauged from the fact that Algebra was originally devoted to the study of polynomial equation solutions. The major obstacles to solving these equations were finally overcome in the 19th century, thanks to the Fundamental Theorem of Algebra, the Abel-Ruffini Theorem, and Galois Theory [1], [12]. The techniques developed for solving polynomial equations include bracketing, interpolation, and iterative methods, often in combination [6]. Needless to say, all these methods rely on algebraic approaches.

However, in 1867, the Austrian engineer Édouard Lill introduced a visual method for solving polynomial equations for real roots using geometry. Lill's method in-

Keywords and phrases: Dynamic geometry, geometric solution, polynomial equation, real root, right angled triangle
2020 AMS Subject Classification: 12E12, 65D18, 51M15, 51N15.

volves constructing a path of straight-line segments with angles corresponding to the polynomial's coefficients. In such a geometric construction, the real roots of the polynomial equation correspond to the slopes of other right-angled paths whose vertices lie on each of the original perpendicular segments, with the final perpendicular just scraping past the top of the last original segment [3].

In 1868, Édouard Lill introduced another visual method for finding the complex roots of a polynomial equation, again using geometric construction [4]. Later, Phillips Verner Bradford and Serge Tabachnikov independently rediscovered these geometric methods [2], [8].

Beyond algebraic and geometric approaches, Origami—the Japanese art of paper folding—has also been applied to solving polynomial equations. This introduced an entirely new perspective to the study of algebraic equations [7], [10].

Despite the extensive work done in solving polynomial equations through various methods, this paper has been written with a singular objective: to provide a simple, easily comprehensible, unique, and standalone geometric method for determining real roots of polynomial equations of any degree. The proposed method employs a right-angled triangle, within which a sequence of similar right-angled triangles is inscribed, corresponding to the degree $n > 1$ of the polynomial equation. A real root of a polynomial equation is a value that, when substituted for the variable in the equation, reduces it to zero. If x_1 is the root of the equation, $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$, then value of $x_1^n + a_{n-1}x_1^{n-1} + a_{n-2}x_1^{n-2} + \cdots + a_2x_1^2 + a_1x + a_0$, will equal zero.
To solve geometrically above polynomial equation, a triangle ABC is constructed with right angle at point B, perpendicular AB of unit length, base BC, hypotenuse AC and angle BCA abbreviated angle C of any arbitrary value. From point B, a perpendicular BD_1 is dawn upon hypotenuse AC meeting it at point D_1, from point D_1, a perpendicular D_1D_2 is drawn upon base BC meeting it at point D_2, from point D_2, a perpendicular D_2D_3 is drawn upon hypotenuse AC and this process of drawing perpendiculars upon base and hypotenuse alternately, is continued till last perpendicular $D_{n-1}D_n$ is drawn. A geometric construction as shown in Figure 1 will, then evolve. This geometric construction will invariably be used in this paper. Length of perpendicular AB will always be kept unity unless it is otherwise specified. If length of AB is omitted mentioning, it will be presumed equal to unity. Perpendiculars $BD_1, D_1D_2,\ D_2D_3,\ \ldots,\ D_{n-1}D_n$ will be considered 1^{st}, 2^{nd}, 3^{rd}, …, n^{th} perpendiculars respectively. Referring to Figure 1 and considering triangle ABD_1, its side $BD_1 = AB\ cos\ C$ and in triangle BD_1D_2, its side $D_1D_2 = BD_1 cos\ C$. Substituting value of BD_1 equal

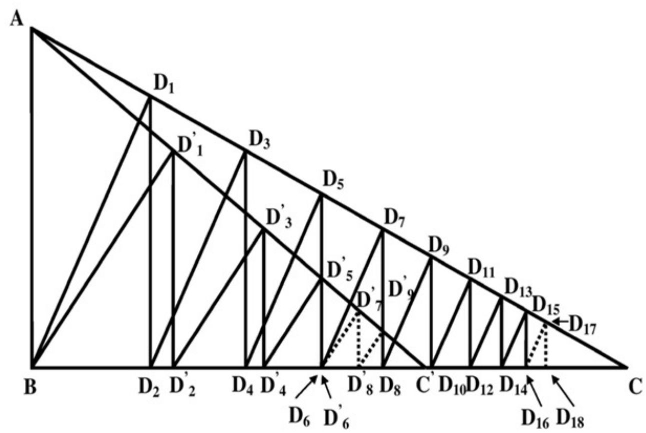


Figure 1: Geometric construction for solution of polynomial equation of any degree.

to $AB\ cos\ C$, makes $D_1D_2 = AB\ cos^2\ C$. Considering triangle $D_1D_2D_3$, its side $D_2D_3 = D_1D_2\ cos\ C$ and substituting value of $D_1D_2 = AB\ cos^2\ C$, yields $D_2D_3 = AB\ cos^3\ C$ [11, 17, 18]. Proceeding in this manner, $D_{n-1}D_n = AB\ cos^n\ C$. Since length AB is assumed equal to unity [11, 17, 18].

$$BD_1 = \cos \mathrm{C}, D_1D_2 = cos^2\ C, D_2D_3 = \ cos^3\ C,$$

$$D_3D_4 = \ cos^4\ C, D_4D_5 = \ cos^5\ C, \ldots, D_{n-1}D_n = \ cos^n\ C. \tag{1.1}$$

These equations will facilitate solution of the polynomial equations. For numerical solutions, algorithm will also be devised, and examples will also be given to prove the veracity of results.

1.1 Dynamic geometry

Dynamic geometry refers to a geometric framework in which an angle or side of a geometric figure is varied continuously to satisfy a given condition. It is particularly useful when constructing a static figure according to predefined conditions is difficult or impractical.

For instance, in Equation (6.1), while the perpendicular segments $BD_1,\ D_1D_2$, $D_2D_3, \ldots D_{n-1}\ D_n$ can represent $x,\ x^2,\ x^3, \ldots, x^n$, the equation is only satisfied for a specific value of x. This required value can be obtained by adjusting angle C dynamically. The variation of angle C results in a continuous change in the geometric parameters, making the construction dynamic rather than static. A static geometric figure, due to its fixed structure, lacks the flexibility to satisfy the polynomial equation exactly. However, dynamic geometry, by allowing controlled variation of angle C, enables the precise satisfaction of the equation, thereby solving the polynomial equation geometrically.

2 Geometric solution of polynomial equations of the form $x^n - Px = -Q$, where P and Q are neither 0 nor ∞

Polynomial equations

$$x^n - Px = -Q, \tag{2.1}$$

$$x^n + Px = Q, \tag{2.2}$$

on substitutions of $y = x/P^{1/(n-1)}$ and $S = Q/P^{n/(n-1)}$, transform to

$$y^n - y = -S, \tag{2.3}$$

$$y^n + y = S \tag{2.4}$$

respectively. Equations (2.3) and (2.4) will be analysed for extracting roots between 0 and 1.

2.1 Geometric solution of polynomail equations of the form $y^n - y = -S$, where $0 < S \le (n-1)/n^{n/(n-1)}$ and $n-1$ is an integer

2.1.1 When $n > 1$ *is an odd integer*

Point of maxima of $y - y^n$ is determined by equating derivative of $y - y^n = -S$ with 0. That yields maximum value of $y - y^n$ at $y_m = 1/n^{1/(n-1)}$ and *maximum* value of $y_m - y_m^n = (n-1)/n^{n/(n-1)}$. At $y = 0$ and $y = 1$, value of S is zero. When value of y increases from 0, value of S increases smoothly and when $y = 1/n^{1/(n-1)}$, S reaches maximum value $(n-1)/n^{n/(n-1)}$. When value of y increases above $1/n^{1/(n-1)}$, then value of S decreases and finally reaches zero. To explain variation of S with y, their figures are given in Table 1. At $y = 1/7^{1/6}$, S has the maximum value $6/7^{6/7}$ and, when $0 < y < 1/7^{1/6}$ or $1 > y > 1/7^{1/6}$, value of S is less than $6/7^{6/7}$. That means, for $S < 6/7^{6/7}$, there are two corresponding values of y, out of which one value of y is less than $1/7^{1/6}$ and other more than $1/7^{1/6}$. When $S = 6/7^{6/7}$, there are two values of y and both equal

Table 1: Displaying variation of S with y when $n = 7$

$S.\,N.$	y	S	$S.N.$	y	S
1	0.0	0.0000000	7	0.6	0.5720064
2	0.1	0.0999999	8	0.7	0.6176457
3	0.2	0.1999872	9	$1/7^{1/6}$	0.61973614
4	0.3	0.2997813	10	0.8	0.5902848
5	0.4	0.3983616	11	0.9	0.4217031
6	0.5	0.4921875	12	1.0	0.0000000

$1/7^{1/6}$. These results can be generalised for n by stating that polynomial equation $y - y^n = S$, where $0 < S < (n-1)/n^{n/(n-1)}$, has two roots between 0 and 1. When $S = (n-1)/n^{n/(n-1)}$, the equation still has two roots but both equal $1/n^{1/(n-1)}$. For extracting roots between 0 and -1, substitution $y = -Y$, is made and that transforms the Equation (2.3) to $Y^n - Y = S$. As $Y > Y^n$, when Y varies from 0 to 1 and S is positive, this equation does not have any root between 0 and 1. For extracting root between $+1$ and plus infinity, substitution of $y = 1/Y$ is made and that transforms the equation to $R - RY^{n-1} = -Y^n$, where $R = 1/S$. From this equation $R = -Y^n/(1 - Y^{n-1})$ and since Y varies from 0 to 1, therefore, R should be negative but given $R = 1/S$, where given S is positive, therefore, R is always positive, hence equation $R - RY^{n-1} = -Y^n$ does not have any root between 0 and 1. For extracting root between -1 and minus infinity, substitution of $y = -1/Y$ is made and that transforms the equation to $R - RY^{n-1} = Y^n$, where $R = 1/S$. From this equation $R = Y^n/(1 - Y^{n-1})$ and since $R > n^{n/(n-1)}/(n-1)$ and $n^{n/(n-1)}/(n-1) > 1$, therefore, equation $R = Y^n/(1 - Y^{n-1})$ is satisfiable, when Y varies from 0 to 1. Hence equation, $R - RY^{n-1} = Y^n$, is solvable for a root between between 0 and 1. In other words, Equation (2.2) has a root between -1 and minus infinity. In this way, Equation (2.2) has three real roots, two roots lie between 0 and 1, whereas third root lies between -1 and minus ∞.

2.1.1.1. *Geometric construction*: A triangle ABC is drawn with right angle at point B, base BC, perpendicular AB, hypotenuse AC and angle BCA abbreviated angle C of any arbitrary value. Perpendicular BD_1 is drawn upon hypotenuse AC meeting it at point D_1, perpendicular D_1D_2 is drawn upon base BC meeting it at point D_2, perpendicular D_2D_3 is drawn upon hypotenuse AC meeting it at point D_3 and perpendiculars, in this way, are drawn successive till last perpendicular $D_{n-1}D_n$ is reached. It is reiteration of method of construction explained in introduction section.

2.1.1.2. *Theory*: Using Equations (1.1), $y - y^n = S$ and $R - RY^{n-1} = Y^n$ can be written by geometric equation, $BD_1 - D_{n-1}D_n = S$ and $R\ (D_{n-2}D_{n-1}) + (D_{n-1}D_n) = R$ respectively.

