Common fixed point results for generalized integral type (I) and (II) $(F - \alpha_s)$-contractions in S-metric spaces with applications

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Abstract

The purpose of this paper is to introduce the notion of generalized integral type (I) and $(II)(F - \alpha_s)$ contractions and establish some common fixed point theorems in the setting of complete S-metric spaces. Additionally, we provide some consequences of the established results and give an example in support of the established result. Furthermore, we use the obtained results to achieve the solution of an integral equation and the bounded solution of a functional equation in dynamic programming. Our results extend, generalize and enrich several previously published results in the existing literature (see, for example [10], [11], [19], [36] and some others).

1 Introduction

In the year 1922, the Polish mathematician, *Banach* [6] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach contraction mapping principle in the framework of metric spaces. This theorem provides a technique for solving a variety of problems of

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applied nature in mathematical science and engineering. Many researchers have extended, generalized and improved Banach fixed point theorem in different ways and directions. In 2012, *Wardowski* [36] defined the notion of F-contraction to generalise Banach fixed point theorem. On the other hand, *Sedghi et al.* [32] introduced the concept of an S-metric space and proved some fixed point theorems by modifying D-metric and G-metric spaces. They also gave some examples of an S-metric space which shows that the S-metric space is different from other spaces. Also, they built up some topological properties in such spaces and proved some fixed point theorems in the setting of S-metric spaces. *Sedghi and Dung* [33] remarked that every S-metric space is topologically equivalent to a metric space. For more results on S-metric spaces, see ([21], [27], [34], [35]) and many others. In 2002, *Branciari* [10] developed the idea of integral type contraction mappings in complete metric spaces and he studied the existence of fixed points for mappings which is defined on complete metric space satisfying integral type contraction. *Samet et al.* [31] introduced the notion of α-admissible mappings. The concept of α-admissible mappings was extended in different directions. *Mlaiki* in [26], introduced the notion of α-admissible mapping in the setting of S-metric spaces. *Khomdram et al.* [23] extended α-admissible mappings to (α, β)-admissible in S-metric like space. *Priyobarta et al.* [28] extended various forms of α-admissible mappings in S-metric space. There are various generalizations of α-admissible mappings as well as F-contractions (see, e.g. [3], [4], [5], [16], [20], [22], [29], [30]). Recently, *Javed et al.* [18] introduced the concept of F_α-contraction which is a generalization of F-contraction and proved a fixed point theorem in the setting of S-metric spaces.

2 Preliminaries

In this part, we recall the following definitions and lemmas in the sequel.

Denote $\mathbb{R}$:= the set of real numbers, $\mathbb{R}_+ := [0, +\infty)$. A point $\eta \in \Gamma$, where Γ is a non-void set, of the mapping $g \colon \Gamma \to \Gamma$ is called a fixed point of operator g if $g(\eta) = \eta$.

Definition 2.1. *([32]) Let $\Gamma \neq \emptyset$ be a set and let $S \colon \Gamma^3 \to \mathbb{R}_+$ be a function satisfying the following conditions:*

(S1) $S(\eta, \theta, \vartheta) = 0 \Leftrightarrow \eta = \theta = \vartheta$;

(S2) $S(\eta, \theta, \vartheta) \leq S(\eta, \eta, \nu) + S(\theta, \theta, \nu) + S(\vartheta, \vartheta, \nu)$,

for all $\eta, \theta, \vartheta, \nu \in \Gamma$.

Then, the function S is called an S-metric on Γ and the pair (Γ, S) is called an S-metric space (in short SMS).

Example 2.1. $(E1)$*([32]) Let $\Gamma = \mathbb{R}^n$ and $\| \cdot \|$ a norm on Γ, then $S(\eta, \theta, \vartheta) = \|\theta + \vartheta - 2\eta\| + \|\theta - \vartheta\|$ is an S-metric on Γ.*

$(E2)$([32]) Let $\Gamma = \mathbb{R}^n$ and $\| \cdot \|$ a norm on Γ, then $S(\eta, \theta, \vartheta) = \|\eta - \vartheta\| + \|\theta - \vartheta\|$ is an S-metric on Γ.

$(E3)$([33]) Let $\Gamma = \mathbb{R}$ be the real line. Then $S(\eta, \theta, \vartheta) = |\eta - \vartheta| + |\theta - \vartheta|$ for all $\eta, \theta, \vartheta \in \mathbb{R}$ is an S-metric on Γ. This S-metric on Γ is called the usual S-metric on Γ.

$(E4)$([21]) Let $\Gamma \neq \emptyset$ be a set and d be an ordinary metric on Γ. Then $S(\eta, \theta, \vartheta) = d(\eta, \vartheta) + d(\theta, \vartheta)$ for all $\eta, \theta, \vartheta \in \Gamma$ is an S-metric on Γ.

Definition 2.2. *Let (Γ, S) be an S-metric space. For $r > 0$ and $\eta \in \Gamma$ we define an open ball $B_S(\eta, r)$ and closed ball $B_S[\eta, r]$ with center η and radius r as follows, respectively:*

$$(1) \quad B_S(\eta, r) = \{\eta' \in \Gamma : S(\eta', \eta', \eta) < r\},$$

$$(2) \quad B_S[\eta, r] = \{\eta' \in \Gamma : S(\eta', \eta', \eta) \leq r\}.$$

Example 2.2. *([33]) Let $\Gamma = \mathbb{R}$. Denote $S(\eta, \theta, \vartheta) = |\theta + \vartheta - 2\eta| + |\theta - \vartheta|$ for all $\eta, \theta, \vartheta \in \mathbb{R}$. Then*

$$
\begin{aligned}
B_S(1, 2) &= \{\eta \in \mathbb{R} : S(\eta, \eta, 1) < 2\} = \{\eta \in \mathbb{R} : |\eta - 1| < 1\} \\
&= \{\eta \in \mathbb{R} : 0 < \eta < 2\} = (0, 2),
\end{aligned}
$$

and

$$
\begin{aligned}
B_S[2, 4] &= \{\eta \in \mathbb{R} : S(\eta, \eta, 2) \leq 4\} = \{\eta \in \mathbb{R} : |\eta - 2| \leq 2\} \\
&= \{\eta \in \mathbb{R} : 0 \leq \eta \leq 4\} = [0, 4].
\end{aligned}
$$

Definition 2.3. *([32], [33]) Let (Γ, S) be an S-metric space and $Y \subset \Gamma$.*

(1) If for every $\eta \in Y$, there exists $r > 0$ such that $B_S(\eta, r) \subset Y$, then the subset Y is called an open subset of Γ.

(2) *A sequence $\{\eta_n\}$ in Γ converges to $\eta \in \Gamma$ if and only if $S(\eta_n, \eta_n, \eta) \to 0$ as $n \to \infty$, that is, for each $\lambda > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, we have $S(\eta_n, \eta_n, \eta) < \lambda$ and we denote this by $\lim_{n \to \infty} \eta_n = \eta$ or $\eta_n \to \eta$ as $n \to \infty$.*

(3) *A sequence $\{\eta_n\}$ in Γ is called a Cauchy sequence if, $S(\eta_n, \eta_n, \eta_m) \to 0$ as $n, m \to \infty$, that is, for each $\lambda > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, we have $S(\eta_n, \eta_n, \eta_m) < \lambda$.*

(4) *The S-metric space (Γ, S) is called complete if every Cauchy sequence in Γ is convergent.*

(5) *Let τ be the set of all $Y \subset \Gamma$ with $\eta \in Y$ and there exists $r > 0$ such that $B_S(\eta, r) \subset Y$. Then, τ is a topology on Γ (induced by the S-metric space).*

(6) *A nonempty subset Y of Γ is S-closed if the closure of Y coincides with Y, that is, $\overline{Y} = Y$.*

Definition 2.4. *([32]) Let (Γ, S) be an S-metric space. A mapping $\mathcal{R} \colon \Gamma \to \Gamma$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that*

$$S(\mathcal{R}\zeta, \mathcal{R}\theta, \mathcal{R}\vartheta) \leq L\, S(\zeta, \theta, \vartheta) \tag{2.1}$$

for all $\zeta, \theta, \vartheta \in \Gamma$.

