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Extending the scope of a minimum-Hardy-Hilbert-type integral inequality

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Abstract

Li and He established an important variation of the Hardy-Hilbert integral inequality in 2007. This version is notable for its dependence on a minimum kernel function, its use of weighted integral norms for the primary functions in the upper bound and its optimality. In this article, we extend the applicability of this inequality by introducing an adjustable parameter that relaxes the integrability conditions, allowing for greater flexibility when selecting the weighted integral norms for the primary functions. Furthermore, we derive the corresponding series analogue and provide full proofs of all the results.

1 Introduction

1.1 Context

Integral inequalities have long served as a link between the abstract theory of functional analysis and its numerous applications in mathematics and physics. One of the most notable examples is the Hardy-Hilbert integral inequality and its many

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variants. These inequalities provide sharp upper bounds that play a central role in solving a wide range of mathematical problems. For comprehensive treatments and historical perspectives, see [2, 14, 15].

Classical variations are usually expressed in terms of a kernel function involving the maximum of two variables, $\max(x, y)$, where x and y are positive real variables. One of the simplest and most elegant results of this type, due to [4], is stated below. Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be two (measurable) functions. Then the following inequality holds:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{\max(x, y)} f(x)g(y)dxdy \leq pq \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \tag{1.1}$$

provided that the two integrals involved in the upper bound converge. This inequality is notable for the way in which it captures the interaction between the kernel function, $1/\max(x, y)$, and the integral norms of f and g. The kernel function is both symmetric and homogeneous (with degree -1), which are important properties for various applications, such as interpolation theory and the study of the boundedness properties of bilinear operators. Furthermore, the sharp constant factor *pq* represents the optimal bound under the stated conditions.

Subsequent research has built upon this classic result in a number of ways, including introducing more general kernel functions, weighted versions and multidimensional analogues. Variations involving the minimum or maximum of variables, combined with other types of transformation, have provided deeper insights into the structure of functional spaces. Further developments in this area can be found in [1, 3, 5–13].

In this article, we focus on two particular results from [5] that enrich the Hardy-Hilbert integral framework by introducing kernel functions based on the minimum function. The first result can be viewed as a natural "minimum" analogue of the inequality in Equation (1.1). More precisely, it replaces $\max(x, y)$ with $\min(x, y)$, and the domain $[0, +\infty)^2$ with $[1 + \infty)^2$, yielding a sharp upper bound under suitable integrability conditions. A formal statement is proposed below.

Theorem 1.1. *[5, Theorem 2.1] Let $p > 1$, $q = p/(p-1)$, and $f, g : [1, +\infty) \to [0, +\infty)$ be two functions. Then we have*

$$\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}f(x)g(y)dxdy$$
$$\leq \frac{1}{p^{1/p}q^{1/q}}\left[\int_1^{+\infty}\left(x^p+\frac{1}{p-1}\right)f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\left(y^q+\frac{1}{q-1}\right)g^q(y)dy\right]^{1/q},$$

provided that the two integrals of this upper bound converge.

This inequality is notable for the way in which it emphasizes the interaction between the kernel function $1/\min(x,y)$ and the weighted integral norms of f and g. Unlike its "maximum" counterpart, the presence of the minimum introduces a different balance to the functional structure, resulting in new weight terms and a distinct sharp constant factor, i.e., $1/(p^{1/p}\,q^{1/q})$.

The second result presented in [5] is the discrete analogue of Theorem 1.1, providing a series version of the integral inequality with a similarly sharp bound. It is stated formally below.

Theorem 1.2. *[5, Theorem 2.4] Let $p>1$, $q=p/(p-1)$, and $(a_n)_{n\in\mathbb{N}\setminus\{0,1\}}$ and $(b_n)_{n\in\mathbb{N}\setminus\{0,1\}}$ be two sequences of non-negative real numbers. Then we have*

$$\sum_{m=2}^{+\infty}\sum_{n=2}^{+\infty}\frac{1}{\min(m,n)}a_m b_n$$
$$\leq \frac{1}{p^{1/p}q^{1/q}}\left[\sum_{m=2}^{+\infty}\left(m^p+\frac{1}{p-1}\right)a_m^p\right]^{1/p}\left[\sum_{n=2}^{+\infty}\left(n^q+\frac{1}{q-1}\right)b_n^q\right]^{1/q},$$

provided that the two series of this upper bound converge.