2.1.1.3. *Geometric Solution:* Geometric Figure 1, is constructed and *angle C is varied, keeping length of perpendicular AB equal to unity and fresh perpendiculars BD_1', $D_1'\ D_2'$, $D_2'\ D_3', \ldots, D_{n-1}'\ D_n'$ are drawn. It is checked whether length $BD_1' - D_{n-1}'D_n'$ equals S or not. If the length equals S, then roots of equation $y - y^n = S$ is (BD_1'). To find second root, angle C is varied and scanned through 90 degrees till $BD_1'' - D_{n-1}''D_n''$ equals S, then second root of equation is (BD_1''). For finding third root, angle C is varied, keeping length of perpendicular $AB = 1$ till length $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''')$ equals R, where $R = 1/S$, then third root of equation is $-1/(BD_1''')$. Thus, equation $y^n - y = -S$, has three real roots, $(BD_1'), (BD_1'')$ and $-1/(BD_1''')$, when $0 < S \leq (n-1)/n^{n/(n-1)}$ and $n > 1$ is an odd integer.*

2.1.2 Solution of equations $y^n - y = S$, *when* $n > 1$, *is an odd integer*

For solving $y^n - y = S$, when $n > 1$ is an odd integer, y is substituted with $-Y$ and that transforms this equation to $Y^n - Y = -S$. Geometric solution of this equation is already given in section 2.1.1 and is not repeated here. However, equation $y^n - y = S$, will have three roots, $-(BD_1'), -(BD_1'')$ and $1/(BD_1''')$, where $0 < S \leq (n-1)/n^{n/(n-1)}$ and n is an odd integer.

2.1.3 Solution of equations $y^n - y = -S$, *when* $n \geq 2$ *is an even integer*

2.1.3.1 *Roots of equation:* This equation $y^n - y = -S$, also has maximum value $(n-1)/n^{n/(n-1)}$ at $y = 1/n^{1/(n-1)}$; its explanation has already been given in section 2.1.1. This equation also has two distinct real roots, when $0 < S < (n-1)/n^{n/(n-1)}$ and when $S = (n-1)/n^{n/(n-1)}$, the equation has two equal roots i.e., $n^{1/(n-1)}$ and $n^{1/(n-1)}$.

For extracting third root, substitution of $y = -Y$ is made and that transforms the equation to $Y^n + Y = -S$, which has left hand side a positive quantity as variation of Y is between 0 to $+1$ and right hand side, a negative quantity, since S is positive. This equation $Y^n + Y = -S$, is not solvable for roots between 0 and 1.

For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$, is made and that transforms the equation to $R + RY^{n-1} = -Y^n$, which has left hand side a positive quantity as Y varies from 0 to $+1$ and right-hand side a negative quantity. This equation $R+RY^{n-1}=-Y^n$, is also not solvable for roots between 0 and 1. Therefore, $y^n-y=-S$, when $n\geq 2$, is an even integer and $0<S\leq (n-1)/n^{n/(n-1)}$, has only two real roots (BD_1') and (BD_1'') .

2.1.3.2 *Geometric construction, Theory and Geometric Solution:* Geometric construction, Theory and Geometric Solution are same as given for $y^n-y=-S$, when S is odd except the fact that this equation has only two real roots between 0 and 1. Therefore, equation $\mathrm{y}^n-y=-S$, has two real roots, (BD_1') and (BD_1'') , where $0<S\leq(n-1)/n^{n/(n-1)}$ and $n\geq 2$ is an even integer.

2.1.4 Solution of equation $y^n-y=S$, *when* $n\geq 2$ *is an even integer*

2.1.4.1. Roots of equation: Equation $\mathrm{y^n-y=S}$, obviously does not have solution for roots between 0 and 1 as its left-hand side is negative, whereas right hand side positive. For extracting roots between 0 and -1, substitution of $y=-Y$, transforms the equation to $Y^n+Y=S$ and this equation has maximum value at $Y=-1/n^{1/(n-1)}$. Since $Y=-1/n^{1/(n-1)}$, is a negative quantity, therefore, it is ignored as variation of Y is being considered from 0 and 1. Right hand side of equation $Y^n+Y=S$, is always positive but less than 1 as $0<S\leq(n-1)/n^{n/(n-1)}$ and left-hand side of this equation is positive and varies from 0 to 2, therefore, equation $Y^n+Y=S$ is solvable for one real root or equation $y^n-y=S$ has one roots between 0 and -1.

For extracting roots between 1 and plus infinity, substitution of $y=1/Y$ is made and that transforms the equation to $R-R\,Y^{n-1}=Y^n$. From this equation, $R=Y^n/(1-Y^{n-1})$, thus R can acquire values more than 1. Also given $R=1/S$, where $S<1$, therefore, equation $R-R\,Y^{n-1}=Y^n$ is solvable for roots between 0 and 1 or equation $y^n-y=S$ has one root between 1 and plus infinity.

For extracting roots between -1 and minus infinity, substitution of $y=-1/Y$, transforms the equation to $R+R\,Y^{n-1}=Y^n$. From this equation, $R=Y^n/(1+Y^{n-1})$ and thus R can acquire value between 0 and $1/2$ whereas given $R=1/S$ where $S<1$. That means given R is more than 1, therefore, the equation $R+R\,Y^{n-1}=Y^n$, can not be satisfied, hence it is not solvable for roots between 0 and 1.

2.1.4.2. *Geometric Solution:* Geometric solution for equation $y^n-y=S$, is same as given in section 2.1.1.3 except the fact that length $D_{n-1}^{''''}D_n^{''''}+BD_1^{''''}$ will be compared with S, value of angle C varied so that the length equals S and also length $D_{n-1}^{'''}D_n^{'''}+R(D_{n-2}^{'''}D_{n-1}^{'''})$ will be compared with R and the value of angle C is varied so that this length equals R, where $R=1/S$, then the equation

$\mathrm{y^n - y = S}$, has two real roots, $-\left(BD_1^{''''}\right)$, and $1/\left(BD_1^{'''}\right)$, where $0 < S \leq (n-1)/\mathrm{n}^{\mathrm{n}/(\mathrm{n}-1)}$ and $n \geq 2$ is an even integer.

2.2 Geometric solution of polynomial equations of the form $Y^n - Y = -S$, where $\infty > S \geq (n-1)/n^{n/(n-1)}$ and $n > 1$ is an integer

2.2.1 Solution of equation $y^n - y = -S$, *where* $\infty > S \geq (n-1)/n^{n/(n-1)}$ *and* $n > 1$ *is an odd integer*

2.2.1.1 Roots of equation $y^n - y = -S$, *when* $n > 1$ *is an odd integer:* Equation $y^n - y = -S$, obviously does not have a solution when y varies between 0 and 1, since it has already been explained that S can have value more than $(n-1)/n^{n/(n-1)}$. For extracting roots between 0 and -1, substitution of $y = -Y$ is made and that transforms the equation to $Y^n - Y = S$. For variation of Y between 0 and 1, this equation has left hand side negative and right hand side positive, therefore, this equation is not solvable for roots between 0 and 1.

For extracting roots between 1 and plus infinity, substitution of $y = 1/Y$ transforms the equation $y^n - y = -S$ to $R - R\,Y^{n-1} = -Y^n$. From this equation, $R = -Y^n/(1 - Y^{n-1})$ and since Y values from 0 to 1, therefore, R is negative but given R is positive, since R being equal to $1/S$, and S is positive, therefore, equation $R = R\,Y^{n-1} - Y^n$, is not solvable for roots between 0 and 1.

For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation $y^n - y = -S$ to $R - R\,Y^{n-1} = Y^n$. From this equation, $R = Y^n/(1 - Y^{n-1})$ and since Y varies from 0 to 1, therefore, R varies from 0 to infinity but it is given $0 < R \leq (n-1)/n^{n/(n-1)}$, where R being equal to $1/S$ and $S < 1$, is more than 1, hence equation $R = R\,Y^{n-1} + Y^n$ is solvable for roots between 0 and 1.

2.2.1.2. *Geometric solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Procedure for extracting roots is same as explained in section 2.1.1.3 except the fact that length $D_{n-1}^{'''}D_n^{'''} + R(D_{n-2}^{'''}D_{n-1}^{'''})$ will be compared with R and when both equal, then root of equations $\mathrm{y^n - y = -S}$, is $-1/BD_1^{'''}$, where $\infty > S \geq (n-1)/\mathrm{n}^{\mathrm{n}/(\mathrm{n}-1)}$ and $\mathrm{n} > 1$ is an odd integer.

2.2.2 Solution of equation $y^n - y = S$, *where* $\infty > S \geq (n-1)/n^{n/(n-1)}$ *and* $n > 1$ *is an odd integer*

If y is substituted as $-Y$ in the above said equation, then the equation transforms to $Y^n - Y = -S$ and this equation is same as solved in section 2.2.1, therefore its solution is not repeated here. However, the equation will have only one real root $1/BD_1^{'''}$,

2.2.3 Solution of equation $y^n - y = -S$, *where* $\infty > S \geq (n-1)/n^{n/(n-1)}$ *and* $n \geq 2$ *is an even integer*

2.2.3.1 Roots of above said equation, $y^n - y = -S$, *when* $n \geq$ *is an even integer:* As it has been explained in section 2.2.1.1, this equation does not have a root between 0 and 1. For extracting roots between 0 and -1, substitution of $y = -Y$, transforms the equation to $Y^n + Y = -S$. This equation is not solvable for roots between 0 and 1 as left-hand side is positive, whereas right hand side is negative.

For extracting roots between 1 and plus infinity, substitution of $y = 1/Y$, transforms the equation to $R - R\,Y^{n-1} = -Y^{n-1}$, this equation is not solvable for roots between 0 and 1 as has already been explained.

For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$, transforms the equation to $R + R\,Y^{n-1} = -Y^{n-1}$, this equation is also not solvable for roots between 0 and 1 as has already been explained. In this way, equation $y^n - y = -S$ is not solvable for real roots when $\infty > S \geq (n-1)/\mathrm{n}^{\mathrm{n}/(\mathrm{n}-1)}$ and $\mathrm{n} \geq 2$ is an even integer.

2.2.4 Solution of equation $y^n - y = S$, *where* $\infty > S \geq (n-1)/n^{n/(n-1)}$ *and* $n \geq 2$ *is an even integer*

2.2.4.1 Roots of above said equation, $y^n - y = S$, *when* $n \geq 2$ *is an even integer:* As it has been explained in section 2.2.1.1, this equation does not have a root between 0 and 1. For extracting roots between 0 and -1, substitution of $y = -Y$, transforms the equation to $Y^n + Y = S$ and this equation is solvable for roots between, 0 and 1 as left-hand side is positive and right-hand side is also positive, when $(n-1)/n^{n/(n-1)} < S \leq 2$. But when $S > 2$, this equation is not solvable. Hence equation $Y^n + Y = S$ is solvable for roots between 0 and 1 when $2 > S \geq (n-1)/n^{n/(n-1)}$.

For extracting roots between 1 and plus infinity, substitution of $y = 1/Y$, transforms the equation to $R - R\,Y^{n-1} = Y^{n-1}$, this equation is solvable when $0 < R < n^{n/(n-1)}/(n-1)$. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation to $R + R\,Y^{n-1} = Y^n$, this equation is solvable for roots between 0 and 1 when R lies between 0 and $1/2$ i.e. when $\infty > S > 2$.

2.2.4.2 *Geometric solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Procedure for extracting roots is same as explained in section 2.1.1.3 except the facts that length $D^{''''}_{n-1}D^{''''}_{n} + BD^{''''}_{1}$ is compared with S, and length $D^{'''}_{n-1}D^{'''}_{n} + R\left(D^{'''}_{n-2}D^{'''}_{n-1}\right)$ is compared with R and length $D^{'''}_{n-1}D^{'''}_{n} - R(D^{'''}_{n-2}D^{'''}_{n-1})$ is compared with R, where $R = 1/S$. When on varying angle C, the length $D^{''''}_{n-1}D^{''''}_{n} + BD^{''''}_{1}$ equals S, then equation, $y^n - y = S$ has one root $-BD^{''''}_{1}$, when $2 > S \geq (n-1)/n^{n/(n-1)}$ and $n \geq 2$ is an even integer. When on varying angle C, the length $D^{'''}_{n-1}D^{'''}_{n} + R(D^{'''}_{n-2}D^{'''}_{n-1})$ equals R, then equation, $y^n - y = S$, has one root $1/BD^{'''}_{1}$, when $\infty > S \geq (n-1)/n^{n/(n-1)}$ and $n \geq 2$ is an even integer. When on varying angle C, the length $D^{'''}_{n-1}D^{'''}_{n} - R(D^{'''}_{n-2}D^{'''}_{n-1})$ equals R, then equation, $y^n - y = S$, has one root $-1/BD^{'''''}_{1}$ when $\infty > S \geq 2$ and $n \geq 2$ is an even integer.