Remark 2.1. *([32]) If the S-metric space (Γ, S) is complete, then the mapping defined as above has a unique fixed point.*

Definition 2.5. *([32]) Let (Γ, S) and (Γ', S') be two S-metric spaces. A function $g \colon \Gamma \to \Gamma'$ is said to be continuous at a point $\eta_0 \in \Gamma$ if for every sequence $\{\eta_n\}$ in Γ with $S(\eta_n, \eta_n, \eta_0) \to 0$, $S'(g(\eta_n), g(\eta_n), g(\eta_0)) \to 0$ as $n \to \infty$. We say that g is continuous on Γ if g is continuous at every point $\eta_0 \in \Gamma$.*

Lemma 2.1. *([32], Lemma 2.5) Let (Γ, S) be an S-metric space. Then, we have $S(\eta, \eta, \theta) = S(\theta, \theta, \eta)$ for all $\eta, \theta \in \Gamma$.*

Lemma 2.2. *([32], Lemma 2.12) Let (Γ, S) be an S-metric space. If $\eta_n \to \eta$ and $\eta'_n \to \eta'$ as $n \to \infty$ then $S(\eta_n, \eta_n, \eta'_n) \to S(\eta, \eta, \eta')$ as $n \to \infty$.*

Remark 2.2. *([33]) Every S-metric space is topologically equivalent to a B-metric space.*

Lemma 2.3. *([15], Lemma 8) Let (Γ, S) be an S-metric space and Y is a nonempty subset of Γ. Then, Y is said to be S-closed if and only if for any sequence $\{\eta_n\}$ in Y such that $\eta_n \to \eta$ as $n \to \infty$, then $\eta \in Y$.*

Lemma 2.4. *([33]) The limit of the sequence $\{\eta_n\}$ in an S-metric space (Γ, S) is unique.*

Lemma 2.5. *([32]) Let (Γ, S) be an S-metric space. Then, the convergent sequence $\{\eta_n\}$ in Γ is Cauchy.*

Corollary 2.1. *([33]) Let $T\colon \Gamma \to \Gamma'$ be a map from an S-metric space Γ to an S-metric space Γ'. Then, T is said to be continuous at $\eta \in \Gamma$ if and only if $T\eta_n \to T\eta$ whenever $\eta_n \to \eta$ as $n \to \infty$.*

Definition 2.6. *([31]) Let $\Gamma \neq \emptyset$ be a set. Let $A\colon \Gamma \to \Gamma$ and $\alpha\colon \Gamma \times \Gamma \to \mathbb{R}_+$ be given mappings. We say that A is α-admissible if for all $\eta, \theta \in \Gamma$, we have*

$$\alpha(\eta, \theta) \geq 1 \Rightarrow \alpha(A(\eta), A(\theta)) \geq 1.$$

Lemma 2.6. *([28], Lemma 1.17) Let (Γ, S) be an S-metric space, $A\colon \Gamma \to \Gamma$ be a triangular α_s-admissible mapping. Assume that there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$. Define a sequence $\{\eta_n\}$ by $\eta_{n+1} = A\eta_n$. Then, we have*

$$\alpha_s(\eta_n, \eta_n, \eta_m) \geq 1 \quad \text{for all} \quad m, n \in \mathbb{N} \cup \{0\}.$$

Lemma 2.7. *([28], Lemma 1.18) Let (Γ, S) be an S-metric space, $A, B\colon \Gamma \to \Gamma$ be triangular α_s-admissible mappings. Assume that there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$. Define sequences*

$$\eta_{2i+1} = A\eta_{2i} \quad \text{and} \quad \eta_{2i+2} = B\eta_{2i+1}, \quad \text{where } i = 0, 1, 2, \ldots.$$

Then we have $\alpha_s(\eta_n, \eta_n, \eta_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

In [2], *Aydi* generalized Definition 2.6 and introduced the following.

Definition 2.7. *([2]) Let $\Gamma \neq \emptyset$ be a set, let $A, B\colon \Gamma \to \Gamma$ and $\alpha\colon \Gamma \times \Gamma \to \mathbb{R}_+$ be mappings. We say that (A, B) is a generalized α-admissible pair if for all $\eta, \theta \in \Gamma$, we have*

$$\alpha(\eta, \theta) \geq 1 \Rightarrow \alpha(A(\eta), B(\theta)) \geq 1 \quad \text{and} \quad \alpha(B(\theta), A(\eta)) \geq 1.$$

Definition 2.8. *([26]) Let (Γ, S) be an S-metric space, $A\colon \Gamma \to \Gamma$ and $\alpha\colon \Gamma \times \Gamma \times \Gamma \to \mathbb{R}_+$. We say that A is α-admissible if $\eta, \theta, \vartheta \in \Gamma$, $\alpha(\eta, \theta, \vartheta) \geq 1$ implies that $\alpha(A(\eta), A(\theta), A(\vartheta)) \geq 1$.*

Example 2.3. *Let $\Gamma = \mathbb{R}_+$ and define $T\colon \Gamma \to \Gamma$ and $\alpha\colon \Gamma^3 \to \mathbb{R}_+$ by $A(\eta) = 4\eta$ for all $\eta, \theta, \vartheta \in \Gamma$ and*

$$\alpha(\eta, \theta, \vartheta) = \begin{cases} e^{\frac{\vartheta}{\eta\theta}}, & \text{if } \eta \geq \theta \geq \vartheta, \ \eta, \theta \neq 0, \\ 0, & \text{if } \eta < \theta < \vartheta. \end{cases}$$

Then, A is α-admissible.

In [28], *Priyobarta et al.* generalized Definition 2.8 and introduced the following.

Definition 2.9. *([28]) Let (Γ, S) be an S-metric space, $A, B\colon \Gamma \to \Gamma$ and $\alpha_s\colon \Gamma \times \Gamma \times \Gamma \to \mathbb{R}_+$. We say that the pair (A, B) is an α_s-admissible if $\eta, \theta, \vartheta \in \Gamma$ such that $\alpha_s(\eta, \theta, \vartheta) \geq 1$, then we have $\alpha_s(A(\eta), A(\theta), B(\vartheta)) \geq 1$ and $\alpha_s(B(\eta), B(\theta), A(\vartheta)) \geq 1$.*

Definition 2.10. *([28]) Let (Γ, S) be an S-metric space, $A\colon \Gamma \to \Gamma$ and $\alpha_s\colon \Gamma \times \Gamma \times \Gamma \to \mathbb{R}_+$. We say that A is a triangular α_s-admissible mapping if*
(i) *$\alpha_s(\eta, \theta, \vartheta) \geq 1$ implies $\alpha_s(A(\eta), A(\theta), A(\vartheta)) \geq 1$, $\eta, \theta, \vartheta \in \Gamma$;*
(ii) *$\alpha_s(\eta, \theta, t) \geq 1$ and $\alpha_s(t, t, \vartheta) \geq 1$ imply $\alpha_s(\eta, \theta, \vartheta) \geq 1$, $\eta, \theta, \vartheta, t \in \Gamma$.*

The notion of F-contraction was introduced by Wardowski [36] defined as follows.

Definition 2.11. *([36]) (F-contraction) Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying the following axioms:*
$(F1)$ F is strictly increasing, that is, $\eta < \theta$ implies that $F(\eta) < F(\theta)$ for all $\eta, \theta \in \mathbb{R}_+$.
$(F2)$ For every sequence $\{\eta_n\}$ in $\mathbb{R}_+$, we have $\lim_{n\to\infty} \eta_n = 0$ if and only if $\lim_{n\to\infty} F(\eta_n) = -\infty$.
$(F3)$ There exists a number $k \in (0, 1)$ such that $\lim_{\eta\to 0^+} \eta^k F(\eta) = 0$.

In what follows, $\mathcal{F}$ stands for the family of all functions F which satisfies the above three conditions.

Definition 2.12. *([36]) A mapping $T\colon \Gamma \to \Gamma$ is said to be an F-contraction if there exists a number $\tau > 0$ such that*

$$\forall\, \eta, \theta \in \Gamma,\ d(T(\eta), T(\theta)) > 0 \Rightarrow \tau + F(d(T(\eta), T(\theta))) \le F(d(\eta, \theta)),$$

for all $\eta, \theta \in \Gamma$.

Some examples related to F-contraction [36] are:

Example 2.4. (1) *Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\eta) = ln(\eta)$, it is clear that F satisfies $(F1) - (F3)$ $((F3)$ for any $k \in (0, 1))$.*

(2) *Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\eta) = \eta + ln(\eta)$, $\eta > 0$. Then, F satisfies $(F1) - (F3)$.*

(3) *Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\eta) = -\frac{1}{\sqrt{\eta}}$, $\eta > 0$. Then, F satisfies $(F1) - (F3)$ $((F3)$ for any $k \in (1/2, 1))$.*

(4) *Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\eta) = ln(\eta^2 + \eta)$, $\eta > 0$. Then, F satisfies $(F1) - (F3)$.*

The following theorem was developed by Wardowski [36].

Theorem 2.1. *([36]) Let (Γ, d) be a complete metric space and let $T\colon \Gamma \to \Gamma$ be an F-contraction. Then, T has a unique fixed point $\zeta \in \Gamma$ and for each $\eta_0 \in \Gamma$, the sequence $\{T^n \eta_0\}$ converges to ζ.*

Remark 2.3. *([36]) If we assume that $F(\eta) = ln(\eta)$, an F-contraction mapping becomes the Banach contraction mapping.*

Remark 2.4. *Clearly Definition 2.12 and $(F1)$ implies that $d(T(\eta), T(\theta)) < d(\eta, \theta)$ for all $\eta, \theta \in \Gamma$ with $T(\eta) \ne T(\theta)$. Hence, every F-contraction mapping is continuous.*

Definition 2.13. *([11]) Let (Γ, S) be an S-metric space. A mapping $T\colon \Gamma \to \Gamma$ is said to be an F-contraction if there exists a number $\tau > 0$ such that*

$$S(T(\eta), T(\theta), T(\vartheta)) > 0 \Rightarrow \tau + F(S(T(\eta), T(\theta), T(\vartheta))) \le F(S(\eta, \theta, \vartheta)),$$

for all $\eta, \theta, \vartheta \in \Gamma$.

Definition 2.14. *([18]) Let (Γ, S) be an S-metric space and let $\alpha\colon \Gamma^3 \to \mathbb{R}_+$ be a function. A mapping $T\colon \Gamma \to \Gamma$ is called an F_α-contraction if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that*

$$S(T(\eta), T(\theta), T(\vartheta)) > 0 \Rightarrow \tau + F(\alpha(\eta, \theta, \vartheta)S(T(\eta), T(\theta), T(\vartheta)) \leq F(S(\eta, \theta, \vartheta)),$$

for all $\eta, \theta, \vartheta \in \Gamma$.