Similar to its continuous counterpart, Theorem 1.2 demonstrates how the kernel function $1/\min(m,n)$ interacts with the weighted series norms of the primary sequences. Together, Theorems 1.1 and 1.2 extend the Hardy-Hilbert integral framework to continuous and discrete settings, offering a unified perspective on inequalities driven by "minimum-type" kernel functions.

1.2 Contributions

Examining Theorems 1.1 and 1.2 reveals a certain rigidity in their convergence requirements. For example, the validity of the inequality in Theorem 1.1 depends on the following integrability conditions:

$$\int_1^{+\infty} \left(x^p + \frac{1}{p-1}\right) f^p(x)dx < +\infty, \qquad \int_1^{+\infty} \left(y^q + \frac{1}{q-1}\right) g^q(y)dy < +\infty.$$

As these conditions may be too restrictive in certain situations, one area of research focuses on making them more flexible. This article discusses this topic in more detail. In particular, we introduce an adjustable parameter that extends the scope of Theorems 1.1 and 1.2. This generalization is achieved by carefully modifying the original proofs to accommodate the increased flexibility. Furthermore, the resulting upper bounds remain sharp, thereby confirming that the generalized inequalities preserve their optimal nature even under the relaxed conditions.

1.3 Organization

The remainder of the article is as follows: Section 2 contains the integral results, including the optimality of the upper bound. The series result is developed in Section 3. Section 4 provides a conclusion.

2 Integral results

2.1 Main inequality

Our main Hardy-Hilbert-type integral inequality is presented in the theorem below. We highlight the introduction of the parameter α, which constitutes the central novelty of the result and provides a flexible framework for extending the classical setting.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [1, +\infty) \to [0, +\infty)$ be two functions. Then, for any $\alpha > \max(1/p, 1/q)$, we have*

$$\int_1^{+\infty} \int_1^{+\infty} \frac{1}{\min(x, y)} f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha p^{1/p} q^{1/q}} \left[\int_1^{+\infty} \left(x^{\alpha p} + \frac{1}{\alpha p - 1}\right) f^p(x)dx\right]^{1/p} \left[\int_1^{+\infty} \left(y^{\alpha q} + \frac{1}{\alpha q - 1}\right) g^q(y)dy\right]^{1/q},$$

provided that the two integrals of this upper bound converge.

Proof of Theorem 2.1. By appropriately decomposing the integrand through the identities $1 = (x/y)^\alpha (y/x)^\alpha$ and $1/p + 1/q = 1$, and then applying the Hölder integral inequality followed by the Fubini-Tonelli integral theorem, we obtain

$$\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}f(x)g(y)dxdy$$
$$=\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[\min(x,y)]^{1/p}}\left(\frac{x}{y}\right)^{\alpha}f(x)\times\frac{1}{[\min(x,y)]^{1/q}}\left(\frac{y}{x}\right)^{\alpha}g(y)dxdy$$
$$\leq\left[\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}\left(\frac{x}{y}\right)^{\alpha p}f^p(x)dxdy\right]^{1/p}\times$$
$$\left[\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}\left(\frac{y}{x}\right)^{\alpha q}g^q(y)dxdy\right]^{1/q}$$
$$=\left[\int_1^{+\infty}\omega(x,p)f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\omega(y,q)g^q(y)dy\right]^{1/q},\tag{2.1}$$

where

$$\omega(t,r)=\int_1^{+\infty}\frac{1}{\min(t,s)}\left(\frac{t}{s}\right)^{\alpha r}ds,\tag{2.2}$$

with $\alpha>1/r$. Let us now express this integral term.

Performing the change of variables $s=ut$, applying the Chasles integral theorem and using $\alpha>1/r$, we get

$$\omega(t,r)=\int_1^{+\infty}\frac{1}{\min(t,ut)}\left(\frac{t}{ut}\right)^{\alpha r}tdu=\int_{1/t}^{+\infty}\frac{1}{\min(1,u)}\left(\frac{1}{u}\right)^{\alpha r}du$$
$$=\int_{1/t}^{1}\frac{1}{\min(1,u)}\left(\frac{1}{u}\right)^{\alpha r}du+\int_1^{+\infty}\frac{1}{\min(1,u)}\left(\frac{1}{u}\right)^{\alpha r}du$$
$$=\int_{1/t}^{1}u^{-\alpha r-1}du+\int_1^{+\infty}u^{-\alpha r}du=\left[-\frac{1}{\alpha r}u^{-\alpha r}\right]_{u=1/t}^{u=1}+\left[\frac{1}{-\alpha r+1}u^{-\alpha r+1}\right]_{u=1}^{u\to+\infty}$$
$$=\frac{1}{\alpha r}t^{\alpha r}-\frac{1}{\alpha r}+\frac{1}{\alpha r-1}=\frac{1}{\alpha r}\left(t^{\alpha r}+\frac{1}{\alpha r-1}\right).\tag{2.3}$$

