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Existence of solutions of a class of fractional integrodifferential equations in Banach spaces

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Abstract

In this paper, we prove the existence of solutions of fractional integrodifferential equations with nonlocal condition by using the resolvent operators and fixed point theorem. An example is provided to illustrate the theory. As a special case of the equation the controllability problem is studied.

1 Introduction

In the theory of viscoelasticity and in the study of electromagnetism in rigid non-conducting material dielectrics one can encounter the partial integrodifferential equation of the form

$$\begin{aligned} z_t(t,x) &= z_{xx}(t,x) + \sigma\Big(t, z(t,x), \int_0^t \sigma_1(t,\tau,z(\tau,x))d\tau\Big) + \sigma_2(t, z(t,x)), \;\; t>0, \\ z(t,0) &= z(t,1) = 0, \qquad (t,x) \in [0,1]\times[0,1], \\ z(x,0) &= z_0(x), \;\; 0<x<1, \end{aligned}$$

where σ, σ_1 and σ_2 are continuous, continuously differentiable with respect to the first argument and uniformly Lipschitz continuous in t, τ and z respectively.

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The nonlinear function σ of this type with integral term σ_1 occurs in mathematical problems concerned with heat flow in materials with memory and viscoelastic problems in which the integral term represents the viscosity part of the problem [26]. When we introduce the fractional derivative for better effects on the model the above equation takes the following form

$$\begin{aligned} {}^C D^\alpha z(t,x) &= z_{xx}(t,x) + \sigma\Big(t, z(t,x), \int_0^t \sigma_1(t,\tau,z(\tau,x))d\tau\Big) + \sigma_2(t,z(t,x)), \\ & \quad t > 0, \\ z(x,0) &= z_0(x),\ \ 0 < x < 1, \end{aligned}$$

and it can be written in the abstract form as

$$\begin{aligned} {}^C D^\alpha z(t) + Az(t) &= f(t,z(t)) + g\Big(t, z(s), \int_0^t k(t,\tau,z(\tau))d\tau\Big),\ t > 0, \\ z(0) &= z_0, \end{aligned}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative, $-A$ is the infinitesimal generator of a C_0-semigroup in a Banach space X and the nonlinear functions $k,\ g$ and f satisfy appropriate conditions. This equation motivates the study of abstract formulation of partial fractional integrodifferential equations.

The theory of fractional differential equations has been developed by many researchers because of its numerous applications in various fields of science and engineering [3, 16, 22, 24, 27]. In fact, fractional differential equations are considered as alternative models to nonlinear differential equations [13]. The fractional order differential operator is nonlocal, which is the most relevant feature making it a useful tool in applications. The existence problem for abstract fractional differential equations has been studied extensively in the literature [1, 2, 12, 15, 20, 23, 28–30]. Byszewski [14] initiated the study of nonlocal Cauchy problem for semilinear evolution equation. Balachandran and Park [6] extended the results to nonlinear integrodifferential equations by using resolvent operators. It has been observed that the nonlocal initial condition has better effects than the classical condition [19]. The nonlocal Cauchy problem for abstract fractional differential equations was discussed in [7, 8], where as in [4, 9] the authors studied the existence of solutions of fractional impulsive evolution equations and integrodifferential equations in Banach spaces by using fixed point techniques. Hernandez et al. [17] investigated the existence results for abstract fractional differential equations by utilizing the resolvent operator of integral equations and fixed point theorems. Further they [18] established the existence of solutions for a class of abstract fractional differential equations with nonlocal conditions. Balachandran and Kiruthika [5] proved the existence of solutions for fractional integrodifferential equations with nonlocal conditions. It is observed that the concept of mild solutions for abstract fractional differential equations via semigroup theory is not appropriate and not realistic. In order to overcome this drawback, the mild solution is developed by using the resolvent operators of integral equations [17]. Extending this technique to a large class of equations, in this paper we study the existence problem for fractional integrodifferential equations with nonlocal conditions and obtain the results by using resolvent operators and the Krasnoselskii fixed point theorem. An example is provided and the controllability problem for a special case is discussed.