Definition 2.15. *([25]) Let $\{z_n\}_{n\in\mathbb{N}}$ be a non-negative sequence such that $\lim_{n\to\infty} z_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{z_n} \omega(t)\, dt = \int_0^a \omega(t)\, dt,$$

where $\omega\colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that $\forall\, \varepsilon > 0$, $\int_0^\varepsilon \omega(t)dt > 0$.

Definition 2.16. *([25]) Let $\{z_n\}_{n\in\mathbb{N}}$ be a non-negative sequence. Then*

$$\lim_{n\to\infty} \int_0^{z_n} \omega(t)\, dt = 0 \Leftrightarrow \lim_{n\to\infty} z_n = 0,$$

where $\omega\colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that $\forall\, \varepsilon > 0$, $\int_0^\varepsilon \omega(t)dt > 0$.

Denote
$\Phi = \Big\{ \omega\colon \mathbb{R}_+ \to \mathbb{R}_+ \ is\ a$ Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}_+$, non-negative and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \omega(t)dt > 0 \Big\}$.

The following theorem was introduced by Branciari [10] in complete metric space as follows.

Theorem 2.2. *([10]) Let (Γ, d) be a complete metric space, $k \in [0, 1)$, and $T\colon \Gamma \to \Gamma$ is a mapping such that for all $\eta, \theta \in \Gamma$,*

$$\int_0^{d(T\eta, T\theta)} \omega(t)dt \leq k \int_0^{d(\eta, \theta)} \omega(t)dt,$$

where $\omega \in \Phi$. Then, T has a unique fixed point $\zeta \in \Gamma$, such that for each $\eta \in \Gamma$, $\lim_{n\to\infty} T^n(\eta) = \zeta$.

Now, we introduce generalized integral type (I) and (II) $(F-\alpha_s)$-contractions as follows.

Definition 2.17. *Let* (Γ, S) *be an S-metric space and let* $\alpha_s \colon \Gamma^3 \to \mathbb{R}_+$ *be a function. The mappings* $A, B \colon \Gamma \to \Gamma$ *are called generalized integral type* (I) *and* (II) $(F - \alpha_s)$*-contractions respectively, if there exist* $F \in \mathcal{F}$ *and a number* $\tau > 0$ *such that*

$$S(A\eta, A\theta, B\vartheta) > 0 \Rightarrow \tau + F\Big(\alpha_s(\eta, \theta, \vartheta) \int_0^{S(A\eta, A\theta, B\vartheta)} \omega(t)dt\Big)$$
$$\leq F\Big(\int_0^{\Psi_{(I)}(\eta, \theta, \vartheta)} \omega(t)dt\Big), \qquad (2.2)$$

for all $\eta, \theta, \vartheta \in \Gamma$ *and* $\omega \in \Phi$ *where*

$$\Psi_{(I)}(\eta, \theta, \vartheta) = \max\Big\{S(\eta, \theta, \vartheta), S(\eta, \theta, A\eta), S(\vartheta, \vartheta, B\vartheta),$$
$$\frac{S(\eta, \vartheta, A\eta) + S(\vartheta, \vartheta, B\vartheta)}{2}\Big\}$$

and

$$S(A\eta, A\theta, B\vartheta) > 0 \Rightarrow \tau + F\Big(\alpha_s(\eta, \theta, \vartheta) \int_0^{S(A\eta, A\theta, B\vartheta)} \omega(t)dt\Big)$$
$$\leq F\Big(\int_0^{\Psi_{(II)}(\eta, \theta, \vartheta)} \omega(t)dt\Big), \qquad (2.3)$$

for all $\eta, \theta, \vartheta \in \Gamma$, *where*

$$\Psi_{(II)}(\eta, \theta, \vartheta) = l_1 S(\eta, \theta, \vartheta) + l_2 S(\eta, \theta, A\eta) + l_3 S(\vartheta, \vartheta, B\vartheta)$$
$$+ l_4 \frac{S(\vartheta, \theta, B\vartheta)[1 + S(\eta, \theta, A\eta)]}{1 + S(\eta, \theta, \vartheta)}$$
$$+ l_5 \max\big\{S(\eta, \theta, A\eta), S(\vartheta, \vartheta, B\vartheta)\big\},$$

l_1, l_2, l_3, l_4, l_5 *are non-negative reals such that* $l_1 + l_2 + l_3 + 2l_4 + 2l_5 < 1$ *and* $\omega \in \Phi$.

For more other contractions, see (*Ali and Imdad* [1]).
In the case where $\omega(t) = 1$, $F(t) = ln(t)$ for all $t > 0$, Equations (2.2) and

(2.3) become

$$S(A\eta, A\theta, B\vartheta) \leq e^{-\tau}\Psi_{(I)}(\eta, \theta, \vartheta) = k\,\Psi_{(I)}(\eta, \theta, \vartheta) \qquad (*)$$

and

$$S(A\eta, A\theta, B\vartheta) \leq e^{-\tau}\Psi_{(II)}(\eta, \theta, \vartheta) = k\,\Psi_{(II)}(\eta, \theta, \vartheta) \qquad (**)$$

for all $\eta, \theta, \vartheta \in \Gamma$ with $\alpha_s(\eta, \theta, \vartheta) \geq 1$, $A\eta \neq A\theta \neq B\vartheta$ and $k = e^{-\tau} < 1$. Note that Conditions $(*)$ and $(**)$ are also satisfied for all $\eta, \theta, \vartheta \in \Gamma$ with $\alpha_s(\eta, \theta, \vartheta) \geq 1$ and $A\eta = A\theta = B\vartheta$.

Inspired by the work of [10], [18], [26], [32], [36], we derive some common fixed point theorems for generalized integral type (I) and $(II)(F - \alpha_s)$-contractions in the framework of complete S-metric spaces. In addition, we provide some consequences of the established results and give an example in support of the established result. Furthermore, we use the obtained results to achieve the solution of an integral equation and the bounded solution of a functional equation in dynamic programming. The results presented in this paper extend and generalise several previously published results in the existing literature.

3   Main results

In this section, we shall prove some common fixed point theorems for generalized integral type (I) and (II) $(F - \alpha_s)$-contractions in the setting of S-metric spaces.

Theorem 3.1. *Let (Γ, S) be a complete S-metric space, $\alpha_s\colon \Gamma^3 \to \mathbb{R}_+$ be a function. Let $A, B\colon \Gamma \to \Gamma$ be two mappings, then suppose that the following hold:*

(1) (A, B) is a pair of generalized integral type (I) $(F - \alpha_s)$ (2.2) contraction mapping,

(2) (A, B) is triangular α_s-admissible,

(3) there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,

(4) A and B are continuous.

Then (A, B) has a common fixed point.

Proof. Let $\eta_1 \in \Gamma$ be such that $\eta_1 = A\eta_0$ and $\eta_2 = B\eta_1$. By induction, we construct a sequence $\{\eta_n\}$ of points in Γ such that

$$\eta_{2i+1} = A\eta_{2i} \quad \text{and} \quad \eta_{2i+2} = B\eta_{2i+1}, \qquad (3.1)$$

where $i = 0, 1, 2, \ldots$.

By the assumption $\alpha_s(\eta_0, \eta_0, \eta_1) \geq 1$ and the pair (A, B) is α_s-admissible, by Lemma 2.7, we have

$$\alpha_s(\eta_n, \eta_n, \eta_{n+1}) \geq 1 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Then,

$$
\begin{aligned}
S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) &= S(A\eta_{2i}, A\eta_{2i}, B\eta_{2i+1}) \\
&\leq \alpha_s(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) S(A\eta_{2i}, A\eta_{2i}, B\eta_{2i+1}).
\end{aligned}
$$

By Equation (2.2), this implies that

$$F\Big(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt \Big) \leq F\Big(\alpha_s(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) \int_0^{S(A\eta_{2i}, A\eta_{2i}, B\eta_{2i+1})} \omega(t)dt \Big).$$

So, we have

$$F\Big(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt \Big) \leq F\Big(\int_0^{\Psi_{(I)}(\eta_{2i}, \eta_{2i}, \eta_{2i+1})} \omega(t)dt \Big) - \tau, \quad \text{(3.3)}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$
\begin{aligned}
\Psi_{(I)}(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) &= \max\Big\{ S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i}, \eta_{2i}, A\eta_{2i}), S(\eta_{2i+1}, \eta_{2i+1}, B\eta_{2i+1}), \\
&\qquad \frac{1}{2}\big[S(\eta_{2i}, \eta_{2i+1}, A\eta_{2i}) + S(\eta_{2i+1}, \eta_{2i+1}, B\eta_{2i+1}) \big] \Big\} \\
&= \max\Big\{ S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}), \\
&\qquad \frac{1}{2}\big[S(\eta_{2i}, \eta_{2i+1}, \eta_{2i+1}) + S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) \big] \Big\} \\
&\leq \max\Big\{ S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}), \\
&\qquad \frac{1}{2}\big[S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) + S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) \big] \Big\} \\
&\qquad \text{(by } (S2) \text{ and } Lemma\, 2.1) \\
&= \max\Big\{ S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) \Big\}.
\end{aligned}
$$

If $\max\left\{S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})\right\} = S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})$, then from Equation (3.3), we obtain

$$F\left(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt\right) \leq F\left(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt\right) - \tau,$$

which is a contradiction since $\tau > 0$. Hence, we conclude that

$$F\left(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt\right) \leq F\left(\int_0^{S(\eta_{2i}, \eta_{2i}, \eta_{2i+1})} \omega(t)dt\right) - \tau. \qquad (3.4)$$