It follows from Equations (2.1) and (2.3) with $r=p$ and $r=q$, and $\alpha>\max(1/p,1/q)$, and the identity $1/p+1/q=1$ that

$$
\begin{aligned}
&\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}f(x)g(y)dxdy\\
&\leq\left[\int_1^{+\infty}\frac{1}{\alpha p}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\frac{1}{\alpha q}\left(y^{\alpha q}+\frac{1}{\alpha q-1}\right)g^q(y)dy\right]^{1/q}\\
&=\frac{1}{\alpha p^{1/p}q^{1/q}}\left[\int_1^{+\infty}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\left(y^{\alpha q}+\frac{1}{\alpha q-1}\right)g^q(y)dy\right]^{1/q}.
\end{aligned}
$$

This concludes the proof of Theorem 2.1. □

When applied to $\alpha = 1$, Theorem 2.1 reduces to Theorem 1.1. Thus the parameter α interpolates between known cases and also allows the inequality to adapt to a broader range of functional scenarios, thereby enriching its analytical significance. Furthermore, as will be demonstrated in the next subsection, the obtained upper bound is optimal.

2.2 Optimality

The optimality of the upper bound in Theorem 2.1 is formalized in the proposition below.

Proposition 2.1. *In the framework of Theorem 2.1, the upper bound is optimal.*

Proof of Proposition 2.1. Let us consider the two following functions: $f_\star(x) = x^{-\alpha q}$ for $x \in [1, +\infty)$ and $g_\star(x) = x^{-\alpha p}$ for $x \in [1, +\infty)$. Let us express the lower and upper bounds established in Theorem 2.1 defined with these functions.

For the lower bound, we have

$$
\begin{aligned}
&\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}f_\star(x)g_\star(y)dxdy\\
&=\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}x^{-\alpha q}y^{-\alpha p}dxdy\\
&=\int_1^{+\infty}x^{-\alpha(p+q)}\int_1^{+\infty}\frac{1}{\min(x,y)}\left(\frac{x}{y}\right)^{\alpha p}dydx\\
&=\int_1^{+\infty}x^{-\alpha(p+q)}\omega(x,p)dx, \qquad (2.4)
\end{aligned}
$$

where $\omega(t, r)$ is given by Equation (2.2). Using Equation (2.3) and standard power primitives combined with $\alpha > \max(1/p, 1/q)$, we get

$$\int_1^{+\infty} x^{-\alpha(p+q)}\omega(x,p)dx = \int_1^{+\infty} x^{-\alpha(p+q)}\frac{1}{\alpha p}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)dx$$
$$= \frac{1}{\alpha p}\left[\int_1^{+\infty} x^{-\alpha q}dx + \frac{1}{\alpha p-1}\int_1^{+\infty} x^{-\alpha(p+q)}dx\right]$$
$$= \frac{1}{\alpha p}\left\{\left[\frac{1}{-\alpha q+1}x^{-\alpha q+1}\right]_{x=1}^{x\to+\infty} + \frac{1}{\alpha p-1}\left[\frac{1}{-\alpha(p+q)+1}x^{-\alpha(p+q)+1}\right]_{x=1}^{x\to+\infty}\right\}$$
$$= \frac{1}{\alpha p}\left\{\frac{1}{\alpha q-1}+\frac{1}{(\alpha p-1)[\alpha(p+q)-1]}\right\}$$
$$= \frac{\alpha(p+q)-2}{(\alpha q-1)(\alpha p-1)[\alpha(p+q)-1]}. \tag{2.5}$$

It follows from Equations (2.4) and (2.5) that

$$\int_1^{+\infty}\int_1^{+\infty}\frac{1}{\min(x,y)}f_\star(x)g_\star(y)dxdy = \frac{\alpha(p+q)-2}{(\alpha q-1)(\alpha p-1)[\alpha(p+q)-1]}.$$