2 Preliminaries

We need some basic definitions and properties of fractional calculus and operator theory for establishing our results. Let X be a Banach space and $C(J;X)$ denote the space of all continuous functions from $J := [0,b]$ into a Banach space X with supnorm denoted by $\|\cdot\|_{C(J;X)}$. The notation X_A denotes the domain of A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$. In addition, $B_r(x,X)$ represents the closed ball with center at x and radius r in X.

Definition 2.1. *[22]*(Riemann-Liouville Fractional Integral). *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}}, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2.2. *[22]*(Riemann-Liouville Fractional Derivative). *The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in N$, is defined as*

$$D^\alpha f(t) = D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3. *[22]*(Caputo Fractional Derivative). *The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, is defined as*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{2.3}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$. If $0 < \alpha < 1$, then

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)ds}{(t-s)^{\alpha}}, \tag{2.4}$$

where $f'(s) = Df(s) = \dfrac{df(s)}{ds}$ and f is an abstract function with values in X.

Consider the fractional differential equation

$$\left.\begin{array}{l} ^{C}D^{\alpha} u(t) = Au(t) + f(t), \quad t \in J, \\ u(0) = u_0, \end{array}\right\} \tag{2.5}$$

where $0 < \alpha < 1$, A is a closed linear unbounded operator in X and $f \in C(J; X)$. Equation (2.5) is equivalent to the following integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Au(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \qquad t \in J. \tag{2.6}$$

This equation can be written in the following form of integral equation

$$u(t) = w(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Au(s)}{(t-s)^{1-\alpha}} ds, \qquad t \geq 0, \tag{2.7}$$

where $w(t) = u_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}}$. We assume that the integral equation (2.7) has an associated resolvent operator $\{S(t)\}$ on X and it is analytic [25], and there exists a function φ_A in $L^1_{loc}([0,\infty); \mathbb{R}^+)$ such that

$$\|S'(t)x\| \leq \varphi_A(t) \|x\|_{X_A} \qquad \text{for all } \ t > 0.$$

Definition 2.4. *A one parameter family of bounded linear operators $\{S(t)\}$ on X is called a resolvent operator for (2.7) if the following conditions hold:*

(i) $S(\cdot)x \in C([0,\infty); X)$ *and S(0) x = x for all* $x \in X$*,*

(ii) $S(t)D(A) \subset D(A)$ *and AS(t) x = S(t)Ax for all* $x \in D(A)$ *and every* $t \geq 0$*,*

(iii) for every $x \in D(A)$ *and* $t \geq 0$*,*

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{AS(s)x}{(t-s)^{1-\alpha}} ds. \tag{2.8}$$

Definition 2.5. *A function* $u \in C(J; X)$ *is called a mild solution of the integral equation (2.7) on J if* $\int_0^t (t-s)^{\alpha-1} u(s) ds \in D(A)$ *for all* $t \in J,\ w(t) \in C(J; X)$ *and*

$$u(t) = \frac{A}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds + w(t) \text{ for all } t \in J.$$

With the above conditions and if $w \in C(J; X_A)$, then the function $u : J \to X$ defined by [25]

$$u(t) = \int_0^t S'(t-s) w(s) ds + w(t),\ t \in J, \tag{2.9}$$

is a mild solution of (2.7) on J.

Theorem 2.1. *[31](Krasnoselskii fixed point theorem) Let* K *be a nonempty closed convex subset of a Banach space* X*. If* A *and* B *are two operators such that*

(i) $Ax + By \in K$ *for any* $x, y \in K$*,*

(ii) A *is compact and continuous,*

(iii) B *is a contraction mapping,*

then there exists $z \in K$ *such that* $z = Az + Bz$*.*

3 Existence results

Consider the following class of abstract fractional integrodifferential equations with nonlocal condition

$${}^C D^\alpha [u(t) - \int_0^t a(t-s) g(s, u(s)) ds] = Au(t) + f(t, u(t), \int_0^t k(t, s, u(s)) ds), t \in J, \tag{3.1}$$

$$u(0) + h(u) = u_0, \tag{3.2}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, A is a closed linear operator in a Banach space X with dense domain $D(A)$, $u_0 \in X$ and $f : J \times X \times X \to X$, $g : J \times X \to X$, $a : J \to R$, $k : \Delta \times X \to X$, $h : C(J; X) \to X$ are continuous. Here $\Delta = \{(t,s) : 0 \le s \le t \le b\}$. For brevity, let us take $Ku(t) = \int_0^t k(t,s,u(s))ds$.