This implies that

$$F\left(\int_0^{S(\eta_n, \eta_n, \eta_{n+1})} \omega(t)dt\right) \leq F\left(\int_0^{S(\eta_{n-1}, \eta_{n-1}, \eta_n)} \omega(t)dt\right) - \tau \qquad (3.5)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Repeating the same process as above, we obtain

$$F\left(\int_0^{S(\eta_{n-1}, \eta_{n-1}, \eta_n)} \omega(t)dt\right) \leq F\left(\int_0^{S(\eta_{n-2}, \eta_{n-2}, \eta_{n-1})} \omega(t)dt\right) - \tau. \qquad (3.6)$$

Using Equations (3.5) and (3.6), we get

$$
\begin{aligned}
F\left(\int_0^{S(\eta_n, \eta_n, \eta_{n+1})} \omega(t)dt\right) &\leq F\left(\int_0^{S(\eta_{n-1}, \eta_{n-1}, \eta_n)} \omega(t)dt\right) - \tau \\
&\leq F\left(\int_0^{S(\eta_{n-2}, \eta_{n-2}, \eta_{n-1})} \omega(t)dt\right) - 2\tau \\
&\leq F\left(\int_0^{S(\eta_{n-3}, \eta_{n-3}, \eta_{n-2})} \omega(t)dt\right) - 3\tau \\
&\leq \ \ldots \\
&\leq F\left(\int_0^{S(\eta_0, \eta_0, \eta_1)} \omega(t)dt\right) - n\tau. \qquad (3.7)
\end{aligned}
$$

Let

$$a_n = S(\eta_n, \eta_n, \eta_{n+1}).$$

Therefore, from Equation (3.7), we obtain

$$F\left(\int_0^{a_n} \omega(t)dt\right) \leq F\left(\int_0^{a_0} \omega(t)dt\right) - n\tau. \tag{3.8}$$

Then, it follows that

$$\lim_{n\to\infty} F\left(\int_0^{a_n} \omega(t)dt\right) = -\infty.$$

By $F \in \mathcal{F}$ and $(F2)$, we obtain

$$\lim_{n\to\infty} \int_0^{a_n} \omega(t)dt = 0,$$

and so, by Definition 2.16, we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S(\eta_n, \eta_n, \eta_{n+1}) = 0. \tag{3.9}$$

Now, by $F \in \mathcal{F}$ and $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (a_n)^k F(a_n) = 0. \tag{3.10}$$

Now, using Equation (3.8), we have

$$(a_n)^k \left[F\left(\int_0^{a_n} \omega(t)dt\right) - F\left(\int_0^{a_0} \omega(t)dt\right)\right] \leq -n(a_n)^k \tau \leq 0. \tag{3.11}$$

Using Equations (3.11) and (3.10), we obtain

$$\lim_{n\to\infty} n(a_n)^k = 0 = \lim_{n\to\infty} n(S(\eta_n, \eta_n, \eta_{n+1}))^k. \tag{3.12}$$

Therefore, there exists a positive integer $N_1 \in \mathbb{N}$ such that $n(a_n)^k < 1$ for all $n > N_1$, or

$$a_n = S(\eta_n, \eta_n, \eta_{n+1}) < \frac{1}{n^{1/k}}. \tag{3.13}$$

Let $m, n \in \mathbb{N}$ with $m > n > N_1$. Then, we have

$$
\begin{aligned}
S(\eta_m, \eta_m, \eta_n) &\leq a_{m-1} + a_{m-2} + \cdots + a_n \\
&\leq \sum_{p=n}^{m-1} a_p \leq \sum_{p=n}^{\infty} a_p \\
&= \sum_{p=n}^{\infty} \frac{1}{p^{1/k}}.
\end{aligned}
\tag{3.14}
$$

As $k \in (0, 1)$ and the series $\sum_{p=n}^{\infty} \frac{1}{p^{1/k}}$ is convergent, so

$$
\lim_{m,n \to \infty} S(\eta_m, \eta_m, \eta_n) = 0.
\tag{3.15}
$$

Thus, $\{\eta_n\}$ is a Cauchy sequence in an S-metric space (Γ, S). Since, by assumption (Γ, S) is complete, so there exists a $z \in \Gamma$ such that $\eta_n \to z$ implies that $\eta_{2i+1} \to z$ and $\eta_{2i+2} \to z$. As A and B are continuous, so we get $B\eta_{2i+1} \to Bz$ and $A\eta_{2i+2} \to Az$. Thus, $z = Az$. Similarly, $z = Bz$, we have $Az = Bz = z$. Then, (A, B) has a common fixed point. The proof is completed. $\qquad\square$

In the following theorem, we dropped continuity.

Theorem 3.2. *Let (Γ, S) be a complete S-metric space, $\alpha_s \colon \Gamma^3 \to \mathbb{R}_+$ be a function. Let $A, B \colon \Gamma \to \Gamma$ be two mappings, then suppose that the following hold:*

(1) (A, B) is a pair of generalized integral type (I) $(F - \alpha_s)$ (2.2) contraction mapping,

(2) (A, B) is triangular α_s-admissible,

(3) there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,

(4) If $\{\eta_n\}$ is a sequence in Γ such that $\alpha_s(\eta_n, \eta_n, \eta_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\eta_n \to \eta \in \Gamma$ as $n \to \infty$, then there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\alpha_s(\eta_{n_k}, \eta_{n_k}, \eta) \geq 1$ for all k.

Then (A, B) has a common fixed point.

Proof. Follows similar lines of proof of Theorem 3.1. Define a sequence $\eta_{2i+1} = A\eta_{2i}$ and $\eta_{2i+2} = B\eta_{2i+1}$, where $i = 0, 1, 2, \ldots$ converges to $\eta \in \Gamma$. By the assumption (4), there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\alpha_s(\eta_{n_k}, \eta_{n_k}, \eta) \geq$

1 for all k. Now, by using Equation (2.2) for all k, we have

$$
\begin{aligned}
S(\eta_{2n_k+1}, \eta_{2n_k+1}, B\eta) &= S(A\eta_{2n_k}, A\eta_{2n_k}, B\eta) \\
&\leq \alpha_s(\eta_{2n_k}, \eta_{2n_k}, \eta) S(A\eta_{2n_k}, A\eta_{2n_k}, B\eta).
\end{aligned}
$$

By Equation (2.2), this implies that

$$
F\Big(\int_0^{S(\eta_{2n_k+1}, \eta_{2n_k+1}, B\eta)} \omega(t)dt \Big) \leq F\Big(\alpha_s(\eta_{2n_k}, \eta_{2n_k}, \eta) \int_0^{S(A\eta_{2n_k}, A\eta_{2n_k}, B\eta)} \omega(t)dt \Big).
$$

So, we have

$$
F\Big(\int_0^{S(\eta_{2n_k+1}, \eta_{2n_k+1}, B\eta)} \omega(t)dt \Big) \leq F\Big(\int_0^{\Psi_{(I)}(\eta_{2n_k}, \eta_{2n_k}, \eta)} \omega(t)dt \Big) - \tau \quad (3.16)
$$

where

$$
\begin{aligned}
\Psi_{(I)}(\eta_{2n_k}, \eta_{2n_k}, \eta) &= \max\Big\{ S(\eta_{2n_k}, \eta_{2n_k}, \eta), S(\eta_{2n_k}, \eta_{2n_k}, A\eta_{2n_k}), S(\eta, \eta, B\eta), \\
&\qquad \frac{1}{2}\big[S(\eta_{2n_k}, \eta, A\eta_{2n_k}) + S(\eta, \eta, B\eta) \big] \Big\} \\
&= \max\Big\{ S(\eta_{2n_k}, \eta_{2n_k}, \eta), S(\eta_{2n_k}, \eta_{2n_k}, \eta_{2n_k+1}), S(\eta, \eta, B\eta), \\
&\qquad \frac{1}{2}\big[S(\eta_{2n_k}, \eta, \eta_{2n_k+1}) + S(\eta, \eta, B\eta) \big] \Big\}.
\end{aligned}
$$

Letting $k \to \infty$ in the above inequality, we obtain

$$
\begin{aligned}
\lim_{k\to\infty} \Psi_{(I)}(\eta_{2n_k}, \eta_{2n_k}, \eta) &= \max\Big\{ S(\eta, \eta, \eta), S(\eta, \eta, A\eta), S(\eta, \eta, B\eta), \\
&\qquad \frac{1}{2}\big[S(\eta, \eta, A\eta) + S(\eta, \eta, B\eta) \big] \Big\} \\
&= \max\Big\{ 0, S(\eta, \eta, A\eta), S(\eta, \eta, B\eta), \frac{1}{2}\big[S(\eta, \eta, A\eta) + S(\eta, \eta, B\eta) \big] \Big\} \\
&= \max\big\{ S(\eta, \eta, A\eta), S(\eta, \eta, B\eta) \big\}. \quad (3.17)
\end{aligned}
$$

Case I:

$$
\lim_{k\to\infty} \Psi_{(I)}(\eta_{2n_k}, \eta_{2n_k}, \eta) = S(\eta, \eta, B\eta).
$$

Suppose that $S(\eta, \eta, B\eta) > 0$. From Equation (3.17), for large k, $\Psi_{(I)}(\eta_{2n_k}, \eta_{2n_k}, \eta) > 0$, which implies from (3.16) that

$$F\Big(\int_0^{S(\eta_{2n_k+1}, \eta_{2n_k+1}, B\eta)} \omega(t)dt\Big) \leq F\Big(\int_0^{S(\eta, \eta, B\eta)} \omega(t)dt\Big) - \tau. \qquad (3.18)$$

Letting $k \to \infty$ in Equation (3.18), we obtain

$$F\Big(\int_0^{S(\eta, \eta, B\eta)} \omega(t)dt\Big) \leq F\Big(\int_0^{S(\eta, \eta, B\eta)} \omega(t)dt\Big) - \tau,$$

which is a contradiction, since $\tau > 0$. Hence, $S(\eta, \eta, B\eta) = 0$ and so by condition $(S1)$, we find that $\eta = B\eta$.