For the upper bound, more technical developments are necessary. Using $p(1-q) = -q$, $q(1-p) = -p$, standard power primitives combined with $\alpha > \max(1/p, 1/q)$, and $pq = p+q$, we have

$$\frac{1}{\alpha p^{1/p}q^{1/q}}\left[\int_1^{+\infty}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\left(y^{\alpha q}+\frac{1}{\alpha q-1}\right)g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha p^{1/p}q^{1/q}}\left[\int_1^{+\infty}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)x^{-\alpha pq}dx\right]^{1/p}\left[\int_1^{+\infty}\left(y^{\alpha q}+\frac{1}{\alpha q-1}\right)y^{-\alpha pq}dy\right]^{1/q}$$
$$= \frac{1}{\alpha p^{1/p}q^{1/q}}\left[\int_1^{+\infty} x^{\alpha p(1-q)}dx+\frac{1}{\alpha p-1}\int_1^{+\infty} x^{-\alpha pq}dx\right]^{1/p}\times$$
$$\left[\int_1^{+\infty} y^{\alpha q(1-p)}dy+\frac{1}{\alpha q-1}\int_1^{+\infty} y^{-\alpha pq}dy\right]^{1/q}$$

$$= \frac{1}{\alpha p^{1/p} q^{1/q}} \left[\int_1^{+\infty} x^{-\alpha q} dx + \frac{1}{\alpha p - 1} \int_1^{+\infty} x^{-\alpha p q} dx \right]^{1/p} \times$$
$$\left[\int_1^{+\infty} y^{-\alpha p} dy + \frac{1}{\alpha q - 1} \int_1^{+\infty} y^{-\alpha p q} dy \right]^{1/q}$$
$$= \frac{1}{\alpha p^{1/p} q^{1/q}} \left\{ \left[\frac{1}{-\alpha q + 1} x^{-\alpha q + 1} \right]_{x=1}^{x \to +\infty} + \frac{1}{\alpha p - 1} \left[\frac{1}{-\alpha p q + 1} x^{-\alpha p q + 1} \right]_{x=1}^{x \to +\infty} \right\}^{1/p} \times$$
$$\left\{ \left[\frac{1}{-\alpha p + 1} y^{-\alpha p + 1} \right]_{y=1}^{y \to +\infty} + \frac{1}{\alpha q - 1} \left[\frac{1}{-\alpha p q + 1} y^{-p \alpha q + 1} \right]_{y=1}^{y \to +\infty} \right\}^{1/q}$$
$$= \frac{1}{\alpha p^{1/p} q^{1/q}} \left[\frac{1}{\alpha q - 1} + \frac{1}{(\alpha p - 1)(\alpha p q - 1)} \right]^{1/p} \left[\frac{1}{\alpha p - 1} + \frac{1}{(\alpha q - 1)(\alpha p q - 1)} \right]^{1/q}$$
$$= \frac{1}{\alpha p^{1/p} q^{1/q}} \left[\frac{\alpha(\alpha p^2 q - pq - p + q)}{(\alpha q - 1)(\alpha p - 1)(\alpha p q - 1)} \right]^{1/p} \left[\frac{\alpha(\alpha p q^2 - pq + p - q)}{(\alpha p - 1)(\alpha q - 1)(\alpha p q - 1)} \right]^{1/q}$$
$$= \frac{1}{\alpha p^{1/p} q^{1/q}} \left[\frac{\alpha(\alpha p^2 q - 2p)}{(\alpha q - 1)(\alpha p - 1)[\alpha(p+q) - 1]} \right]^{1/p} \left[\frac{\alpha(\alpha p q^2 - 2q)}{(\alpha p - 1)(\alpha q - 1)[\alpha(p+q) - 1]} \right]^{1/q}$$
$$= \left[\frac{\alpha(p+q) - 2}{(\alpha q - 1)(\alpha p - 1)[\alpha(p+q) - 1]} \right]^{1/p} \left[\frac{\alpha(p+q) - 2}{(\alpha p - 1)(\alpha q - 1)[\alpha(p+q) - 1]} \right]^{1/q}$$
$$= \frac{\alpha(p+q) - 2}{(\alpha q - 1)(\alpha p - 1)[\alpha(p+q) - 1]}.$$

We thus obtain the same result as for the lower bound, meaning that $f_\star$ and $g_\star$ are extremal functions and that the upper bound is optimal. This completes the proof of Proposition 2.1.