2.3 Geometric solution of polynomial equations of the form $Y^n + Y = S$, where $0 < S \leq 2$ and $n > 1$ is an integer

2.3.1 Solution of equation $y^n + y = S$, *where* $0 < S \leq 2$ *and* $n > 1$ *is an odd integer*

2.3.1.1. *Roots of equation* $y^n + y = S$: When y varies from 0 to 1, equation $y^n + y = S$, is solvable for one real root as S varies from 0 to 2. For extracting roots between 0 and -1, substitution of $y = -Y$ transforms the equation to $Y^n + Y = -S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is positive and right-hand side negative. For extracting roots between 1 and infinity, substitution of $y = 1/Y$ transforms the equation to $R + RY^{n-1} = Y^n$, where $R = 1/S$. This equation is not solvable for roots between 0 and 1 as value of R obtained from equation, $R = Y^n/(1 + Y^{n-1})$ varies from 0 to $1/2$ where as given value of R is more than and less than infinity. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation to $R + RY^{n-1} = -Y^n$, where $R = 1/S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is negative and right-hand side positive.

2.3.1.2. *Geometric Solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Procedure for extracting roots is same as explained in section 2.1.1.3 except the facts that length $D^{''''}_{n-1}D^{''''}_{n} + BD^{''''}_{1}$ is compared with S and when on varying the value of angle C, the length $D^{''''}_{n-1}D^{''''}_{n} + BD^{''''}_{1}$ equals S, then equation, $y^n + y = S$ has one real root $BD^{''''}_{1}$, when $0 < S \leq 2$ and $n > 1$ is an odd integer.

2.3.2 Solution of equation $y^n + y = -S$, *where* $0 < S \leq 2$ *and* $n > 1$ *is an odd integer*

Substitution of $y = -Y$, transforms the equation $y^n + y = -S$ to equation $Y^n + Y = S$ and this equation has already been solved in section 2.3.1. Its solution is not repeated here.

2.3.3 Solution of equation $y^n + y = S$, *where* $0 < S \leq 2$ *and* $n \geq 2$, *is an even integer*

2.3.3.1. *Roots of equation* $y^n + y = S$: When y varies from 0 to 1, equation $y^n + y = S$, is solvable for one real root as S varies from 0 to 2. For extracting roots between 0 and -1, substitution of $y = -Y$ transforms the equation to $Y^n - Y = S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is negative and right hand side positive. For extracting roots between 1 and infinity, substitution of $y = 1/Y$ is made and that transforms the equation to $R + RY^{n-1} = Y^n$, where $R = 1/S$. This equation is not solvable for roots between 0 and 1 as value of R obtained from equation $R = Y^n/(1 + Y^{n-1})$ varies from 0 to $1/2$ whereas given value of R is more than $1/2$ and less than infinity. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation to $R - RY^{n-1} = Y^n$, where $R = 1/S$. From this equation, $= Y^n/(1 - Y^{n-1})$, therefore, R varies from 0 to infinity and given R varies from ½ to infinity, therefore, equation $R = RY^{n-1} + Y^n$, is solvable for roots between 0 and 1.

2.3.3.2 *Geometric Solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Procedure for extracting roots is same as explained in section 2.1.1.3 except the facts that length $D^{''''}_{n-1}D^{''''}_{n} + BD^{''''}_{1}$ is compared with, S and length $D^{'''}_{n-1}D^{'''}_{n} + R(D^{'''}_{n-2}D^{'''}_{n-1})$ is compared with R . When length $D^{''''}_{n-1}D^{''''}_{n} + BD^{''''}_{1}$ equals S, on variation of value of angle C, then one root of equation $y^n + y = S$, is $BD^{''''}_{1}$, where $0 < S \leq 2$ *and* $n \geq 2$, is an even integer. When length $D^{'''}_{n-1}D^{'''}_{n} + R(D^{'''}_{n-2}D^{'''}_{n-1})$ equals R on variation of value of angle C, then one root of equation $y^n + y = S$, *is* $-1/BD^{'''}_{1}$, *where* $0 < S \leq 2$ *and*

$n \geq 2,$ *is an even integer.*

2.3.4 Solution of equation $y^n + y = -S$, *where* $0 < S \leq 2$ *and* $n \geq 2$, *is an even integer*

2.3.4.1 *Roots of equation* $y^n + y = -S$: when y varies from 0 to 1, equation $y^n + y = -S$, is not solvable for real roots between 0 and 1 as left hand side of this equation is positive, whereas right hand side negative. For extracting roots between 0 and -1, substitution of $y = -Y$ is made and that transforms the equation to $Y^n - Y = -S$. This equation is solvable for two roots between 0 and 1, when $0 < S \leq (n-1)/n^{n/(n-1)}$. Its complete solution is given in section 2.1.3. For extracting roots between 1 and infinity, substitution of $y = 1/Y$, transforms the equation to $R + RY^{n-1} = -Y^n$. This equation is not solvable for roots between 0 and 1 as left hand side of this equation is positive, whereas its right hand side is negative. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation to $R - RY^{n-1} = -Y^n$, where $R = 1/S$. From this equation, $R = -Y^n/(1 - Y^{n-1})$, therefore, R is negative whereas given R is positive, therefore, equation $R = RY^{n-1} - Y^n$, is not solvable for roots between 0 and 1.

2.3.4.2 *Geometric Solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Its complete solution is given in section 2.1.3, when $0 < S \leq (n-1)/n^{n/(n-1)}$. Roots of $y^n + y = -S$ are $-BD'_1$ and $-BD''_1$, when $0 < S \leq (n-1)/n^{n/(n-1)}$ and there is no real root when $2 > S > (n-1)/n^{n/(n-1)}$.

2.4 Geometric solution of polynomial equations of the form $y^n + y = S$, *where* $\infty > S > 2$ *and* $\mathrm{n} > 1$ *is an integer*

2.4.1 Solution of equation $y^n + y = S$, *where* $\infty > S > 2$ *and* $n > 1$ *is an odd integer*

2.4.1.1 *Roots of equation* $y^n + y = S$: When y varies from 0 to 1, equation $y^n + y = S$, is not solvable for roots between 0 and 1 since $\infty > S > 2$. For extracting roots between 0 and -1, substitution of $y = -Y$ transforms the equation to $Y^n + Y = -S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is positive and right-hand side negative. For extracting roots between 1 and infinity, substitution of $y = 1/Y$, transforms the equation to $R + R\,Y^{n-1} = Y^n$, where $R = 1/S$. This equation is solvable for roots between 0 and 1 as value of R obtained from equation $R = Y^n/(1 + Y^{n-1})$ varies from 0 to 1/2 and given value of R is less than 1/2 and more than 0. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ is made and that transforms the equation to $R + RY^{n-1} = -Y^n$, where $R = 1/S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is negative and right-hand side positive.

2.4.1.2 *Geometric Solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Procedure for extracting roots is same as explained in section 2.1.1.3 except the facts that length $D''''''_{n-1}D''''''_n - R(D''''''_{n-1}D''''''_n)$ is compared with R and when on varying value of angle C, the length $D''''''_{n-1}D''''''_n - R(D''''''_{n-1}D''''''_n)$ equals R, then equation, $y^n + y = S$ has one real root $1/BD''''''_1$, when $\infty > S > 2$ *and* $n > 1$ *is an odd integer.*

2.4.2 Solution of equation $y^n + y = -S$, *where* $\infty > S > 2$ *and* $n > 1$ *is an odd integer*

On substituting $y = -Y$, the equation $y^n + y = -S$ transforms to $Y^n + Y = S$ and this equation has already been solved in section 2.4.1, therefore, its solution is not repeated here. However, its roots will be opposite in sign to those of equation $y^n + y = S$.

2.4.3 Solution of equation $y^n + y = S$, *where* $\infty > S > 2$ *and* $n \geq 2$ *is an even integer*

2.4.3.1 *Roots of equation* $y^n + y = S$: When y varies from 0 to 1, equation $y^n + y = S$, is not solvable for roots between 0 and 1 since $\infty > S > 2$. For extracting roots between 0 and -1, substitution of $y = -Y$ transforms the equation to $Y - Y^n = -S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is positive and right-hand side negative. For extracting roots between 1 and infinity, substitution of $y = 1/Y$ transforms the equation to $R + R\,Y^{n-1} = Y^n$, where $R = 1/S$. This equation is solvable for roots between 0 and 1 as value of R obtained from the equation is $R = Y^n/(1+Y^{n-1})$ and R varies from 0 to and given value of R is less than and more than 0. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation to $R - RY^{n-1} = Y^n$, where $R = 1/S$. From this equation, $R = Y^n/(1 - Y^{n-1})$ and thus R varies from 0 to infinity but it is given $0 < R < 1/2$, therefore this equation $R - RY^{n-1} = Y^n$ is solvable for roots between 0 and 1.

2.4.3.2. *Geometric Solution:* Geometric construction is same as explained in section 2.1.1.3 and pertains to Figure 1. Procedure for extracting roots is same as explained in section 2.1.1.3 except the facts that length $D_{n-1}^{'''''}D_n^{'''''} - R(D_{n-1}^{'''''}D_n^{'''''})$ is compared with R and also length $D_{n-1}^{'''}D_n^{'''} + R\left(D_{n-2}^{'''}D_{n-1}^{'''}\right)$ is compared with R. When on varying the value of angle C, the length $D_{n-1}^{'''''}D_n^{'''''} - R(D_{n-1}^{'''''}D_n^{'''''})$ equals R, then equation, $y^n + y = S$ has one real root $1/BD_1^{'''''}$, when $\infty > S > 2$ *and* $n > 1$ *is an odd integer.* When on varying the value of angle C, the length $D_{n-1}^{'''}D_n^{'''} + R\left(D_{n-2}^{'''}D_{n-1}^{'''}\right)$ equals R, then equation, $y^n + y = S$ has one real root $-1/BD_1^{'''}$, when $\infty > S > 2$ *and* $n > 1$ *is an odd integer.*

2.4.4 Solution of equation $y^n + y = -S$, *where* $\infty > S > 2$ *and* $n \geq 2$ *is an even integer*

2.4.4.1. *Roots of equation* $y^n + y = -S$: When y varies from 0 to 1, equation $y^n + y = S$, is not solvable for roots between 0 and 1 since its left-hand side is positive and right-hand side is negative. For extracting roots between 0 and -1, substitution of $y = -Y$ transforms the equation to $Y - Y^n = S$. This equation is not solvable for roots between 0 and 1 as $S > 2$ as S can achieve maximum value $(n-1)/n^{n/(n-1)}$ which is less than 1. For extracting roots between 1 and infinity, substitution of $y = 1/Y$ transforms the equation to $R + R\,Y^{n-1} = -Y^n$, where $R = 1/S$. This equation is not solvable for roots between 0 and 1 as its left-hand side is positive and right-hand side negative. For extracting roots between -1 and minus infinity, substitution of $y = -1/Y$ transforms the equation to $R - RY^{n-1} = -Y^n$, where $R = 1/S$. From this equation, $R = -Y^n/(1 - Y^{n-1})$ and thus R is negative but it is given $0 < R < 1/2$, therefore, this equation $R - RY^{n-1} = -Y^n$ is not solvable for roots between 0 and 1. Thus equation $y^n + y = -S$, when $\infty > S > 2$ and $\mathrm{n} \geq 2$ is an even integer, does not have real root.