Now we introduce the concept of mild solution for the equations (3.1)-(3.2). This equation is equivalent to the following integral equation

$$\begin{aligned} u(t) &= u_0 - h(u) + \int_0^t a(t-s)g(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{Au(s)}{(t-s)^{1-\alpha}}ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,u(s),Ku(s))}{(t-s)^{1-\alpha}}ds \text{ for all } t \in J. \end{aligned} \tag{3.3}$$

Definition 3.1. *A function $u \in C(J; X)$ is said to be a mild solution of (3.1)-(3.2) on J, if $\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds \in D(A)$ for all $t \in J$ and satisfies the integral equation (3.3).*

Suppose there exists a resolvent operator $\{S(t)\}$ which is differentiable and the functions f, g, k and h are continuous in X_A, then we have

$$\begin{aligned} u(t) = &\ u_0 - h(u) + \int_0^t a(t-s)g(s,u(s))ds + \tfrac{1}{\Gamma(\alpha)}\int_0^t \tfrac{f(s,u(s),Ku(s))}{(t-s)^{1-\alpha}}ds \\ &+ \int_0^t S'(t-s)\Bigg(u_0 - h(u) + \int_0^s a(s-\tau)g(\tau,u(\tau))d\tau \\ &+ \tfrac{1}{\Gamma(\alpha)}\int_0^s \tfrac{f(\tau,u(\tau),Ku(\tau))}{(s-\tau)^{1-\alpha}}d\tau\Bigg)ds. \end{aligned} \tag{3.4}$$

Let $x_1, x_2, y_1, y_2 \in X$, $t \in J$, $(t,s) \in \Delta$ and assume the following conditions.

(H1) The function $f : J \times X \times X \to X_A$ is continuous and there exist constants $L_1, N > 0$ such that

$$\|f(t,x_1,x_2) - f(t,y_1,y_2)\| \le L_1\left(\|x_1 - y_1\| + \|x_2 - y_2\|\right),$$

and $N = \max_{t\in J} f(t,0,0)$.

(H2) The function $k : \Delta \times X \to X_A$ is continuous and there exist constants

$B, B' > 0$ such that

$$\left\| \int_0^t [k(t,s,x) - k(t,s,y)]ds \right\| \leq B\|x-y\|$$

and $B' = \max_{t\in J} \int_0^t k(t,s,0)ds$.

(H3) The function $g : J \times X \to X_A$ is continuous and there exist constants $L_2, N_1 > 0$ such that

$$\left\| \int_0^t a(t-s)[g(s,x(s)) - g(s,y(s))]ds \right\| \leq L_2\|x-y\|.$$

and $N_1 = \max_{t\in J} \int_0^t a(t-s)g(s,0)ds$.

(H4) The function $h : C(J;X) \to X_A$ is continuous and there exists a constant $G > 0$ such that

$$\|h(x) - h(y)\| \leq G\|x-y\|.$$

(H5) $(G + L_2)(1 + b\|\varphi_A\|) < 1$.

For brevity, let us take $L = \|u_0\| + \|h(0)\| + N_1 + L_1\gamma B' + \gamma L_1 N,\ L^* = G + L_2 + \gamma L_1 B + \gamma L_1,\ \gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}$.

Theorem 3.1. *Assume the conditions $(H1)-(H5)$ hold. Then there exists a mild solution of (3.1)-(3.2) on J.*

Proof. First we transform the problem of existence of solutions of (3.1)-(3.2) into a fixed point problem. For that, by considering the equation (2.7), we introduce the map $\Phi : C(J;X) \to C(J;X)$ by

$$\begin{aligned}
\Phi u(t) &= u_0 - h(u) + \int_0^t a(t-s)g(s,u(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s,u(s),Ku(s))}{(t-s)^{1-\alpha}} ds \\
&+ \int_0^t S'(t-s)\bigg(u_0 - h(u) + \int_0^s a(s-\tau)g(\tau,u(\tau))d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^s \frac{f(\tau,u(\tau),Ku(\tau))}{(s-\tau)^{1-\alpha}} d\tau \bigg) ds.
\end{aligned}$$