3 A series result

The series analogue of Theorem 2.1 is presented below.

Theorem 3.1. *Let $p > 1$, $q = p/(p-1)$, and $(a_m)_{m \in \mathbb{N} \setminus \{0,1\}}$ and $(b_n)_{n \in \mathbb{N} \setminus \{0,1\}}$ be two sequences of non-negative real numbers. Then, for any $\alpha > \max(1/p, 1/q)$, we have*

$$\sum_{m=2}^{+\infty} \sum_{n=2}^{+\infty} \frac{1}{\min(m,n)} a_m b_n$$
$$\leq \frac{1}{\alpha p^{1/p} q^{1/q}} \left[\sum_{m=2}^{+\infty} \left(m^{\alpha p} + \frac{1}{\alpha p - 1} \right) a_m^p \right]^{1/p} \left[\sum_{n=2}^{+\infty} \left(n^{\alpha q} + \frac{1}{\alpha q - 1} \right) b_n^q \right]^{1/q},$$

provided that the two series of this upper bound converge.

Proof ofl Theorem 3.1. For any $m,\ n \in \mathbb{N}\backslash\{0,1\}$, let us set $f_\dagger(x) = a_m$ for $x \in [m-1,m)$ and $g_\dagger(y) = b_n$ for $y \in [n-1,n)$. Since $1/\min(x,y)$ is a non-increasing function with respect to x and y, we have

$$\frac{1}{\min(m,n)}a_m b_n \leq \int_{n-1}^{n}\int_{m-1}^{m}\frac{1}{\min(x,y)}f_\dagger(x)g_\dagger(y)dxdy.$$

This and Theorem 2.1 applied to the functions $f_\dagger$ and $g_\dagger$ give

$$\sum_{m=2}^{+\infty}\sum_{n=2}^{+\infty}\frac{1}{\min(m,n)}a_m b_n \leq \sum_{m=2}^{+\infty}\sum_{n=2}^{+\infty}\int_{n-1}^{n}\int_{m-1}^{m}\frac{1}{\min(x,y)}f_\dagger(x)g_\dagger(y)dxdy$$
$$= \int_{1}^{+\infty}\int_{1}^{+\infty}\frac{1}{\min(x,y)}f_\dagger(x)g_\dagger(y)dxdy$$
$$\leq \frac{1}{\alpha p^{1/p}q^{1/q}}\left[\int_{1}^{+\infty}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)f_\dagger^p(x)dx\right]^{1/p}\left[\int_{1}^{+\infty}\left(y^{\alpha q}+\frac{1}{\alpha q-1}\right)g_\dagger^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha p^{1/p}q^{1/q}}\left[\sum_{m=2}^{+\infty}\int_{m-1}^{m}\left(x^{\alpha p}+\frac{1}{\alpha p-1}\right)a_m^p dx\right]^{1/p}\left[\sum_{n=2}^{+\infty}\int_{n-1}^{n}\left(y^{\alpha q}+\frac{1}{\alpha q-1}\right)b_n^q dy\right]^{1/q}$$
$$\leq \frac{1}{\alpha p^{1/p}q^{1/q}}\left[\sum_{m=2}^{+\infty}\left(m^{\alpha p}+\frac{1}{\alpha p-1}\right)a_m^p\right]^{1/p}\left[\sum_{n=2}^{+\infty}\left(n^{\alpha q}+\frac{1}{\alpha q-1}\right)b_n^q\right]^{1/q}.$$

This concludes the proof of Theorem 3.1. □

When applied to $\alpha = 1$, Theorem 3.1 reduces to Theorem 1.2. Like Theorem 2.1, the parameter α interpolates between known cases and also allows the inequality to adapt to a broader range of functional scenarios.

4 Conclusion

In this article, we extended two fundamental Hardy-Hilbert-type integral inequalities involving minimum-based kernel functions. This was achieved by introducing an adjustable parameter that relaxes the original convergence conditions while maintaining the sharpness of the constant factors. This generalization broadens the class of admissible functions and sequences, enhancing the applicability of these inequalities to diverse analytical settings. Beyond their intrinsic theoretical value, these results could form the basis for further developments in areas such as weighted norm inequalities, fractional integral operators and discrete-continuous interpolation techniques. Future research could investigate multidimensional extensions, nonlinear analogues or connections with probabilistic inequalities and potential theory, opening up new avenues of exploration within the Hardy-Hilbert integral framework.

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