For finding roots of equation, $x^n - Px = -Q$ and $x^n + Px = Q$, where real numbers $P \neq 0$, $P \neq \infty, Q \neq 0$ and $Q \neq \infty$, roots of $\mathrm{y^n - y = -S}$ and $\mathrm{y^n + y = S}$, are multiplied with $P^{1/(n-1)}$.

3 Roots at a glance of polynomial equations $y^n - y = -S$ and $y^n + y = S$, where real number $S \neq 0$ and also $S \neq \infty$

Table 2 showing solution of $y^n - y = -S$, $y^n + y = -S$ and $R = 1/S$

n	Equation	Value of S	Geometric Equation	Real Roots
Odd	$y^n - y = -S$	$0 < S \leq (n-1)/n^{n/(n-1)}$	$BD_1' - D_{n-1}'D_n' = S$, $BD_1'' - D_{n-1}''D_n'' = S$, $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	BD_1', BD_1'', $-1/BD_1'''$
Odd	$y^n - y = S$	$0 < S \leq (n-1)/n^{n/(n-1)}$	$BD_1' - D_{n-1}'D_n' = S$, $BD_1'' - D_{n-1}''D_n'' = S$, $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	$-BD_1'$, $-BD_1''$, $1/BD_1'''$
Even	$y^n - y = -S$	$0 < S \leq (n-1)/n^{n/(n-1)}$	$BD_1' - D_{n-1}'D_n' = S$, $BD_1'' - D_{n-1}''D_n'' = S$,	BD_1', BD_1''
Even	$y^n - y = S$	$0 < S \leq (n-1)/n^{n/(n-1)}$	$D_{n-1}''''D_n'''' + BD_1''''$ $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	$-BD_1''''$, $1/BD_1'''$
Odd	$y^n - y = -S$	$\infty > S \geq (n-1)/n^{n/(n-1)}$	$D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	$-1/BD_1'''$
Odd	$y^n - y = S$	$\infty > S \geq (n-1)/n^{n/(n-1)}$	$D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	$1/BD_1'''$
Even	$y^n - y = -S$	$\infty > S \geq (n-1)/n^{n/(n-1)}$	Not applicable	No real root
Even	$y^n - y = S$	$2 > S \geq (n-1)/n^{n/(n-1)}$ $\infty > S \geq 2$ $\infty > S \geq (n-1)/n^{n/(n-1)}$	$D_{n-1}''''D_n'''' + BD_1'''' = S$ $D_{n-1}'''''D_n''''' - R(D_{n-1}'''''D_n''''') = R$ $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	$-BD_1''''$, $-1/BD_1'''''$ $1/BD_1'''$
Odd	$y^n + y = S$	$0 < S \leq 2$	$D_{n-1}''''D_n'''' + BD_1'''' = S$	BD_1''''
Odd	$y^n + y = -S$	$0 < S \leq 2$	$D_{n-1}''''D_n'''' + BD_1'''' = S$	$-BD_1''''$
Even	$y^n + y = S$	$0 < S \leq 2$	$D_{n-1}''''D_n'''' + BD_1'''' = S$ $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	BD_1'''' $-1/BD_1'''$
Even	$y^n + y = -S$	$0 < S \leq (n-1)/n^{n/(n-1)}$, $2 > S > (n-1)/n^{n/(n-1)}$	$BD_1' - D_{n-1}'D_n' = S$, $BD_1'' - D_{n-1}''D_n'' = S$, Not applicable	$-BD_1'$, $-BD_1''$, No real root
Odd	$y^n + y = S$	$\infty > S > 2$	$D_{n-1}'''''D_n''''' - R(D_{n-1}'''''D_n''''') = R$	$1/BD_1'''''$
Odd	$y^n + y = -S$	$\infty > S > 2$	$D_{n-1}'''''D_n''''' - R(D_{n-1}'''''D_n''''') = R$	$-1/BD_1'''''$
Even	$y^n + y = S$	$\infty > S > 2$	$D_{n-1}'''''D_n''''' - R(D_{n-1}'''''D_n''''') = R$ $D_{n-1}'''D_n''' + R(D_{n-2}'''D_{n-1}''') = R$	$1/BD_1'''''$ $-1/BD_1'''$
Even	$y^n + y = -S$	$\infty > S > 2$	Not applicable	No real root

4 Equations reducible to the forms $y^n - y = -S$ and $y^n + y = S$, where real number $S \neq 0$ and also $S \neq \infty$

Let the given equation be $x\left(1-x^2\right)^n = S^n$. Substitution of $x = (1 - S/y)^{1/2}$, transforms this equation to $y^{2n+1} - y = -S$. If the given equation is $z(1-z)^{2n} = S^{2n}$, it can be transformed to $x\left(1-x^2\right)^n = S^n$ by substituting $z = x^2$.

Let the given equation be $x\left(1-x^2\right)^{(2n-1)/2} = S^{(2n-1)/2}$. Substitution of $x = (1 - S/y)^{1/2}$, transforms this equation to $y^{2n} - y = -S$. If the given equation is $z(1-z)^{2n-1} = S^{2n-1}$, it can be transformed to $x\left(1-x^2\right)^{(2n-1)/2} = S^{(2n-1)/2}$ by substituting $z = x^2$.

Let the given equation be $x\left(1+x^2\right)^n = S^n$. Substitution of $x = (S/y-1)^{1/2}$, transforms this equation to $y^{2n+1}+y=S$. If the given equation is $z(1+z)^{2n} = S^{2n}$, it can be transformed to $x\left(1+x^2\right)^n = S^n$ by substituting $z=x^2$.

Let the given equation be $x\left(1+x^2\right)^{(2n-1)/2} = S^{(2n-1)/2}$. Substitution of $x = (S/y-1)^{1/2}$, transforms this equation to $y^{2n}+y=-S$. If the given equation is $z(1-z)^{2n-1} = S^{2n-1}$, it can be transformed to $x\left(1-x^2\right)^{(2n-1)/2} = S^{(2n-1)/2}$ by substituting $z=x^2$.

All these equations are solvable by geometric method described in section 2.

5 Algorithm for geometric solution of equations of the forms $y^n - y = -S$ and $y^n + y = S$, where Integer $n > 1$ real number $S \neq 0$ and also $S \neq \infty$

1. Let given equation be $y^n-y=-S$ or $y^n-y=S$ or $y^n+y=S$ or $y^n+y=-S$, where y is a variable and S is a real number such that $S\neq 0,\ S\neq\infty$ and integer $n>1$ but not equal to ∞.

2. Draw a triangle ABC with right angle at point B, base BC, hypotenuse AC and perpendicular AB of unit length. Draw perpendicular BD_1, from point B upon hypotenuse AC meeting it at point D_1, draw perpendicular D_1D_2, from point D_1 upon base BC meeting it at point D_2, draw perpendicular D_2D_3, from point D_2 and continue drawing perpendiculars in this fashion till last perpendicular $D_{(n-1)}D_n$ is drawn. Measure lengths BD_1, $D_{n-1}D_n$ from geometric figure and compare length $BD_1 - D_{n-1}D_n$ with given S, if it is not equal to S, then vary the value of angle C, keeping the length $AB=1$ and redraw perpendiculars BD_1', $D_1'D_2'$, $D_2'D_3'$,…,$D_{n-1}'D_n'$, so that length BD_1'-$D_{n-1}'D_n'$ equals S. Note length BD_1'. Again vary the value of angle C, redraw perpendiculars keeping length $AB=1$ till the length $BD_1'' - D_{n-1}''D_n''$, equals S. Measure the length BD_1''. Again vary the value of angle C, redraw perpendiculars, keeping length $AB=1$ till the length $D_{n-1}'''D_n''' + R\left(D_{n-2}'''D_{n-1}'''\right)$ equals R, where $R=1/S$. Measure the length BD_1'''. Again, vary the value of angle C, redraw perpendiculars, keeping the length $AB=1$ till the length $D_{n-1}''''D_n'''' + BD_1''''$ equals S. Measure the length BD_1''''. Again vary the value of angle C, redraw perpendiculars, keeping the length $AB=1$ till the length $D_{n-1}'''''D_n''''' -$

$R\left(D''''''_{n-1}D''''''_{n}\right)$ equals R. Measure the length BD''''''_{1}.

3. If given equation $y^n - y = -S$, where n is odd and $0 < S \le (n-1)/n^{n/(n-1)}$ then go to 21.
4. If given equation $y^n - y = S$, where n is odd and $0 < S \le (n-1)/n^{n/(n-1)}$ then go to 22.
5. If given equation $y^n - y = -S$, where n is even and $0 < S \le (n-1)/n^{n/(n-1)}$ then go to 23.
6. If given equation $y^n - y = S$, where n is even and $0 < S \le (n-1)/n^{n/(n-1)}$ then go to 24.
7. If given equation $y^n - y = -S$, where n is odd and $\infty > S \ge (n-1)/n^{n/(n-1)}$, then go to 25.
8. If given equation $y^n - y = S$, where n is odd and $\infty > S \ge (n-1)/n^{n/(n-1)}$, then go to 26.
9. If given equation $y^n - y = -S$, where n is even and $\infty > S \ge (n-1)/n^{n/(n-1)}$, then go to 35.
10. If given equation $y^n - y = S$, where n is even and $2 > S \ge (n-1)/n^{n/(n-1)}$, then go to 24.
11. If given equation $y^n - y = S$, where n is even and $\infty > S > 2$ then go to 27.
12. If given equation $y^n + y = S$, where n is odd and $0 < S \le 2$, then go to 28.
13. If given equation $y^n + y = -S$, where n is odd and $0 < S \le 2$, then go to 29.
14. If given equation $y^n + y = S$, where n is even and $0 < S \le 2$, then go to 30.
15. If given equation $y^n + y = -S$, where n is even and $0 < S \le (n-1)/n^{n/(n-1)}$,then go to 31.
16. If given equation $y^n + y = -S$, where n is even and $2 > S > (n-1)/n^{n/(n-1)}$,then go to 35.
17. If given equation $y^n + y = S$, where n is odd and $\infty > S > 2$, then go to 32.

18. If given equation $y^n + y = -S$, where n is odd and $\infty > S > 2$, then go to 33.
19. If given equation $y^n + y = S$, where n is even and $\infty > S > 2$, then go to 34.
20. If given equation $y^n + y = -S$, where n is even and $\infty > S > 2$, then go to 35.
21. Print real of roots of given equation, BD_1', BD_1'', $-1/BD_1'''$. Go to 36.
22. Print real of roots of given equation, $-BD_1'$, $-BD_1''$, $1/BD_1'''$. Go to 36.
23. Print real of roots of given equation, BD_1', BD_1''. Go to 37.
24. Print real of roots of given equation, $-BD_1^{''''}$, $1/BD_1^{'''}$. Go to 36.
25. Print real of root of given equation, $-1/BD_1^{'''}$. Go to 36.
26. Print real of root of given equation, $1/BD_1^{'''}$. Go to 36.
27. Print real of root of given equation, $-1/BD_1^{'''''}$,$1/BD_1^{'''}$ Go to 36.
28. Print real of root of given equation, $BD_1^{''''}$. Go to 36.
29. Print real of root of given equation, $-BD_1^{''''}$. Go to 36.
30. Print real of roots of given equation, $BD_1^{''''} -1/BD_1^{'''}$. Go to 36.
31. Print real of roots of given equation, $-BD_1'$, $-BD_1''$. Go to 36.
32. Print no real of root of given equation, $1/BD_1^{'''''}$. Go to 36.
33. Print no real of root of given equation, $-1/BD_1^{'''''}$. Go to 36.
34. Print no real of roots of given equation, $1/\mathrm{BD}_1^{'''''}$, $-1/\mathrm{BD}_1^{'''}$. Go to 36.
35. Print no real of root of given equation.
36. Stop.