Now we decompose Φ as $\Phi_1 + \Phi_2$ where

$$\Phi_1 u(t) = u_0 - h(u) + \int_0^t a(t-s)g(s,u(s))ds \\ + \int_0^t S'(t-s)(u_0 - h(u) + \int_0^s a(s-\tau)g(\tau,u(\tau))d\tau)ds,$$

and

$$\Phi_2 u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,u(s),Ku(s))}{(t-s)^{1-\alpha}}ds \\ + \int_0^t S'(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s \frac{f(\tau,u(\tau),Ku(\tau))}{(s-\tau)^{1-\alpha}}d\tau ds.$$

Here, $w(t) = u_0 - h(u) + \int_0^t a(t-s)g(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,u(s),Ku(s))}{(t-s)^{1-\alpha}}ds \in C(J;X_A)$. Let $Z = C(J;X)$ and $B_r = \{z \in Z : \|z\| \le r\}$. Choose $r \ge \frac{L(1+b\|\varphi_A\|)}{(1-L^*(1+b\|\varphi_A\|)}$. For any $u, v \in Z$, we have

$$\begin{aligned}
&\|\Phi_1 u(t) + \Phi_2 u(t)\| \\
&\le \|u_0\| + \|h(u) - h(0)\| + \|h(0)\| + \int_0^t a(t-s)\|g(s,u(s)) - g(s,0)\|ds \\
&+ |\int_0^t a(t-s)g(s,0)ds| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\|f(s,u(s),Ku(s)) - f(s,0,0)\| + \|f(s,0,0)\|}{(t-s)^{1-\alpha}}ds \\
&+ \int_0^t \|S'(t-s)\|\Big(\|u_0\| + \|h(u) - h(0)\| + \|h(0)\| \\
&+ \int_0^s a(s-\tau)\|g(\tau,u(\tau)) - g(\tau,0)\|ds + |\int_0^s a(s-\tau)g(\tau,0)ds| \\
&+ \frac{1}{\Gamma(\alpha)}\Big[\int_0^s \frac{\|f(\tau,u(\tau),Ku(\tau)) - f(\tau,0,0)\|}{(s-\tau)^{1-\alpha}}d\tau + \int_0^s \frac{\|f(\tau,0,0)\|}{(s-\tau)^{1-\alpha}}d\tau\Big]\Big)ds \\
&\le \|u_0\| + Gr + \|h(0)\| + L_2 r + N_1 + \gamma[L_1(Br + B') + L_1 r + N] \\
&+ \int_0^t \|S'(t-s)\|\Big[\|u_0\| + Gr + \|h(0)\| + L_2 r + N_1 + \gamma[L_1 r + N + +L_1(Br + B')]\Big]ds \\
&\le L + L^* r + \int_0^t \|S'(t-s)\|(L + L^* r)ds \\
&\le L + L^* r + b\|\varphi_A\|(L + L^* r) \\
&\le r.
\end{aligned}$$

Thus, Φ maps B_r into itself and so $\Phi_1 u + \Phi_2 v \in B_r$. From the assumptions (H3) and (H4), we see that, for any $u \in Z$,

$$\left\| \int_0^t S'(t-s)(u_0 + h(u) + \int_0^s a(s-\tau) g(\tau, u(\tau)))d\tau \right\| \\ \leq b\|\varphi_A\|(\|u_0\| + Gr + \|h(0)\| + L_2 r + N_1),$$

which implies that the function $s \to S'(t-s)(u_0 + h(u) + \int_0^s a(s-\tau)g(\tau, u(\tau))d\tau)$ is integrable for all $t \in J$ and $\Phi_1 u \in Z$. Moreover for $u, v \in Z$ and $t \in J$, we get

$$\begin{aligned} \|\Phi_1 u(t) - \Phi_1 v(t)\| &\leq \|h(u) - h(v)\| + \left\| \int_0^t a(t-s)[g(s, u(s)) - g(s, v(s))]ds \right\| \\ &+ \int_0^t \|S'(t-s)\|(\|h(u) - h(v)\| + \left\| \int_0^s a(s-\tau)[g(\tau, u(\tau)) - g(\tau, v(\tau))]d\tau \right\|)ds \\ &\leq G\|u-v\| + L_2\|u-v\| + b\|\varphi_A\|(G\|u-v\| + L_2\|u-v\|) \\ &\leq (1 + b\|\varphi_A\|)(G + L_2)(\|u-v\|). \end{aligned}$$