Case II:

$$\lim_{k \to \infty} \Psi_{(I)}(\eta_{2n_k}, \eta_{2n_k}, \eta) = S(\eta, \eta, A\eta).$$

By similar fashion, we obtain $\eta = A\eta$. Thus, $\eta = B\eta = A\eta$. This completes the proof. $\qquad \square$

Theorem 3.3. *Let (Γ, S) be a complete S-metric space, $\alpha_s \colon \Gamma^3 \to \mathbb{R}_+$ be a function. Let $A, B \colon \Gamma \to \Gamma$ be two mappings, then suppose that the following hold:*

(1) (A, B) is a pair of generalized integral type (II) $(F - \alpha_s)$ (2.3) contraction mapping,

(2) (A, B) is triangular α_s-admissible,

(3) there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,

(4) A and B are continuous.

Then, (A, B) has a common fixed point.

Proof. Follows similar lines of proof of Theorem 3.1. Define a sequence $\eta_{2i+1} = A\eta_{2i}$ and $\eta_{2i+2} = B\eta_{2i+1}$, where $i = 0, 1, 2, \ldots$.

By the assumption $\alpha_s(\eta_0, \eta_0, \eta_1) \geq 1$ and the pair (A, B) is α_s-admissible, by Lemma 2.7, we have

$$\alpha_s(\eta_n, \eta_n, \eta_{n+1}) \geq 1 \quad \text{for all} \ \ n \in \mathbb{N} \cup \{0\}.$$

Then,

$$\begin{aligned}
S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) &= S(A\eta_{2i}, A\eta_{2i}, B\eta_{2i+1}) \\
&\leq \alpha_s(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) S(A\eta_{2i}, A\eta_{2i}, B\eta_{2i+1}).
\end{aligned}$$

By Equation (2.3), this implies that

$$F\Big(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt \Big) \leq F\Big(\alpha_s(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) \int_0^{S(A\eta_{2i}, A\eta_{2i}, B\eta_{2i+1})} \omega(t)dt \Big).$$

Then, we have

$$F\Big(\int_0^{S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})} \omega(t)dt \Big) \leq F\Big(\int_0^{\Psi_{(II)}(\eta_{2i}, \eta_{2i}, \eta_{2i+1})} \omega(t)dt \Big) - \tau \quad (3.19)$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned}
&\Psi_{(II)}(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) \\
&= l_1\, S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) + l_2\, S(\eta_{2i}, \eta_{2i}, A\eta_{2i}) + l_3 S(\eta_{2i+1}, \eta_{2i+1}, B\eta_{2i+1}) \\
&\quad + l_4\, \frac{S(\eta_{2i+1}, \eta_{2i}, B\eta_{2i+1})[1 + S(\eta_{2i}, \eta_{2i}, A\eta_{2i})]}{1 + S(\eta_{2i}, \eta_{2i}, \eta_{2i+1})} \\
&\quad + l_5\, \max\{S(\eta_{2i}, \eta_{2i}, A\eta_{2i}), S(\eta_{2i+1}, \eta_{2i+1}, B\eta_{2i+1})\} \\[2mm]
&= l_1\, S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) + l_2\, S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) + l_3 S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) \\
&\quad + l_4\, \frac{S(\eta_{2i+1}, \eta_{2i}, \eta_{2i+2})[1 + S(\eta_{2i}, \eta_{2i}, \eta_{2i+1})]}{1 + S(\eta_{2i}, \eta_{2i}, \eta_{2i+1})} \\
&\quad + l_5\, \max\{S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})\} \\[2mm]
&= (l_1 + l_2)\, S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) + l_3 S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}) + l_4\, S(\eta_{2i+1}, \eta_{2i}, \eta_{2i+2}) \\
&\quad + l_5\, \max\{S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}), S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2})\} \\
&\leq (l_1 + l_2 + l_4 + l_5) S(\eta_{2i}, \eta_{2i}, \eta_{2i+1}) + (l_3 + l_4 + l_5) S(\eta_{2i+1}, \eta_{2i+1}, \eta_{2i+2}).
\end{aligned}$$

From Equation (3.19), it follows that

$$F\Big(\int_0^{S(\eta_{2i+1},\eta_{2i+1},\eta_{2i+2})} \omega(t)dt\Big)$$
$$\leq F\Big(\int_0^{(l_1+l_2+l_4+l_5)S(\eta_{2i},\eta_{2i},\eta_{2i+1})+(l_3+l_4+l_5)S(\eta_{2i+1},\eta_{2i+1},\eta_{2i+2})} \omega(t)dt\Big)$$
$$- \tau. \tag{3.20}$$

This implies that

$$F\Big(\int_0^{S(\eta_n,\eta_n,\eta_{n+1})} \omega(t)dt\Big)$$
$$\leq F\Big(\int_0^{(l_1+l_2+l_4+l_5)S(\eta_{n-1},\eta_{n-1},\eta_n)+(l_3+l_4+l_5)S(\eta_n,\eta_n,\eta_{n+1})} \omega(t)dt\Big)$$
$$- \tau, \tag{3.21}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Continuing the above process, we get

$$
\begin{aligned}
F\Big(\int_0^{d_n} \omega(t)dt\Big) \;&\leq\; F\Big(\int_0^{(l_1+l_2+l_4+l_5)d_{n-1}+(l_3+l_4+l_5)d_n} \omega(t)dt\Big) - \tau \\
&\leq\; F\Big(\int_0^{(l_1+l_2+l_4+l_5)d_{n-2}+(l_3+l_4+l_5)d_{n-1}} \omega(t)dt\Big) - 2\tau \\
&\leq\; F\Big(\int_0^{(l_1+l_2+l_4+l_5)d_{n-3}+(l_3+l_4+l_5)d_{n-2}} \omega(t)dt\Big) - 3\tau \\
&\;\;\vdots \\
&\leq\; F\Big(\int_0^{(l_1+l_2+l_4+l_5)d_0+(l_3+l_4+l_5)d_1} \omega(t)dt\Big) - n\tau,
\end{aligned}
$$

where $d_n = S(\eta_n, \eta_n, \eta_{n+1})$.

Then, it follows that

$$\lim_{n\to\infty} F\Big(\int_0^{d_n} \omega(t)dt\Big) = -\infty.$$

By $F \in \mathcal{F}$ and $(F2)$, we obtain

$$\lim_{n \to \infty} \int_0^{d_n} \omega(t)dt = 0,$$

and so, by Definition 2.16, we obtain

$$\lim_{n \to \infty} d_n = \lim_{n \to \infty} S(\eta_n, \eta_n, \eta_{n+1}) = 0.$$

Now, by $F \in \mathcal{F}$ and $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n \to \infty} (d_n)^k F(d_n) = 0.$$

Rest of the proof follows from Theorem 3.1. $\square$

Theorem 3.4. *Let* (Γ, S) *be a complete S-metric space,* $\alpha_s \colon \Gamma^3 \to \mathbb{R}_+$ *be a function. Let* $A, B \colon \Gamma \to \Gamma$ *be two mappings, then suppose that the following hold:*

(1) (A, B) *is a pair of generalized integral type* (II) $(F - \alpha_s)$ *(2.3) contraction mapping,*

(2) (A, B) *is triangular* α_s*-admissible,*

(3) there exists $\eta_0 \in \Gamma$ *such that* $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,

(4) If $\{\eta_n\}$ *is a sequence in* Γ *such that* $\alpha_s(\eta_n, \eta_n, \eta_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\eta_n \to \eta \in \Gamma$ *as* $n \to \infty$, *then there exists a subsequence* $\{\eta_{n_k}\}$ *of* $\{\eta_n\}$ *such that* $\alpha_s(\eta_{n_k}, \eta_{n_k}, \eta) \geq 1$ *for all* k.

Then (A, B) *has a common fixed point.*

Proof. It follows from Theorem 3.1 - Theorem 3.3. $\square$

For the uniqueness, we need the following additional condition.

(U) For all $u, v \in CF(A, B)$, we have $\alpha_s(u, u, v) \geq 1$, where $CF(A, B)$ represents the set of common fixed points of A and B.

Theorem 3.5. *Adding condition* (U) *to the hypotheses of Theorem 3.1 (resp. Theorem 3.3), we obtain that* η *is the unique common fixed point of* A *and* B.