6 Geometric solution for real roots of general polynomial equation

It is not always true that a polynomial equation will be of the forms $x^n - x = -S$ or $x^n - x = S$ or $x^n + x = S$ or $x^n + x = -S$, where x is a variable, S is a real number such that $S \neq 0,\ S \neq \infty$ and integer $n > 1$ but not equal to ∞. A polynomial equation may be of the general form

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0, \qquad (6.1)$$

where $a_{n-1},\ a_{n-2}, a_{n-3},\ \ldots, a_2,\ a_1,\ a_0$, coefficients of $x^{n-1},\ x^{n-2},\ x^{n-3},\ \ldots,$ $x^2,\ x$, constant term, are real quantities including zero but excluding infinity. According to Fundamental Theorem of algebra, if n is an even integer, this equation has n roots, which may be all complex or all real or some real and some complex, If n happens to be an odd integer, then except one real root, remaining $(n - 1)$ roots can be all real or all complex or some real and some complex. My endeavour is to extract all real roots of this equation, using geometric method of right angled triangles by equating $cos\ C$ with variable x and that puts the embargo that the roots of the Equation must lie between -1 and $+1$ in conformity with variation of $cos\ C$. Since negative quantities, in this paper, are not considered graphical in the geometric figure, roots lying between -1 to 0 will be brought in the range of 1 to 0 by substituting $x\ = -X$. That will transform the equation (6.1) to

$$X^n - a_{n-1}X^{n-1} + a_{n-2}X^{n-2} - \cdots - a_2X^2 + a_1X - a_0 = 0, \qquad (6.2)$$

when n is odd. When n is even, the equation transforms to

$$X^n - a_{n-1}X^{n-1} + a_{n-2}X^{n-2} - \cdots + a_2X^2 - a_1X + a_0 = 0. \qquad (6.3)$$

However, there may be some roots having values more than 1 or less than -1, these will also be brought in the range of 0 to $+1$ by substitutions of $x\ = 1/X$ or $x = -1/X$ as the case may be. Substitution of $x = 1/X$, transforms the Equation (6.1) to

$$a_0X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_{n-2}X^2 + a_{n-1}X + 1 = 0. \qquad (6.4)$$

If n is an odd integer, then substitution of $x\ =\ -1/X$ transforms the Equation (6.1) to

$$a_0X^n - a_1X^{n-1} + a_2X^{n-2} - \cdots - a_{n-2}X^2 + a_{n-1}X - 1 = 0. \qquad (6.5)$$

If n is an even integer, then substitution of $x = -1/X$ transforms the Equation (6.1) to

$$a_0X^n - a_1X^{n-1} + a_2X^{n-2} - \cdots + a_{n-2}X^2 - a_{n-1}X + 1 = 0 \tag{6.6}$$

Geometric construction for solution of these equations is same as Figure 1 and method of construction is not being repeated here, since it has already been explained at number of places in this paper. In this case, total number of perpendiculars will count n, assuming $BD_1, D_1D_2, D_2D_3, \ldots, D_{n-1}D_n$ as 1^{st}, 2^{nd}, 3^{rd},..., nth perpendicular. For extracting roots between 0 and -1 for odd n, geometric equation $D^*_{n-1}D^*_n - a_{n-1}\left(D^*_{n-2}D^*_{n-1}\right) + a_{n-2}\left(D^*_{n-3}D^*_{n-2}\right)$ $-\cdots - a_2\left(D^*_1D^*_2\right) + a_1\left(BD^*_1\right) - a_0 = 0$ corresponding to Equation (6.2) will have to be satisfied by varying the value use of angle C. Between 0 and 1, value of angle C is varied, perpendiculars BD'_1, $D'_1D'_2$, $D'_2D'_3$, $\ldots$, $D'_{n-1}D'_n$ are drawn as already explained and their lengths are measured till length $D'_{n-1}D'_n + a_{n-1}(D'_{n-2}$ $D'_{n-1}) + a_{n-2}(D'_{n-3}D'_{n-2}) + \cdots + a_2(D'_1D'_2) + a_1(BD'_1) + a_0$ equals 0, then one real root of the equation is BD'_1. Angle C is varied from 0 to 90 degrees and perpendiculars are redrawn in each case, remeasured till geometric equation $D_{n-1}D_n + a_{n-1}(D_{n-2}D_{n-1}) + a_{n-2}(D_{n-3}D_{n-2}) + \cdots + a_2(D_1D_2) + a_1(BD_1) + a_0 = 0$ is satisfied. If this geometric equation is satisfied N_1 times by varying angle C, then the equation will have N_1 real roots as BD'_1, BD''_1, BD'''_1, $\ldots$ between 0 and 1. For extracting roots between 0 and -1 for odd n, geometric equation $D^*_{n-1}D^*_n - a_{n-1}(D^*_{n-2}D^*_{n-1}) + a_{n-2}(D^*_{n-3}D^*_{n-2}) - \cdots - a_2(D^*_1D^*_2) + a_1(BD^*_1) - a_0 = 0$. Geometric Figure 1, is drawn, value of angle C is varied till the above said geometric equation is satisfied. Then one real root of Equation (6.1) will be$-BD^*_{1.}$. By scanning angle C through 90 degrees and satisfying the equation, all the roots $-BD^*_{1.}, -BD^{**}_1,\ -BD^{***}_1, \ldots$ (say N_2 in number) of Equation (6.1) lying between 0 and -1 can be extracted.

If n is even integer, then geometric equation $D_{n-1}D_n - a_{n-1}\left(D_{n-2}D_{n-1}\right) + a_{n-2}\left(D_{n-3}D_{n-2}\right) - \cdots + a_2\left(D_1D_2\right) - a_1\left(BD_1\right) + a_0 = 0$, corresponding to Equation (6.3) will have to be satisfied by varying the value of angle C through 90 degrees. If N_2 are real roots of this equation, then these are given by $-BD^*_{1.}, -BD^{**}_{1.}$, $-BD^{***}_1, \ldots$

For extracting roots between 1 and infinity, geometric Figure 1, is drawn, value of angle C is varied till geometric equation $a_0(D^{\#}_{n-1}D^{\#}_n) + a_1(D^{\#}_{n-2}D^{\#}_{n-1}) + a_2(D^{\#}_{n-3}D^{\#}_{n-2}) + \cdots + a_2(D^{\#}_1D^{\#}_2) + a_1(BD^{\#}_1) + a_0 = 0$, corresponding to Equation (6.4) is satisfied. Then one real root of Equation (6.1) will be $1/BD^{\#}_{1.}$. By scanning angle C through 90 degrees and satisfying the said equation, the roots

$1/BD_1^{\#}, 1/BD_1^{\#\#},\ \ 1/BD_1^{\#\#\#}, \ldots$ (say N_3 in number) of Equation 6.1 lying between 1 and ∞ can be extracted.

For extracting roots between -1 and minus infinity, when n is odd integer, geometric Figure 1, is drawn, value of angle C is varied till geometric equation $a_0\left(D_{n-1}^{\circ} D_n^{\circ}\right) - a_1\left(D_{n-2}^{\circ} D_{n-1}^{\circ}\right) + a_2\left(D_{n-3}^{\circ} D_{n-2}^{\circ}\right) - \cdots - a_{n-2}\left(D_1^{\circ} D_2^{\circ}\right) + a_{n-1}\left(BD_1^{\circ}\right) - 1 = 0$ corresponding to Equation (6.5) is satisfied. Then one real root of Equation (6.1) is$-1/BD_{1.}^{\circ}$. By scanning angle C through 90 degrees and satisfying the said equation, all the roots $-1/BD_1^{\circ}, -1/BD_1^{\circ\circ},\ \ -1/BD_1^{\circ\circ\circ}, \ldots$ (say N_4 in number) of Equation (6.1) lying between -1 and $-\infty$ can be extracted. For extracting roots between -1 and minus infinity, when n is even integer, geometric Figure 1, is drawn, value of angle C is varied till geometric equation $a_0\left(D_{n-1}^{\circ} D_n^{\circ}\right) - a_1\left(D_{n-2}^{\circ} D_{n-1}^{\circ}\right) + a_2\left(D_{n-3}^{\circ} D_{n-2}^{\circ}\right) - \cdots + a_{n-2}\left(D_1^{\circ} D_2^{\circ}\right) - a_{n-1}\left(BD_1^{\circ}\right) + 1 = 0$, corresponding to Equation (6,6) is satisfied. Then one real root of Equation (6.1) is$-1/BD_1^{\circ}$. By scanning angle C through 90 degrees and satisfying the said equation, all the roots $-1/BD_1^{\circ}, -1/BD_1^{\circ\circ},\ \ -1/BD_1^{\circ\circ\circ}, \ldots$ (say N_4 in number) of Equation (6.1) lying between -1 and $-\infty$ can be extracted.

It is also worthwhile to state that $N_1 + N_2 + N_3 + N_4$ will be either equal to or less than n. And $n-(N_1+N_2+N_3+N_4)$ will either be zero or even integer. It is prudent to state, two steps out of four described above, can be avoided if the polynomial equation happens to have no positive real roots. Similarly, if the polynomial equation happens to have no negative real roots, two steps out of four described above, can be avoided. Whether the equation does not have negative or positive roots, can be checked, using Descartes Rule of Signs [9], [5]. According to this rule, the equation, $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$, will not have a positive root if $a_{n-1},\ a_{n-2}, \ldots, a_2,\ a_1,\ a_0$ all are positive, resulting in no change of signs in the nonzero terms $x^n,\ \ a_{n-1}x^{n-1},\ \ a_{n-2}x^{n-2},\ \ldots,\ \ a_2x^2,\ \ a_1x,\ \ a_0$ which will all be positive or negative. If on substituting $x = -X$,there is no change of signs in the nonzero terms, then the equation will not have any negative root [5, 9].

7 Algorithm for solving polynomial of any degree for real roots

1. Let given equation be $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$, where $a_{n-1},\ a_{n-2}, \ldots, a_2,\ a_1,\ a_0$, all are real number but not equal to infinity and integer $n > 1$ but $n \neq \infty$.

2. Check whether n is odd or even. If n is even go to 4.

3. A triangle ABC is constructed with right angle at point B, base BC, perpendicular AB of unit length, hypotenuse AC and and angle BCA abbreviated angle C of any arbitrary value. From point B, a perpendicular BD_1 is dawn upon hypotenuse AC meeting it at point D_1, from point D_1, a perpendicular D_1D_2 is drawn upon base BC meeting it at point D_2, from point a D_2 perpendicular is drawn upon hypotenuse AC and this process of drawing perpendiculars alternatively upon hypotenuse and base is continued till last perpendicular $D_{n-1}D_n$ is drawn. Value of angle C is varied, fresh perpendiculars $BD_1,\ D_1D_2,\ D_2D_3,\ \ldots,\ D_{n-1}D_n$ are dawn and the process of variation of angle C and drawing perpendiculars is continued till $\left(D_{n-1}^{'}D_n^{'}\right) + a_{n-1}\left(D_{n-2}^{'}D_{n-1}^{'}\right) + a_{n-2}\left(D_{n-3}^{'}D_{n-2}^{'}\right) + \cdots + a_2\left(D_1^{'}D_2^{'}\right) + a_1\left(BD_1^{'}\right) + a_0$, equals zero, then measure length of perpendicular $\left(BD_1^{'}\right)$. One of the roots will be $\left(BD_1^{'}\right)$. Keep varying angle C and drawing perpendiculars till angle from 0 to 90 degrees is scanned, all such stages where,

$$\left(D'_{n-1}D'_n\right) + a_{n-1}\left(D'_{n-2}D'_{n-1}\right) + a_{n-2}\left(D'_{n-3}D'_{n-2}\right) + \cdots + a_1\left(BD'_1\right) + a_0,$$
$$\left(D''_{n-1}D''_n\right) + a_{n-1}\left(D''_{n-2}D''_{n-1}\right) + a_{n-2}\left(D''_{n-3}D''_{n-2}\right) + \cdots + a_1\left(BD''_1\right) + a_0.$$

..., so on each equals zero, are found and lengths $\left(BD_1^{'}\right)\left(BD_1^{''}\right),\left(BD_1^{'''}\right)$, ..., are measured.