By (H5), Φ_1 is a contraction on B_r. Next we show that the operator Φ_2 is completely continuous. Note that the function $s \to \int_0^t S'(t-s) \int_0^s \frac{f(\tau, u(\tau), Ku(\tau))}{(s-\tau)^{1-\alpha}} d\tau ds$ is integrable from the assumptions on $f(\cdot)$ and $k(\cdot)$. First we show that Φ_2 is uniformly bounded. Now, for $t \in J$,

$$\begin{aligned} \|\Phi_2 u(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\|f(s, u(s), Ku(s))\|}{(t-s)^{1-\alpha}} ds \\ &+ \int_0^t \|S'(t-s)\| \frac{1}{\Gamma(\alpha)} \int_0^s \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-\alpha}} d\tau ds \\ &\leq \gamma(1 + b\|\varphi_A\|)[r(L_1 + B) + B' + N]. \end{aligned}$$

This shows that Φ_2 is uniformly bounded. Let $\{u_n\}$ be a sequence in B_r such that $u_n \to u$ in B_r. Since the functions f and k are continuous, we have

$$f(s, u_n(s), Ku_n(s)) \to f(s, u(s), Ku(s)), \ as\ n \to \infty.$$

Now for each $t \in J$, we have

$$
\begin{aligned}
&\|\Phi_2 u_n(t) - \Phi_2 u(t)\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\|f(s,u_n(s),Ku_n(s)) - f(s,u(s),Ku(s))\|}{(t-s)^{1-\alpha}} ds \\
&\quad + \int_0^t \|S'(t-s)\| \Big[\frac{1}{\Gamma(\alpha)} \int_0^s \frac{\|f(\tau,u_n(\tau),Ku_n(\tau)) - f(\tau,u(\tau),Ku(\tau))\|}{(s-\tau)^{1-\alpha}} d\tau \\
&\quad \to 0 \ \ as\, n \to \infty.
\end{aligned}
$$

From the above it is clear that Φ_2 is continuous. Next we prove that the set $\{\Phi_2 u(t) : u \in B_r\}$ is relatively compact in Z for all $t \in J$. For that we have to prove that Φ_2 is equicontinuous. For $t < t+h \leq b,\ h > 0$ we have

$$
\begin{aligned}
&\|\Phi_2 u(t+h) - \Phi_2 u(t)\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t+h} \frac{f(s,u(s),Ku(s))}{(t+h-s)^{1-\alpha}} ds - \int_0^t \frac{f(s,u(s),Ku(s))}{(t-s)^{1-\alpha}} ds \right\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t+h} S'(t+h-s) \int_0^s \frac{f(\tau,u(\tau),Ku(\tau))}{(s-\tau)^{1-\alpha}} d\tau ds \right. \\
&\quad \left. - \int_0^t S'(t-s) \int_0^s \frac{f(\tau,u(\tau),Ku(\tau))}{(s-\tau)^{1-\alpha}} d\tau ds \right\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t \left[\frac{1}{(t+h-s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}} \right] \|f(s,u(s),Ku(s))\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+h} \frac{\|f(s,u(s),Ku(s))\|}{(t+h-s)^{1-\alpha}} ds \\
&\quad + \int_0^t \|S'(t+h-s) - S'(t-s)\| \frac{1}{\Gamma(\alpha)} \int_0^s \frac{\|f(\tau,u(\tau),Ku(\tau))\|}{(s-\tau)^{1-\alpha}} d\tau ds \\
&\quad + \int_t^{t+h} \|S'(t+h-s)\| \frac{1}{\Gamma(\alpha)} \int_0^s \frac{\|f(\tau,u(\tau),Ku(\tau))\|}{(s-\tau)^{1-\alpha}} d\tau ds,
\end{aligned}
$$

which tends to zero as $h \to 0$ [25], and the set $\{\Phi_2 u : u \in B_r\}$ is equicontinuous. Thus it is relatively compact and by Arzela-Ascoli's theorem, Φ_2 is compact. Hence, the Krasnoselskii fixed point theorem guarantee that there exists a fixed point $u \in Z$ such that $\Phi u = u$ which is a mild solution of (3.1) with the nonlocal condition (3.2).