Proof. We argue by contradiction, that is, suppose there exist $\eta, \zeta \in \Gamma$ such that $\eta = A\eta = B\eta$ and $\zeta = A\zeta = B\zeta$ with $\eta \neq \zeta$. By assumption (U), we have

$\alpha_s(\eta, \eta, \zeta) \geq 1$. First, we assume that hypotheses of Theorem 3.1 hold. Since, $S(\eta, \eta, \zeta) = S(A\eta, A\eta, B\zeta) > 0$, by Equation (2.2), we have

$$
\begin{aligned}
\tau + F\Big(\int_0^{S(\eta,\eta,\zeta)} \omega(t)dt \Big) &= \tau + F\Big(\int_0^{S(A\eta,A\eta,B\zeta)} \omega(t)dt \Big) \\
&\leq F\Big(\int_0^{\Psi_{(I)}(\eta,\eta,\zeta)} \omega(t)dt \Big) = F\Big(\int_0^{S(\eta,\eta,\zeta)} \omega(t)dt \Big).
\end{aligned}
$$

Then, by $(F1)$, we have

$$
0 < \int_0^{S(\eta,\eta,\zeta)} \omega(t)dt < \int_0^{S(\eta,\eta,\zeta)} \omega(t)dt,
$$

which is a contradiction. Hence, $\eta = \zeta$. By a similar fashion, we can prove uniqueness of common fixed point of Theorem 3.3. □

4   Consequences

If

$$
\begin{aligned}
\Psi_{(I)}(\eta, \theta, \vartheta) &= \max \Big\{ S(\eta, \theta, \vartheta), S(\eta, \theta, A\eta), S(\vartheta, \vartheta, B\vartheta), \\
& \qquad \frac{S(\eta, \vartheta, A\eta) + S(\vartheta, \vartheta, B\vartheta)}{2} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{(II)}(\eta, \theta, \vartheta) &= l_1\, S(\eta, \theta, \vartheta) + l_2\, S(\eta, \theta, A\eta) + l_3\, S(\vartheta, \vartheta, B\vartheta) \\
& \quad + l_4\, \frac{S(\vartheta, \theta, B\vartheta)[1 + S(\eta, \theta, A\eta)]}{1 + S(\eta, \theta, \vartheta)} \\
& \quad + l_5\, \max \big\{ S(\eta, \theta, A\eta), S(\vartheta, \vartheta, B\vartheta) \big\}
\end{aligned}
$$

and $A = B$ in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, then we have the following corollaries.

Corollary 4.1. *Let (Γ, S) be a complete S-metric space and let $A \colon \Gamma \to \Gamma$ be an α_s-admissible mapping such that the following hold:*
   (1) *A is a generalized integral type (I) $(F - \alpha_s)$ contraction mapping,*
   (2) *there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,*

(3) *A is continuous.*

Then, A has a fixed point $\eta \in \Gamma$ and A is a Picard operator, that is, $\{A^n \eta_0\}$ converges to η.

Corollary 4.2. *Let (Γ, S) be a complete S-metric space and let $A \colon \Gamma \to \Gamma$ be an α_s-admissible mapping such that the following hold:*

(1) *A is a generalized integral type (I) $(F - \alpha_s)$ contraction mapping,*

(2) *there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,*

(3) *If $\{\eta_n\}$ is a sequence in Γ such that $\alpha_s(\eta_n, \eta_n, \eta_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\eta_n \to \eta \in \Gamma$ as $n \to \infty$, then there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\alpha_s(\eta_{n_k}, \eta_{n_k}, \eta) \geq 1$ for all k.*

Then, A has a fixed point $\eta \in \Gamma$ and A is a Picard operator, that is, $\{A^n \eta_0\}$ converges to η.

Corollary 4.3. *Let (Γ, S) be a complete S-metric space and let $A \colon \Gamma \to \Gamma$ be an α_s-admissible mapping such that the following hold:*

(1) *A is a generalized integral type (II) $(F - \alpha_s)$ contraction mapping,*

(2) *there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,*

(3) *A is continuous.*

Then, A has a fixed point $\eta \in \Gamma$ and A is a Picard operator, that is, $\{A^n \eta_0\}$ converges to η.

Corollary 4.4. *Let (Γ, S) be a complete S-metric space and let $A \colon \Gamma \to \Gamma$ be an α_s-admissible mapping such that the following hold:*

(1) *A is a generalized integral type (II) $(F - \alpha_s)$ contraction mapping,*

(2) *there exists $\eta_0 \in \Gamma$ such that $\alpha_s(\eta_0, \eta_0, A\eta_0) \geq 1$,*

(3) *If $\{\eta_n\}$ is a sequence in Γ such that $\alpha_s(\eta_n, \eta_n, \eta_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\eta_n \to \eta \in \Gamma$ as $n \to \infty$, then there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\alpha_s(\eta_{n_k}, \eta_{n_k}, \eta) \geq 1$ for all k.*

Then, A has a fixed point $\eta \in \Gamma$ and A is a Picard operator, that is, $\{A^n \eta_0\}$ converges to η.

If we take $\alpha_s(\eta, \theta, \vartheta) = 1$ and $\omega(t) = 1$ for all $t > 0$ in Theorem 3.1, then we have the following result.

Corollary 4.5. *Let* (Γ, S) *be a complete S-metric space and let* $A, B \colon \Gamma \to \Gamma$ *be two mappings, then suppose that the following hold:*

(1) $S(A\eta, A\theta, B\vartheta) > 0 \Rightarrow \tau + F\big(S(A\eta, A\theta, B\vartheta)\big) \leq F\big(\Psi_{(I)}(\eta, \theta, \vartheta)\big)$ $(*')$

for all $\eta, \theta, \vartheta \in \Gamma$, *where*

$$\Psi_{(I)}(\eta, \theta, \vartheta) = \max \Big\{ S(\eta, \theta, \vartheta), S(\eta, \theta, A\eta), S(\vartheta, \vartheta, B\vartheta), \\ \frac{S(\eta, \vartheta, A\eta) + S(\vartheta, \vartheta, B\vartheta)}{2} \Big\},$$

(2) A and B are continuous,

(3) Condition (U) *holds.*

Then, (A, B) *has a unique common fixed point.*

Example 4.1. *Let* $\Gamma = \{1, 2, 3\}$ *and S be an S-metric. Let* $S(1, 1, 3) = S(3, 3, 1) = 1$, $S(1, 1, 1) = S(2, 2, 2) = S(3, 3, 3) = 0$, $S(1, 1, 2) = S(2, 2, 1) = \frac{3}{7}$, $S(2, 2, 3) = S(3, 3, 2) = \frac{2}{7}$ *and* $S(2, 3, 1) = \frac{1}{7}$.

Define the mappings $A, B \colon \Gamma \to \Gamma$ *as follows:* $A(\eta) = 1$ *for all* $\eta \in \Gamma$, $B(1) = B(3) = 1$, $B(2) = 3$. *Let* $\omega(t) = 1$, $F(t) = ln(t)$ *for all* $t > 0$, $\alpha_s(\eta, \theta, \vartheta) = 1$ *and* $e^{-\tau} = k < 1$. *Now, we prove that Theorem 3.5 (Corollary 4.5) is satisfied.*

Let $\eta, \theta \in \Gamma$ *and let* $\eta = 2$ *and* $\theta = 3$, *then from Condition* $(*')$, *we have*

$$S(A\eta, A\eta, B\theta) \leq k\, \Psi_{(I)}(\eta, \eta, \theta)$$

where

$$\Psi_{(I)}(\eta, \eta, \theta) = \max \Big\{ S(2, 2, 3), S(2, 2, A2), S(3, 3, B3), \frac{S(2, 3, A2) + S(3, 3, B3)}{2} \Big\}$$

$$= \max \Big\{ S(2, 2, 3), S(2, 2, 1), S(3, 3, 1), \frac{S(2, 3, 1) + S(3, 3, 1)}{2} \Big\}$$

$$= \max \Big\{ \frac{2}{7}, \frac{3}{7}, 1, \frac{\frac{1}{7} + 1}{2} \Big\}$$

$$= \max \Big\{ \frac{2}{7}, \frac{3}{7}, 1, \frac{4}{7} \Big\} = 1,$$

and

$$S(A2, A2, B3) = S(1, 1, 1) = 0.$$

Thus, we see that

$$S(A\eta, A\eta, B\theta) \le k\,\Psi_{(I)}(\eta, \eta, \theta)$$
$$\Rightarrow 0 \le k.1 = k,$$

that is, $k \ge 0$. If we take $0 < k < 1$, then all the assumptions of Theorem 3.5 (Corollary 4.5) are satisfied. So, A and B have a unique common fixed point. Here, 1 is a unique common fixed point of A and B.

5   Application on an integral equation

In this section, we apply the result given by Corollary 4.5 to study the existence of a solution to a system of integral equations.

For instance, we consider the nonlinear integral equations

$$\eta(t) = g(t) + \int_0^1 \mathcal{M}_1(t, s, \eta(s))ds, \quad t \in [0, 1], \tag{5.1}$$

and

$$\eta(t) = g(t) + \int_0^1 \mathcal{M}_2(t, s, \eta(s))ds, \quad t \in [0, 1], \tag{5.2}$$

where $g\colon [0, 1] \to \mathbb{R}$ and $\mathcal{M}_1, \mathcal{M}_2\colon [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $\Gamma = C([0, 1], \mathbb{R})$ represents the set of all continuous real-valued functions on $[0, 1]$. Define an S-metric $S\colon \Gamma^3 \to \mathbb{R}$ by

$$S(\eta, \theta, \vartheta) = \|\eta - \vartheta\| + \|\theta - \vartheta\| = \sup\left\{(|\eta(t) - \vartheta(t)| + |\theta(t) - \vartheta(t)|) : t \in [0, 1]\right\},$$

for all $\eta, \theta, \vartheta \in \Gamma$.

It is well-known that (Γ, S) is a complete S-metric space (see, [24]). Now, we prove the following result.