Repeat this exercise and find all such stages where

$$\left(D^{\circ}_{n-1}D^{\circ}_n\right) - a_{n-1}\left(D^{\circ}_{n-2}D^{\circ}_{n-1}\right) + a_{n-2}\left(D^{\circ}_{n-3}D^{\circ}_{n-2}\right) - \cdots + a_1\left(BD^{\circ}_1\right) - a_0,$$
$$\left(D^{\circ\circ}_{n-1}D^{\circ\circ}_n\right) - a_{n-1}\left(D^{\circ\circ}_{n-2}D^{\circ\circ}_{n-1}\right) + a_{n-2}\left(D^{\circ\circ}_{n-3}D^{\circ\circ}_{n-2}\right) - \cdots + a_1\left(BD^{\circ\circ}_1\right) - a_0.$$

..., so on each equals zero, are found and lengths $\left(BD_1^{\circ}\right)\left(BD_1^{\circ\circ}\right),\ \left(BD_1^{\circ\circ\circ}\right),\ \ldots,$ are measured.

Repeat this exercise and find all such stages where

$$a_0\left(D^{*}_{n-1}D^{*}_n\right) + a_1\left(D^{*}_{n-2}D^{*}_{n-1}\right) + a_2\left(D^{*}_{n-3}D^{*}_{n-2}\right) + \cdots + a_{n-1}\left(BD^{*}_1\right) + 1,$$
$$a_0\left(D^{**}_{n-1}D^{**}_n\right) + a_1\left(D^{**}_{n-2}D^{**}_{n-1}\right) + a_2\left(D^{**}_{n-3}D^{**}_{n-2}\right) + \cdots + a_{n-1}\left(BD^{**}_1\right) + 1.$$

..., so on each equals zero, are found and lengths $\left(BD^{*}_1\right)\left(BD^{**}_1\right),\ \left(BD^{***}_1\right),\ \ldots,$ are measured.

Repeat this exercise and find all such stages, where

$$a_0\left(D^{\#}_{n-1}D^{\#}_{n}\right)-a_1\left(D^{\#}_{n-2}D^{\#}_{n-1}\right)+a_2\left(D^{\#}_{n-3}D^{\#}_{n-2}\right)-\cdots+a_{n-1}\left(BD^{\#}_{1}\right)-1,$$
$$a_0\left(D^{\#\#}_{n-1}D^{\#\#}_{n}\right)-a_1\left(D^{\#\#}_{n-2}D^{\#\#}_{n-1}\right)+a_2\left(D^{\#\#}_{n-3}D^{\#\#}_{n-2}\right)-\cdots+a_{n-1}\left(BD^{\#\#}_{1}\right)-1.$$

..., so on each equals zero, are found and lengths $\left(BD^{\#}_{1}\right)$, $\left(BD^{\#\#}_{1}\right)$, $\left(BD^{\#\#\#}_{1}\right)$, ..., are measured. Then go to 5.

4. A triangle ABC is constructed with right angle at point B, base BC, perpendicular AB of unit length, hypotenuse AC and and angle BCA abbreviated angle C of any arbitrary value. From point B, a perpendicular BD_1 is dawn upon hypotenuse AC meeting it at point D_1, from point D_1, a perpendicular D_1D_2 is drawn upon base BC meeting it at point D_2, from point a D_2 perpendicular is drawn upon hypotenuse AC and this process of drawing perpendiculars alternatively upon hypotenuse and base is continued till last perpendicular $D_{n-1}D_n$ is drawn. Value of angle C is varied, fresh perpendiculars $BD_1,\ D_1D_2,\ D_2D_3,\ \ldots,\ D_{n-1}D_n$ are dawn and the process of variation of angle C and drawing perpendiculars is continued till $\left(D^{'}_{n-1}D^{'}_{n}\right)+a_1\left(D^{'}_{n-2}D^{'}_{n-1}\right)+a_2\left(D^{'}_{n-3}D^{'}_{n-2}\right)+\cdots+a_2\left(D^{'}_{1}D^{'}_{2}\right)+a_1\left(BD^{'}_{1}\right)+a_0$, equals zero, then measure length of perpendicular $\left(BD^{'}_{1}\right)$. One of the roots will be $\left(BD^{'}_{1}\right)$. Keep varying angle C and drawing perpendiculars till angle from 0 to 90 degrees is scanned, all such stages, where

$$\left(D'_{n-1}D'_{n}\right)+a_{n-1}\left(D'_{n-2}D'_{n-1}\right)+a_{n-2}\left(D'_{n-3}D'_{n-2}\right)+\cdots+a_1\left(BD'_1\right)+a_0,$$
$$\left(D''_{n-1}D''_{n}\right)+a_{n-1}\left(D''_{n-2}D''_{n-1}\right)+a_{n-2}\left(D''_{n-3}D''_{n-2}\right)+\cdots+a_1\left(BD''_1\right)+a_0.$$

..., so on each equals zero, are found and lengths $\left(BD^{'}_{1}\right)\left(BD^{''}_{1}\right)$, $\left(BD^{'''}_{1}\right)$, ..., are measured.

Repeat this exercise and find all such stages where

$$\left(D^{\circ}_{n-1}D^{\circ}_{n}\right)-a_{n-1}\left(D^{\circ}_{n-2}D^{\circ}_{n-1}\right)+a_{n-2}\left(D^{\circ}_{n-3}D^{\circ}_{n-2}\right)-\cdots-a_1\left(BD^{\circ}_1\right)+a_0,$$
$$\left(D^{\circ\circ}_{n-1}D^{\circ\circ}_{n}\right)-a_{n-1}\left(D^{\circ\circ}_{n-2}D^{\circ\circ}_{n-1}\right)+a_{n-2}\left(D^{\circ\circ}_{n-3}D^{\circ\circ}_{n-2}\right)-\cdots-a_1\left(BD^{\circ\circ}_1\right)+a_0.$$

..., so on each equals zero, are found and lengths $\left(BD_1^{\circ}\right)\left(BD_1^{\circ\circ}\right)$, $\left(BD_1^{\circ\circ\circ}\right)$, ..., are measured.

Repeat this exercise and find all such stages where

$$a_0\left(D^{\#}_{n-1}D^{\#}_{n}\right)+a_1\left(D^{\#}_{n-2}D^{\#}_{n-1}\right)+a_2\left(D^{\#}_{n-3}D^{\#}_{n-2}\right)+\cdots+a_{n-1}\left(BD^{\#}_{1}\right)+1,$$
$$a_0\left(D^{\#\#}_{n-1}D^{\#\#}_{n}\right)+a_1\left(D^{\#\#}_{n-2}D^{\#\#}_{n-1}\right)+a_2\left(D^{\#\#}_{n-3}D^{\#\#}_{n-2}\right)+\cdots+a_{n-1}\left(BD^{\#\#}_{1}\right)+1.$$

$\ldots$, so on each equals zero, are found and lengths $\left(BD_1^{*}\right)\left(BD_1^{**}\right),\ \left(BD_1^{***}\right),\ \ldots,$ are measured.

Repeat this exercise and find all such stages, where

$$a_0\left(D^{*}_{n-1}D^{*}_{n}\right)-a_1\left(D^{*}_{n-2}D^{*}_{n-1}\right)+a_2\left(D^{*}_{n-3}D^{*}_{n-2}\right)-\cdots-a_{n-1}\left(BD^{*}_{1}\right)+1,$$
$$a_0\left(D^{**}_{n-1}D^{**}_{n}\right)-a_1\left(D^{**}_{n-2}D^{**}_{n-1}\right)+a_2\left(D^{**}_{n-3}D^{**}_{n-2}\right)-\cdots-a_{n-1}\left(BD^{**}_{1}\right)+1.$$

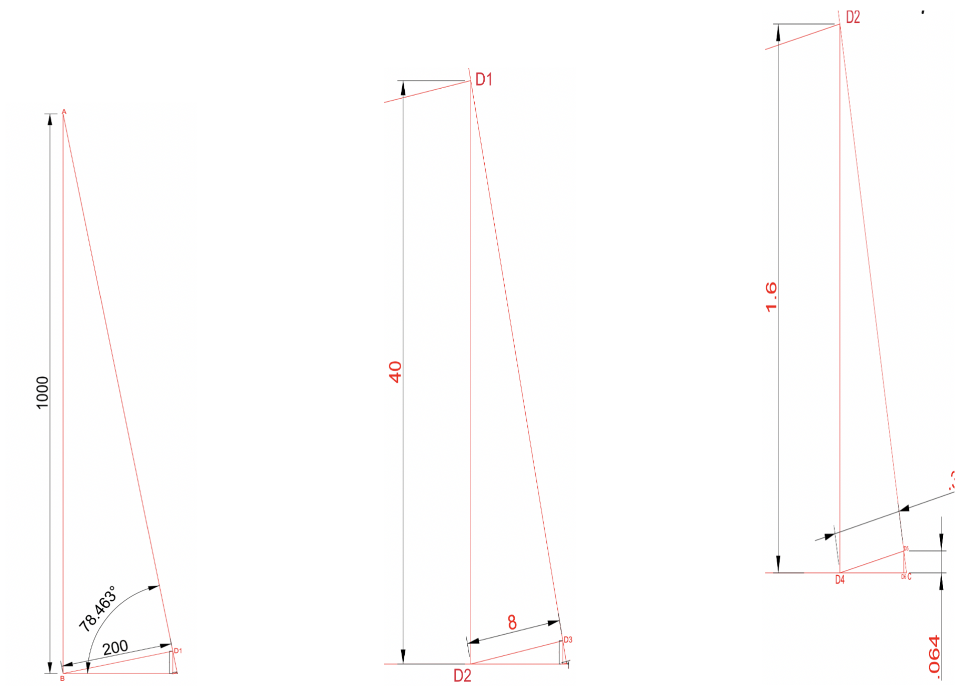
$\ldots$, so on each equals zero, are found and lengths $\left(BD_1^{\#}\right)\left(BD_1^{\#\#}\right),\left(BD_1^{\#\#\#}\right),$ $\ldots$, are measured.

5. Print real roots of the given equation

$$\left(BD_1'\right),\left(BD_1''\right),\left(BD_1'''\right),\ldots,-\left(BD_1^{\circ}\right),-\left(BD_1^{\circ\circ}\right),-\left(BD_1^{\circ\circ\circ}\right),\ldots,$$
$$\left(\frac{1}{BD_1^{*}}\right),\left(\frac{1}{BD_1^{**}}\right),\left(\frac{1}{BD_1^{***}}\right),\ldots,-\left(\frac{1}{BD_1^{\#}}\right),-\left(\frac{1}{BD_1^{\#\#}}\right),-\left(\frac{1}{BD_1^{\#\#\#}}\right),\ldots$$

8 Practicability of drawing geometric construction

There may be situations, where angle C in geometric figure is either approaches 0 or 90 degrees. In such cases, the length of perpendiculars will be too short or too congested to draw or measure, then it would be preferable to consider length AB more than unity so as to facilitate drawability of geometric figure. If AB is considered equal to m units, then each perpendicular will have to be multiplied with m. For roots of equation, (BD_1/m) will be considered in place of BD_1. Kindly refer to Figures 2, 3 and 4, where length of AB is considered more than 1.