4 Example

Consider the following fractional partial integrodifferential equation of the form

$$\left.\begin{array}{l}\dfrac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left(u(t,x)-\displaystyle\int_0^t a_1(t-s)u(s,x)ds\right)=\dfrac{\partial^2}{\partial x^2}u(t,x)+a_2(t)\,sin\,u(t,x)\\ +\displaystyle\int_0^t c(t-s)e^{-u(s,x)}ds,\ t>0,\\ u(t,0)=u(t,\pi)=0,\qquad (t,x)\in[0,1]\times[0,\pi],\\ u(0,x)+\displaystyle\int_0^1 a(s)u(s,x)ds=z(x),\end{array}\right\} \tag{4.1}$$

where $z \in L^2[0,\pi]$ and $a, a_1, a_2, c \in C[0,1]$. Take $X = L^2[0,\pi]$ and let A be the operator given by $Aw = w''$ with domain

$$D(A) := \{w \in X : w'' \in X, w(0) = w(\pi) = 0\}.$$

It is well known that A is an infinitesimal generator of an analytic semigroup $T(t)$ on X. Clearly A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenfunctions are given by $w_n(x) := \sqrt{(2/\pi)}sin\,(nx)$. In addition, $\{w_n : n \in \mathbb{N}\}$ is an orthogonal basis for X and

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}\langle w, w_n\rangle\, w_n \text{ for all } w \in X \text{ and for every } t > 0.$$

From these expressions, it follows that $\{T(t)\}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda I - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$. From [25], we know that the integral equation

$$u(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{Au(s)}{(t-s)^{\frac{1}{2}}}ds, \qquad t \geq 0,$$

has an associated analytic resolvent operator $\{S(t)\}$ on X given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda^{\frac{1}{2}} - A)^{-1}d\lambda, & t > 0,\\ I, & t = 0,\end{cases} \tag{4.2}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the ray $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in \left(\frac{\pi}{2}, \pi\right)$. It is easy to see that {S(t)} is differentiable, and there exists a constant $N > 0$ such that $\|S'(t)x\| \leq N\|x\|$, for $x \in D(A),\ t > 0$. To represent the differential equation (4.1) in the abstract form (3.1)-(3.2), we take

$$\begin{aligned}
&a(t-s)g(s,u)(x) = a_1(t-s)u(x),\\
&f(t,u,Ku)(x) = u(x) + a_2(t)sin\,u(x) + Ku,\\
&h(u(x)) = \int_0^1 a(s)u(s,x)ds,\\
&Ku = \int_0^t k(t,s,u(x))ds = \int_0^t c(t-s)e^{-u(x)}ds.
\end{aligned}$$

Note that $\|h(u(x))-h(v(x))\| \leq |a|\;\|u-v\|$ and $L_1 = 1+max|a_2(t)|+\max|c(t)|, L_2 = \max|a_1(t)|$. Here $\|\varphi_A\| = N$ and $b = 1$. It is possible to make $(|a| + L_2)(1 + N) < 1$. Thus the conditions (H1)-(H5) of Theorem 3.2 are satisfied. Hence there is a function $u \in C([0,1]; L^2[0,\pi])$ which is a mild solution of (4.1) on [0,1].