Theorem 5.1. *Suppose the following hypotheses hold:*
There exists $k \in (0, 1)$ where $k = e^{-\tau} < 1$ and $\beta\colon \Gamma \times \Gamma \to [0, \infty)$ such that

for all $\eta, \theta \in \Gamma$ and $s \in [0, 1]$, we have

$$0 \leq |\mathcal{M}_1(t, s, \eta(s)) - \mathcal{M}_2(s, t, \theta(s))| \leq \beta(t, s)\,|\eta(s) - \theta(s)|, \qquad (5.3)$$

and

$$\sup_{t \in [0,1]} \int_0^1 \beta(t, s)ds = k.$$

Then, the integral Equations (5.1) and (5.2) have a unique common solution η^* in Γ.

Proof. For $\eta \in \Gamma$ and $t \in [0, 1]$, define the mappings

$$A\eta(t) = g(t) + \int_0^1 \mathcal{M}_1(t, p, \eta(p))dp,$$

and

$$B\eta(t) = g(t) + \int_0^1 \mathcal{M}_2(t, p, \eta(p))dp.$$

Thus, by Condition (5.3) of Theorem 5.1, we have

$$
\begin{aligned}
|A\eta(t) - B\theta(t)| \quad &\leq \quad \int_0^1 |\mathcal{M}_1(t, s, \eta(s)) - \mathcal{M}_2(t, s, \theta(s))|ds \\
&\leq \quad \int_0^1 \beta(t, s)(|\eta(s) - \theta(s)|)ds \\
&\leq \quad k\,\|\eta - \theta\|, \qquad\qquad\qquad\qquad (5.4)
\end{aligned}
$$

or equivalently,

$$2|A\eta(t) - B\theta(t)| \leq 2k\,\|\eta - \theta\|.$$

We deduce that for all $\eta, \theta \in \Gamma$

$$S(A\eta, A\eta, B\theta) \leq k\,S(\eta, \eta, \theta) \leq k\,\Psi_{(I)}(\eta, \eta, \theta).$$

Condition (1) of Corollary 4.5 holds by taking $\tau = -ln(k)$ and $F(t) = ln(t)$.

Hence, Corollary 4.5 is applicable and so the mappings A and B have a unique common fixed point, that is, the functional equations (FE) (5.1) and (5.2) have a unique common solution $\eta^* \in \Gamma$. $\square$

6   Application on a dynamic programming

The problem of dynamic programming related to a multistage process reduces to the subject of solving functional equations. The existence of solutions of functional equations and system of functional equations arising in dynamic programming which have been studied by using various fixed point theorems (for more details one can see [7], [8], [9]). Suppose that $\mathcal{U}$ and $\mathcal{V}$ are Banach spaces, $\mathcal{W} \subset \mathcal{U}$ is a state space and $\mathcal{D} \subset \mathcal{V}$ is a decision space. It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well. In this part, we are interested in solving the following functional equations arising in dynamic programming by Corollary 4.5, we set $r \colon \mathcal{W} \to \mathbb{R}$

$$r(x) = \sup_{y \in \mathcal{D}} \big\{ f(x, y) + \mathcal{G}(x, y, r(\tau(x, y))) \big\}, \quad x \in \mathcal{W}, \tag{6.1}$$

$$r(x) = \sup_{y \in \mathcal{D}} \big\{ f(x, y) + \mathcal{K}(x, y, r(\tau(x, y))) \big\}, \quad x \in \mathcal{W}, \tag{6.2}$$

where $\tau \colon \mathcal{W} \times \mathcal{D} \to \mathcal{W}$, $f \colon \mathcal{W} \times \mathcal{D} \to \mathbb{R}$, $\mathcal{G}, \mathcal{K} \colon \mathcal{W} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$. Here, we study the existence and uniqueness of $\eta_* \in B(\mathcal{W})$, where $B(\mathcal{W})$ denote the space of all bounded real-valued functions on $\mathcal{W}$, a common solution of the functional Equations (6.1) and (6.2).

Now, we equip $B(\mathcal{W})$ with the following S-metric, which is obviously a complete S-metric space,

$$S(\eta, \theta, \vartheta) = \sup \big\{ \left(|\eta(x) - \vartheta(x)| + |\theta(x) - \vartheta(x)| \right) : x \in \mathcal{W} \big\},$$

for all $\eta, \theta, \vartheta \in B(\mathcal{W})$ (see, [24]).

Now, we state the main result of this part.

Theorem 6.1. *Let* $f \colon \mathcal{W} \times \mathcal{D} \to \mathbb{R}$ *and* $\mathcal{G}, \mathcal{K} \colon \mathcal{W} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ *be two bounded functions and also let* $\tau \colon \mathcal{W} \times \mathcal{D} \to \mathcal{W}$ *be a function. Let* $A, B \colon B(\mathcal{W}) \to B(\mathcal{W})$

be defined by

$$A(h)(x) = \sup_{y \in \mathcal{D}} \big\{ f(x,y) + \mathcal{G}(x,y,h(\tau(x,y))) \big\}, \quad x \in \mathcal{W}, \qquad (6.3)$$

$$B(h)(x) = \sup_{y \in \mathcal{D}} \big\{ f(x,y) + \mathcal{K}(x,y,h(\tau(x,y))) \big\}, \quad x \in \mathcal{W}, \qquad (6.4)$$

for all $h \in B(\mathcal{W})$ and $x \in \mathcal{W}$. Suppose that the following condition holds:

$$|\mathcal{G}(x,y,h_1(\tau(x,y))) - \mathcal{K}(x,y,h_2(\tau(x,y)))| \leq k\,|h_1(x) - h_2(x)|, \qquad (6.5)$$

for all $h_1, h_2 \in B(\mathcal{W})$, $x \in \mathcal{W}$ and $y \in \mathcal{D}$. Then, the functional Equations (6.1) and (6.2) have a unique common solution in $B(\mathcal{W})$.

Proof. We know that $(B(\mathcal{W}), S)$ is a complete S-metric space. Let $\mu > 0$ be an arbitrary positive real number, $x \in \mathcal{W}$ and $h_1, h_2 \in B(\mathcal{W})$, then there exist $y_1, y_2 \in \mathcal{D}$ such that

$$A(h_1)(x) < f(x,y_1) + \mathcal{G}(x,y_1,h_1(\tau(x,y_1))) + \frac{\mu}{2}, \qquad (6.6)$$

$$B(h_2)(x) < f(x,y_2) + \mathcal{K}(x,y_2,h_2(\tau(x,y_2))) + \frac{\mu}{2}, \qquad (6.7)$$

$$A(h_1)(x) \geq f(x,y_2) + \mathcal{G}(x,y_2,h_1(\tau(x,y_2))), \qquad (6.8)$$

and

$$B(h_2)(x) \geq f(x,y_1) + \mathcal{K}(x,y_1,h_2(\tau(x,y_1))). \qquad (6.9)$$

Then, by Equations (6.5), (6.6) and (6.9), we have

$$
\begin{aligned}
A(h_1)(x) - B(h_2)(x) \;&<\; \mathcal{G}(x,y_1,h_1(\tau(x,y_1))) - \mathcal{K}(x,y_1,h_2(\tau(x,y_1))) + \frac{\mu}{2} \\
&\leq\; \big|\mathcal{G}(x,y_1,h_1(\tau(x,y_1))) - \mathcal{K}(x,y_1,h_2(\tau(x,y_1)))\big| + \frac{\mu}{2} \\
&\leq\; k\,\big|h_1(x) - h_2(x)\big| + \frac{\mu}{2}.
\end{aligned}
$$

Thus, we get

$$A(h_1)(x) - B(h_2)(x) \le k\left|h_1(x) - h_2(x)\right| + \frac{\mu}{2}. \tag{6.10}$$

Similarly, by Equations (6.5), (6.7) and (6.8), we get

$$B(h_2)(x) - A(h_1)(x) \le k\left|h_1(x) - h_2(x)\right| + \frac{\mu}{2}. \tag{6.11}$$

Therefore, by Equations (6.10) and (6.11), we obtain that

$$\max\left\{A(h_1)(x) - B(h_2)(x), -(A(h_1)(x) - B(h_2)(x))\right\} \le k\left|h_1(x) - h_2(x)\right| + \frac{\mu}{2},$$

that is,

$$\left|A(h_1)(x) - B(h_2)(x)\right| \le k\left|h_1(x) - h_2(x)\right| + \frac{\mu}{2}.$$

Hence, we have

$$2\left|A(h_1)(x) - B(h_2)(x)\right| \le 2k\left|h_1(x) - h_2(x)\right| + \mu, \tag{6.12}$$

which implies

$$S(A(h_1)(x), A(h_1)(x), B(h_2)(x)) \le k\,S(h_1(x), h_1(x), h_2(x)) + \mu. \tag{6.13}$$

Since the Inequality (6.13) is true for any $x \in \mathcal{W}$, we get

$$S(A(h_1), A(h_1), B(h_2)) \le k\,S(h_1, h_1, h_2) + \mu. \tag{6.14}$$

Finally, since $\mu > 0$ is arbitrary, so we can deduce that

$$S(A(h_1), A(h_1), B(h_2)) \le k\,S(h_1, h_1, h_2) \le k\,\Psi_{(I)}(h_1, h_1, h_2).$$

Thus, all the hypotheses of Corollary 4.5 are satisfied with $e^{-\tau} = k < 1$ and $F(t) = ln(t)$ for all $t > 0$. Hence by application of Corollary 4.5, the functional equations (6.1) and (6.2) have a unique common solution $\eta_* \in B(\mathcal{W})$. $\square$

Example 6.1. *Consider the following functional equations:*

$$A(h)(x) = \sup_{y \in \mathcal{D}} \left\{ arctan(x + 3|y|) + \frac{1}{2}\left(x + \frac{1}{1+|y|} + |h(x)|\right)\right\}, \quad (6.15)$$

$$B(j)(x) = \sup_{y \in \mathcal{D}} \left\{ arctan(x + 3|y|) + \frac{1}{2}\left(x + \frac{1}{1+|y|} + |j(x)|\right)\right\}, \quad (6.16)$$

for $x \in [0, 1]$, where $\mathcal{W} = [0, 1]$, $\mathcal{D} = \mathbb{R}$. Then,

$f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ is defined by $f(x, y) = arctan(x + 3|y|)$,

$h\colon [0, 1] \times \mathbb{R} \to [0, 1]$ is defined by $h(x, y) = x$,

$j\colon [0, 1] \times \mathbb{R} \to [0, 1]$ is defined by $j(x, y) = x$,

$\mathcal{G}\colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $\mathcal{G}(x, y, t) = \frac{1}{2}\left(x + \frac{1}{1+|y|} + |t|\right)$, and

$\mathcal{K}\colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $\mathcal{K}(x, y, u) = \frac{1}{2}\left(x + \frac{1}{1+|y|} + |u|\right)$.