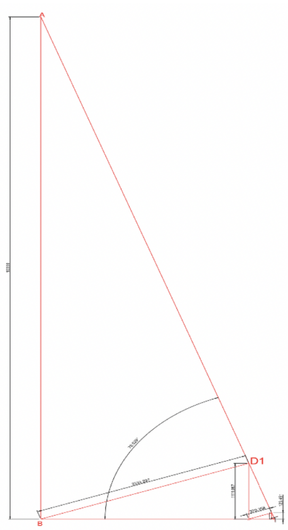


(a) Figure 2A (b) Figure 2B (c) Figure 2C

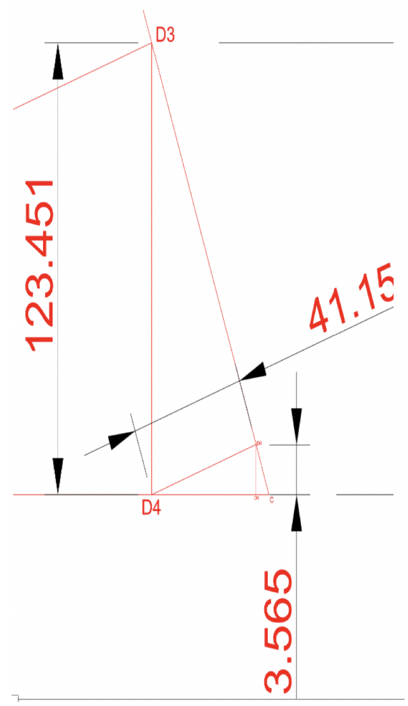
Figure 2: Displaying the situation, when $D_5D_6 + (19/5)D_4D_5 - (54/5)D_3D_4 - (24)D_2D_3 - (29/5)D_1D_2 - (139/5)BD_1 + 6 = 0$ and $(6)\,D_5D_6 + (139/5)D_4D_5 - (29/5)D_3D_4 + (24)D_2D_3 - (54/5)D_1D_2 - (19/5)BD_1 + 1 = 0$.

Example: Solve $x^6 + (19/5)x^5 - (54/5)x^4 - 24x^3 - (29/5)x^2 - (139/5)x + 6 = 0$.

In this equation, $a_5 = 19/5$,$a_4 = -54/5, a_3 = -24, a_2 = -29/5, a_1 = -139/5$ and $a_0 = 6$. For extracting roots between 0 and 1, geometric Figure 1 is drawn and value of angle C is varied till length $D_5D_6 + (19/5)D_4D_5 - (54/5)D_3D_4 - (24)D_2D_3 - (29/5)D_1D_2 - (139/5)BD_1 + 6$, reduces to zero. That situation is shown in Figure 2 which comprises of part Figures 2A, 2B and 2C. Part Figure 2A displays magnified view of perpendiculars AB, BD_1, Part Figure 2B displays magnified view of perpendiculars D_1D_2, D_2D_3 whereas Part Figure 2C displays magnified view of perpendiculars D_3D_4, D_4D_5 and D_6D_6. For better visibility, length of perpendicular AB is taken as 1000 units, therefore, length of each perpendicular mentioned in Figure 2 will be divided by 1000. Kindly refer to section 8. One root real root of given equation is length of $BD_1/1000$, or $200/1000$ or $1/5$. Value of angle C is scanned through 90 degrees, but length $D_5D_6 + (19/5)D_4D_5 - (54/5)D_3D_4 - (24)D_2D_3 - (29/5)D_1D_2 - (139/5)BD_1 + 6$, is not found to equal zero at any stage except the one stated above. Hence there is only no root between 0 and 1 except root equal to $1/5$.



(a) Figure 3A



(b) Figure 3B

Figure 3: Displaying the situation when $(6)D_5D_6 - (139/5)D_4D_5 - (29/5)D_3D_4 - (24)D_2D_3 - (54/5)D_1D_2 + (19/5)BD_1 + 1 = 0$.

For extracting roots between between 0 and -1, substitution of $x = -X$ is made and that transforms the given equation to $X^6 - (19/5)X^5 - (54/5)X^4 + 24X^3 - (29/5)X^2 + (139/5)X + 6 = 0$ and its corresponding geometric equation is $D_5D_6 - (19/5)D_4D_5 - (54/5)D_3D_4 + (24)a_2D_3 - (29/5)D_1D_2 + (139/5)BD_1 + 6 = 0$. Again value of angle C is varied lengths AB, $BD_1, \ldots, D_5D_6$ are measured and length $D_5D_6 - (19/5)D_4D_5 - (54/5)D_3D_4 + (24)D_2D_3 - (29/5)D_1D_2 +$

$(139/5)BD_1 + 6$ is calculated. Using AutoCad application, when angle C was varied from 0 to 90 degrees, it was found, the length $D_5D_6 - (19/5)D_4D_5 - (54/5)D_3D_4 + (24)D_2D_3 - (29/5)D_1D_2 + (139/5)BD_1 + 6$ never reduces to zero. Hence, the given equation does not have any root between 0 and -1.

For extracting roots between between $+1$ and $+\infty$, substitution of $x = 1/X$ is made and that transforms the given equation to $6X^6 - (139/5)X^5 - (29/5)X^4 - 24X^3 - (54/5)X^2 + +(19/5)X + 1$ and its corresponding geometric equation is $(6)D_5D_6 - (139/5)D_4D_5 - (29/5)D_3\ D_4 - (24)D_2D_3 - (54/5)D_1D_2 + (19/5)BD_1{+}1 = 0$. Again value of angle C is varied, lengths AB, BD_1,..., D_5D_6 are measured. Length $(6)D_5D_6 - (139/5)D_4D_5 - (29/5)D_3D_4 - (24)D_2D_3 - (54/5)D_1D_2 + (19/5)BD_1 + 1$, is calculated; when this length equals zero, then the real root of given equation is $1000/BD_1$. The situation when $(6)D_5D_6 - (139/5)D_4D_5 - (29/5)D_3D_4 - (24)D_2D_3 - (54/5)D_1D_2 + (19/5)BD_1 + 1$, equals zero is displayed in Figure 3 which comprises of part Figures 3A and 3B. Part Figure 3A displays magnified view of perpendiculars AB, BD_1, D_1D_2, D_2D_3 and Part Figure 3B displays magnified view of perpendiculars D_3D_4, D_4D_5, D_5D_6. For better visibility, length of perpendicular AB is taken as 10000 units, therefore, length of each perpendicular mentioned in Figure 3 will get divided by 10000. One real root of given equation is length of $10000/BD_1$ or 10000/ 33333.297 or 3. Value of angle C was scanned through 90 degrees, and it was found, length $D_5D_6 + (19/5)D_4D_5 - (54/5)D_3D_4 - (24)D_2D_3 - (29/5)D_1D_2 - (139/5)BD_1 + 6$ does not equal to zero except in one case stated above. Hence the given equation has one real root equal to 3 between +1 and $+\infty$.

For extracting roots between -1 and $-\infty$, substitution of $x = -1/X$, transforms the given equation to $6X^6 + (139/5)X^5 - (29/5)X^4 + 24X^3 - (54/5)X^2 - (19/5)X{+}1$ and its corresponding geometric equation is (6) $D_5D_6{+}(139/5)D_4D_5 - (29/5)D_3D_4 + (24)D_2D_3 - (54/5)D_1D_2 - (19/5)BD_1 + 1 = 0$. Again value of angle C is varied, lengthsAB, BD_1,..., D_5D_6 are measured. Length (6) $D_5D_6 + (139/5)D_4D_5 - (29/5)D_3D_4 + (24)D_2D_3 - (54/5)D_1D_2 - (19/5)BD_1 + 1$, is calculated when this length equals zero, the real root of given equation is $-10000/BD_1$ or $-10000/5000$ or -2. This situation is displayed in Figure 4. Length of perpendicular AB is taken as 10000 units, therefore, length of each perpendicular mentioned in Figure 4 will get divided by 10000. Value of angle C was scanned through 90 degrees, and it was found, the length (6) $D_5D_6 + (139/5)D_4D_5 - (29/5)D_3D_4 + (24)D_2D_3 - (54/5)D_1D_2 - (19/5)BD_1 + 1 = 0$ again equals to zero and the situation is same as displayed in Figure 2. Another real root of the equation is $-1000/200$ or -5. Hence the given equation has two real roots -2 and -5 between 1 and $-\infty$. In this way, it was found by geometric method that

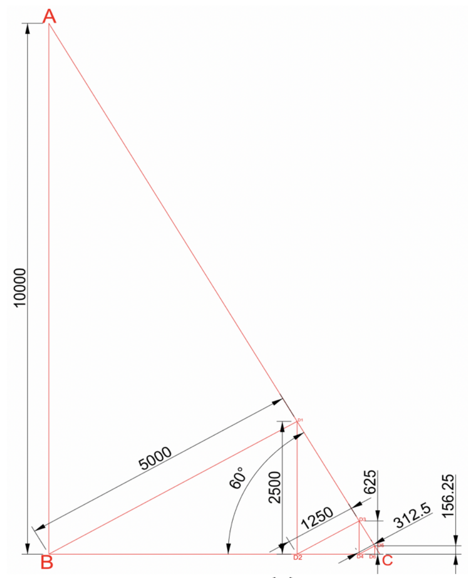


Figure 4: Displaying the situation when $(6)\,D_5D_6 + (139/5)D_4D_5 - (29/5)D_3D_4 + (24)D_2D_3 - (54/5)D_1D_2 - (19/5)BD_1 + 1 = 0$.

$x^6 + (19/5)x^5 - (54/5)x^4 - 24x^3 - (29/5)x^2 - (139/5)x + 6 = 0$ has roots $1/5,\ 3,\ -2$, and -5.

9 Visual representation of real roots

Referring to Figure 1, let $-BD^*_{1.}$, where $0 < BD^*_{1.} < 1$, be one of the roots of the equation, $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$, when n is odd, then the length $D^*_{n-1}D^*_n - a_{n-1}\left(D^*_{n-2}D^*_{n-1}\right) + a_{n-2}\left(D^*_{n-3}D^*_{n-2}\right) - \cdots - a_2\left(D^*_1D^*_2\right) + a_1\left(BD^*_1\right) - a_0$ will equal zero. By plotting positive terms in the direction of +X axis and negative terms in the direction of -X axis, this geometric equation is represented in Figure 5.

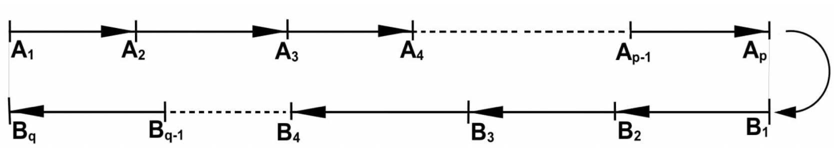


Figure 5: Visual representation of a polynomial equation satisfying a real root.

In this figure, $A_1A_p = B_1B_q, A_1A_2 = D^*_{n-1}D^*_n, A_2A_3 = a_{n-2}\left(D^*_{n-3}D^*_{n-2}\right),$ $A_3A_4 = a_{n-4}\left(D^*_{n-5}D^*_{n-4}\right), \ldots,$ so on and $B_1B_2 = a_{n-1}(D^*_{n-2}D^*_{n-1}), B_2B_3 = a_{n-3}\left(D^*_{n-4}D^*_{n-3}\right), B_3B_4 = a_{n-5}\left(D^*_{n-6}D^*_{n-5}\right), \ \ldots\ ,$ so on. Also from this figure, it is evident, there are p positive terms and q negative terms. In this way, all real roots of the equation can be visualised by such geometric constructions.

10 Salient features of the dynamic geometry method over other methods of approximating roots of a polynomial equation

10.1 Newton's approximation method

- The method requires an initial guess for the root.
- If the initial guess deviates by more than 10–20%, the iteration may diverge or yield inaccurate results.
- Newton's method requires multiple iterations to refine the root approximation.
- Each successive step depends on the accuracy of the previous iteration.
- The method never yields an exact solution, only increasingly accurate approximations.
- Additional iterations improve precision but never provide a closed-form result.
- The process involves tedious calculations, including derivatives and function evaluations.
- As the polynomial degree increases, computational effort grows significantly.
- If the derivative of the polynomial is zero at any step (i.e., a critical point), the method fails or diverges.
- Near saddle points, the method slows down drastically [13].

10.2 Bisection method

- Works only for continuous functions where a root lies between two known values (i.e., sign change).

- Slow convergence (requires many iterations).
- Only provides an approximation—accuracy depends on the number of iterations.
- Fails for multiple roots or if the function does not change sign [14].

10.3 Secant method

- Similar to Newton's method but avoids derivatives.
- Still requires an initial assumption of two close points to start iteration.
- Convergence is not guaranteed-it may fail if points are poorly chosen [15].

10.4 Fixed-Point iteration

- Requires rewriting the equation in the form $x = g(x)$, which is not always possible.
- Low convergence, and divergence is common if the chosen function g(x) is not well-S Behaved [16].

10.5 Dynamic geometry method

- Unlike Newton's method, the solution emerges naturally from geometric construction without needing an initial guess.

- The method provides an exact root once the angle C is adjusted such that the polynomial equation is satisfied.

- The root is found as the length of the perpendicular in the geometric construction.

- Instead of iterative calculations, the method dynamically adjusts the angle C to satisfy the equation.

- No repeated calculations or numerical approximations are required.

- The solution is represented geometrically at the point where the polynomial equation holds true.
- This enhances conceptual understanding and provides an intuitive grasp of the solution.
- The process involves only geometric drawing and measurement, eliminating complex algebraic calculations.
- The final root is measured directly as the length of the perpendicular, in this case BD.
- The method depends on accurate geometric construction and precise instruments to avoid measurement errors.
- Small instrumental inaccuracies may affect the results, requiring careful execution.
- When the desired angle C is near 0° or 90°, the geometric figure may become impractically small or large; this issue is resolved by appropriately scaling the construction.
- Unlike static algebraic methods, the dynamic geometric approach succeeds in solving problems such as trisection, pentasection, and other higher-degree angle divisions, where classical algebraic methods fail.
- No iteration or sign-checking, as required in the bisection method, is needed; the exact root is obtained directly.
- No rewriting is required as in the case of Fixed-Point Iteration; the method naturally finds the solution.

- A comparison between different methods of approximation used in root finding is given in Table 3.

Table 3: Comparison of approximation methods for extraction of roots

Method	**Initial guess needed**	**Iterative**	**Exact/ approx**	**Failure cases**	**Suitability**
Newton Method	Yes, good initial guess is crucial	Yes, it converges quadratically	Approx	Fails if derivative is zero or initial guess is poor	Fastest when close to root
Bisection Method	Yes, it requires sign change	Yes, it has slow convergence	Approx	Fails if function does not change sign	Reliable for continuous functions with known sign change
Secant Method	Yes, two initial points	Yes, it is super linear	Approx	Fails if poorly chosen points lead to divergence	Used in scientific computing when derives are difficult to compute
Fixed point Iteration	Yes, requires transformation	Yes, it is slow	Approx	Diverges if derivative g'(x) too large	Works for simple equations
Dynamic Geometric Method	No	No	Exact	Requires precise drawing	Useful for visualisation

11 Supplementary electronic material

An HTML file titled "Interactive Polynomial Equation Solver" is provided with the manuscript. The interactive figure solves polynomial equations of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \tag{11.1}$$

for degree n. In the current version, $n = 10$, but this limit can be increased by modifying the program. The diagram is constructed similarly to Figure 1 and is reproduced here as Figure 6:

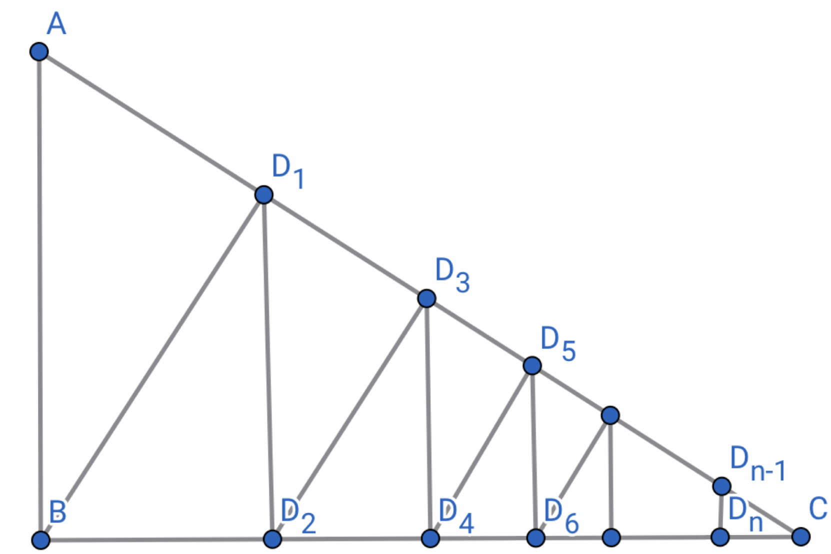


Figure 6: Polynomial equation solve.

Polynomial Equation Solver. Users enter the coefficients $a_0,\ a_1,\ a_2,\ a_3, \ldots,\ a_n$ in decimal forms, in the designated fields. Point C is then moved along CB while monitoring the displayed error, which is reduced to zero to determine the root.This interactive figure not only solves polynomial equations but also demonstrates the correctness of the dynamic geometric method Roots with magnitude greater than 1, less than -1, or between 0 and -1 can be obtained using the same procedure by first transforming the equation through the substitutions x $=\ 1/y,\ x\ =\ -1/y$, or $x\ =\ -y$, respectively, and then reversing the substitution to obtain the root in terms of x.

12 Results and conclusions

Briefly stating, these powers of x $x,\ x^2,\ x^3, \ldots,\ x^n$ can be generated geometrically by a right-angled triangle having n similar right-angled triangles inscribed in it, provided real number $x \neq 0,\ x \neq \infty$ and also integer $n > 1$ and not infinity.

To compute these powers of x, refer to the geometric Figure 1, whose construction has been explained in various sections of the paper and hence not repeated here.

Referring to the set of Equations (1.1) and assuming $cos\ C = x$, the lengths of perpendiculars $x,\ x^2,\ x^3, \ldots,\ x^n$ can be represented by perpendiculars $BD_1, D_1D_2,$ $D_2D_3,\ \ldots,\ D_{n-1}D_n$, thus polynomial equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0,$$

can be written in the form

$D_{n-1}D_n + a_{n-1}\left(D_{n-2}D_{n-1}\right) + a_{n-2}\left(D_{n-3}D_{n-2}\right) + \cdots + a_2\left(D_1D_2\right) + a_1\left(BD_1\right) + a_0 = 0$. Here assumption is that $x = cos\ C$, hence a value of x is related to angle C and is to be taken from interval $[0, 1]$ for ease of geometric construction to avoid negative quantities. This restricts angle C between 0° to 90°.

If x is not in the range [0,1], then substitutions are to be done as follows:

If x belongs to [-1, 0], put $x = -X$,

If x belongs to [1, 8], put $x = 1/X$,

If x belongs to [-1, -8], put $x = -1/X$,

Now to solve a polynomial geometrically, the only requirement is that this expression must become zero. Since the value of each length depends on $\cos C$, i.e. on angle C, so if the polynomial has at least one real root, then there will exist some value of angle C in range [0°, 90°], where this expression becomes zero.

Keeping length $AB = 1$, the value of cosine for that angle C, which satisfies the equation, shall give the root of polynomial as $BD_1 = x$.

This geometric method, unlike Newton's or other approximation techniques, offers an exact, visual determination of the root without requiring iterations or initial guesses. Its reliance on similarity and proportionality ensures a consistent, deterministic approach, particularly powerful for lower-degree polynomials.

If there are N_1 real roots between [0,1], the construction shall produce N_1 distinct values of BD_1 satisfying the equation.

If angle C is too small or too large, triangle becomes too acute or too obtuse, which is not comfortable for construction. In such cases, base length AB may be taken more than unity to spread the diagram.

This shall change the equation as:

$$D_{n-1}D_n + a_{n-1}\left(D_{n-2}D_{n-1}\right) + a_{n-2}\left(D_{n-3}D_{n-2}\right) + \cdots + a_2\left(D_1D_2\right) + a_1\left(BD_1\right) + ABa_0 = 0.$$

And root of polynomial shall be equal to (BD_1/AB) .

In a similar way:

If root lies between [0, -1], root = $-BD_1$:

If root lies between [1, 8], root = $1/BD_1$

If root lies between [-1, -8], root = $-1/BD_1$

This dynamic geometric approach not only simplifies root-finding but enhances its accuracy and clarity through a tangible, visual setup. It gives a physical and visual presentation of roots, lifting them from obscurity to observable reality. The roots of a polynomial equation often remain abstract and intangible, especially to learners. Yet, through this geometric method, their elusive nature becomes tangible — manifesting as specific perpendicular lengths at a dynamically adjusted angle C, precisely when the equation is satisfied **Statements and declarations**.

Conflict-of-interest statements

There is no conflict of interest of any sort related to this research work. Further there are no financial or non financial interests related to this research work.

Competing interests

There are no competing interests directly or indirectly related to this work.

Funds

No funds, grant or any assistance financial or otherwise were taken for this research work.

References

[1] *Algebraic Equations*, Encyclopaedia of Mathematics, EMS Press, 2001 (originally published 1994).

[2] P. V. Bradford, *Visualising solutions to nth-degree algebraic equations using right-angle geometric paths*, www.concentric.net. Archived on 2 May 2010; retrieved 3 February 2012.

[3] M. E. Lill, *Résolution graphique des équations numériques de tous degrés à une seule inconnue*, Nouvelles Annales de Mathématiques, Series 2, **6**(1867), 359–362.

[4] M. E. Lill,. *Résolution graphique des équations algébriques qui ont des racines imaginaires*, Nouvelles Annales de Mathématiques, Series 2, **7**(1868), 363–367.

[5] B. E. Meserve, *Fundamental Concepts of Algebra*, Dover Publications, New York, 1982.

[6] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd ed., Chapter 9, Cambridge University Press, New York, (2007).

[7] C. A. Roger, and J. L. Robert, *One-, Two-, and Multi-Fold Origami Axioms 4OSME*, A K Peters, (2009).

[8] S. Tabachnikov, *Polynomial as polygon*, The Mathematical Intelligencer, **39**(1)(2017), 41–43. doi:10.1007/s00283-016-9681-y.

[9] R. Descartes, *The Geometry of René Descartes*, translated by D. E. Smith and M. L. Latham, Dover Publications, New York, 1954.

[10] C. H. Thomas, *Solving cubics with creases: The work of Beloch and Lill*, American Mathematical Monthly, **118**(4), 307–315. doi:10.4169/amer.math.monthly.118.04.307.

[11] N. K. Wadhawan, *Solution of quintic equation by geometric method and based upon it: A mathematical model*, *Journal of Interdisciplinary Mathematics*, **26**(2023), 775–793. https://doi.org/10.47974/JIM-1538.

[12] E. W. Weierstrass, Algebraic equations. *MathWorld*.

[13] K. Atkinson, *An Introduction to Numerical Analysis*, 2nd ed., SIAM (2013). doi:10.1137/1.9781611973178.

[14] R. L. Burden, and J. D. Faires, *Numerical Analysis*, 10th ed., Cengage Learning (2011).

[15] V. Ruggiero, *A convergence analysis of the secant method.* Numerische Mathematik, **112**(1)(2009), 185–197. doi:10.1007/s00211-009-0251-y.

[16] C. T. Kelley, *Iterative Methods for Linear and Nonlinear Equations*. SIAM(2003). doi:10.1137/1.9780898718836.

[17] N. K. Wadhawan, *Scientific calculator with the aid of geometry and based upon it*: A mechanical calculator. *Science and Technology*, **11**(2)(2023), 8–20.

[18] N. K. Wadhawan, *Solving cubic equations using dynamic geometry.* Resonance, **29**(2024), 919–934. https://doi.org/10.1007/s12045-024-0919-2.