It is clear that $|f(x, y)| \leq \frac{\pi}{2}$, $|\mathcal{G}(x, y, 0)| = \left|\frac{1}{2}\left(x + \frac{1}{1+|y|}\right)\right| < 2$ and $|\mathcal{K}(x, y, 0)| = \left|\frac{1}{2}\left(x + \frac{1}{1+|y|}\right)\right| < 2$ for all $x \in [0, 1]$ and all $y \in \mathbb{R}$.

Hence the first assumption of Theorem 6.1 is satisfied. Now, for all $x \in [0, 1]$ and all $y \in \mathbb{R}$ (we can assume that $|h_1| > |h_2|$ without loss of generality), it follows that:

$$|\mathcal{G}(x, y, h_1(x)) - \mathcal{K}(x, y, h_2(x))|$$

$$\begin{aligned}
&= \left|\frac{1}{2}\left(x + \frac{1}{1+|y|} + |h_1(x)|\right) - \frac{1}{2}\left(x + \frac{1}{1+|y|} + |h_2(x)|\right)\right| \\
&= \left|\frac{1}{2}\left(|h_1(x)| - |h_2(x)|\right)\right| \\
&= \frac{1}{2}\left(|h_1(x)| - |h_2(x)|\right) \\
&= \frac{1}{2}\left(\big||h_1(x)| - |h_2(x)|\big|\right) \\
&\leq \frac{1}{2}\left(|h_1(x) - h_2(x)|\right) \\
&= k\left(|h_1(x) - h_2(x)|\right),
\end{aligned}$$

where $k = \frac{1}{2} < 1$. Thus, Inequality (6.5) of Theorem 6.1 also holds, where $x \in [0, 1]$ and $y \in \mathbb{R}$. Hence, functional Equations (6.15) and (6.16) have a unique

common solution $h_* \in B([0,1])$.

7   Conclusion

In this paper, we prove some common fixed point theorems for generalized integral type (I) and (II) $(F - \alpha_s)$-contractions in the setting of complete S-metric spaces. In addition, we provide some consequences of the established results. An example is provided in support of the established result. Also the results are used to achieve the solution of an integral equation and the bounded solution of a functional equation in dynamic programming. Our results extend and generalise several well-known results in the existing literature (see, for example [10], [36] and many others).

References

[1] J. Ali and M. Imdad, An implicit function implies several contraction conditions, Sarajevo J. Math. **4(2)** (2008), 269-285.

[2] H. Aydi, α-implicit contractive pair of mappings on quasi b-metric spaces and application to integral equations, Accepted in J. Nonlinear Convex Analysis.

[3] H. Aydi and E. Karapinar, Fixed point results for generalized $\alpha - \psi$-contractions in metric-like spaces and applications, Electro. J. Differential Equations, **133(2015)** (2015), 1-15.

[4] H. Aydi and A. Felhi, Fixed points in modular spaces via α-admissible mappings and simulation functions, J. Nonlinear Sci. Appl. **9** (2016), 3686-3701.

[5] H. Aydi, E. Karapinar and H. Yazidi, Modified F-contractions via α-admissible mappings and application to integral equation, Filomat **31** (2017), 1141-1148.

[6] S. Banach, Sur les operation dans les ensembles abstraits et leur application aux equation integrals, Fund. Math. **3**(1922), 133-181.

[7] R. Baskaran and P. V. Subrahmanyam, A note on the solution of a class of functional equations, Appl. Anal. **22** (1986), 235-241.

[8] R. Bellman, Methods of Nonlinear Analysis, vol. II, Academic Press, New York-London, (1973).

[9] P. C. Bhakta and S. Mitra, Some existence theorems for functional equations arising in dynamic programming, J. Math. Anal. Appl. **98** (1984), 348-362.

[10] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. **29(9)** (2002), 531-536.

[11] S. Chaipornjareansri, Fixed point theorems for F_w-contractions in complete S-metric spaces, Thai J. Math. (2016), 98-109.

[12] S. K. Chatterjae, Fixed point theorems, Comptes Rendus de 1' Academic Bulgare des Sciences, **25** (1972), 727-730.

[13] B. C. Dhage, Generalized metric spaces and topological structure, I. Analele Stiintifice ale Universitatii Al. I. Cuza din Lasi, Serie Noua. Matematica **46** (2000), 3-24.

[14] N. Fabiano, Z. Kadelburg, N. Mirkov, V. Šešum Čavić and S. Radenović, On F-contractions: A Survey, Contemporary Math. **3(3)** (2022), 327-342. DOI:https://doi.org/10.37256/cm.3320221517

[15] A. Gupta, Cyclic contraction on S-metric space, Int. J. Anal. Appl. **3(2)** (2013), 119-130.

[16] N. Hussain, E. Karapinar, P. Salimi and F. Akbar, α-admissible mappings and related fixed point theorems, J. Inequal. Appl. **2013** (2013), 114.

[17] N. T. Hieu, N. T. Ly and N. V. Dung, A generalization of Ciric quasi-contractions for maps on S-metric spaces, Thai J. Math. **13(2)** (2015), 369-380.

[18] K. Javed, F. Uddin, F. Adeel, M. Arshad, H. Alaeidizaji and V. Parvaneh, Fixed point results for generalized contractions in S-metric spaces, Math. Anal. Contemp. Appl. **3(2)** (2021), 27-39.

[19] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. **60** (1968), 71-76.

[20] E. Karapinar, $\alpha - \psi$-Geraghty contraction type mappings and some related fixed point results, Filomat **28(1)** (2014), 37-48.

[21] J. K. Kim, S. Sedghi, A. Gholidahneh and M. M. Rezaee, Fixed point theorems in S-metric spaces, East Asian Math. J. **32(5)** (2016), 677-684.

[22] B. Khomdram, Y. Rohen, Y. M. Singh and M. S. Khan, Fixed point theorems of generalized $S - \beta - \psi$-contractive type mappings. Math/ Moravica **22(1)** (2018), 81-92.

[23] B. Khomdram, N. Priyobarta, Y. Rohen and T. Indubala, Remarks on (α, β)-admissible mappings and fixed points under $\mathcal{Z}$-contraction J. Math. 2021. https://doi.org/10.1155/2021/6697739

[24] B. Khomdram, Y. Rohen, M. S. Khan and N. Fabiano, Fixed point results for $\beta - F$-weak contraction mapping in complete S-metric spaces, Vojnotehnicki Glasnik/Miltary Technical Courier, **72(1)** (2024), 13-34. https://doi.org/10.5937/vojtehg72-48268

[25] Z. Liu, J. Li and S. M. Kang, Fixed point theorems of contractive mappings of integral type, Fixed Point Theory Appl. 2013, **2013:300**, 17 pp.

[26] N. M. Mlaiki, $\alpha - \psi$-contraction mapping on S-metric spaces, Math. Sci. Lett. **4(1)** (2015), 9-12.

[27] N. Y. $\ddot{O}zg\ddot{u}r$ and N. Tas, Some new contractive mappings on S-metric spaces and their relationships with the mapping (**S25**), Math. Sci. **11(7)** (2017), 7-16.

[28] N. Priyobarta, Y. Rohen, S. Thounaojam and S. Radenovic, Some remarks on α-admissibility in S-metric spaces, J. Inequ. Appl. 2022, **2022:34**.

[29] H. Qawaqneh, M. S. M. Noorani, W. Shatanawi and H. Alsamir, Common fixed points for pairs of triangular α-admissible mappings, J. Nonlinear Sci. Appl. **10** (2017), 6192-6204.

[30] V. L. A. Rosa and P. Vetro, Common fixed points for α, ψ, ϕ-contractions in generalized metric spaces, Nonlinear Anal. Model. Control **19(1)** (2014), 43-54.

[31] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $(\alpha - \psi)$-contractive type mappings, Nonlinear Anal. **75** (2012), 2154-2165.

[32] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik **64(3)** (2012), 258-266.

[33] S. Sedghi and N. V. Dung, Fixed point theorems on S-metric spaces, Mat. Vesnik **66(1)** (2014), 113-124.

[34] S. Sedghi, M. M. Rezaee, T. Dosenovic and S. Radenovic, Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces, Acta Univ. Sapientiae Math. **8(2)** (2016), 298-311.

[35] S. Sedghi, N. Shobkolaei, M. Shahraki and T. Dosenovic, Common fixed point of four maps in S-metric space, Math. Sci. **12** (2018), 137-143.

[36] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. (2012), **94:2012**.