5 Special case

In Equation (3.1), if we take $a(t-s) = \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1},\ 0 < \alpha < 1,\ h(u) = 0$ and A is a bounded linear operator then we have

$$^{C}D^{\alpha}[u(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,u(s))ds] = Au(t) + f(t,u(t),\int_0^t k(t,s,u(s))ds), \tag{5.1}$$

that is,

$$^{C}D^{\alpha}[u(t) - I^{\alpha}g(t,u(t))] = Au(t) + f(t,u(t),Ku(t)), \tag{5.2}$$

and since $^{C}D^{\alpha}I^{\alpha}u(t) = u(t)$ we have

$$\begin{aligned}
{}^{C}D^{\alpha}u(t) &= Au(t) + g(t,u(t)) + f(t,u(t),Ku(t)), \qquad (5.3)\\
u(0) &= u_0.
\end{aligned}$$

Introducing a control variable $v(t)$ and incorporating g in f the Equation (5.3) takes the form

$$\begin{aligned}
{}^{C}D^{\alpha}u(t) &= Au(t) + Bv(t) + f(t,u(t),Ku(t)),\ t \in J = [0,T], \qquad (5.4)\\
u(0) &= u_0.
\end{aligned}$$

where the control function $v \in L^2(J,V)$, a Banach space of admissible control functions with V as a Banach space and $B : V \to X$ is a bounded linear operator. If $\|A\| \leq \frac{\Gamma(\alpha+1)}{T^{\alpha}}$

then the solution of (5.4) is [10]

$$u(t) = E_\alpha(At^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\big[Bv(s) + f(s,u(s),Ku(s))\big]ds \quad (5.5)$$

where the Mittag-Leffler operator function of a bounded linear operator A is defined as [11]

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha+\beta)}.$$

If $\beta = 1$ then $E_{\alpha,1}(A)$ is $E_\alpha(A)$ and $E_1(A) = e^A$.

Definition 5.1. *[11,21] The system (5.4) is said to be controllable on the interval J if, for every $u_0, u_1 \in X$, there exists a control function $v \in L^2(J,V)$ such that $u(T) = u_1$.*

The controllability Grammian operator $W : X \to X$ is defined as

$$Wz = \int_0^T E_{\alpha,\alpha}(A(T-s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(T-s)^\alpha)zds$$

where the star denotes the adjoint of an operator. If the linear system

$$\begin{aligned} {}^C D^\alpha u(t) &= Au(t) + Bv(t), \\ u(0) &= u_0 \end{aligned}$$

is controllable on J, then the operator W is invertible [21].
Take $M_1 = \sup\limits_{0<t\leq T} \|E_\alpha(At^\alpha)u_0\|$ and $M_2 = \sup\limits_{0<t\leq T} \|(T-t)^{\alpha-1}E_\alpha(A(T-t)^\alpha)\|$.
Let $x_1, x_2, , y_1, y_2 \in B_r, t \in J, (t,s) \in \Delta$ and assume the following conditions.

(A1) The function $f : J \times X \times X \to X$ is continuous and there exist constants $L_1, N > 0$ such that

$$\|f(t,x_1,x_2) - f(t,y_1,y_2)\| \leq L_1\left(\|x_1 - y_1\| + \|x_2 - y_2\|\right)$$

and $N = \max_{t\in J}\ f(t,0,0)$.

(A2) The function $k : \Delta \times X \to X$ is continuous and there exist constants $M, M' > 0$ such that

$$\left\|\int_0^t [k(t,s,x) - k(t,s,y)]ds\right\| \leq M\|x-y\|$$

and $M' = \max_{t\in J} \int_0^t k(t,s,0)ds$.

(A3) $2TM_2L_1(1+M) < 1$.

Theorem 5.1. *Assume that (A1)-(A3) hold and if the linear system is controllable then the nonlinear integrodifferential system (5.4) is controllable on* J.

Proof. Since the linear system is controllable and so the operator W is invertible. Define the control variable $v(t)$ as

$$v(t) = (T-t)^{1-\alpha} B^* E_{\alpha,\alpha}(A^*(T-t)^\alpha) W^{-1}[u_1 - p(T,u)] \tag{5.6}$$

where

$$p(T,u) = E_\alpha(AT^\alpha)u_0 + \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha) f(s,u(s),Ku(s))ds.$$

Choose $r \geq \frac{2M_1 + 2TM_2L_1(N+M') + \|u_1\|}{1 - 2TM_2L_1(1+M)}$ and define the operator $\Psi : Z \to Z$ by

$$\begin{aligned}
&\Psi u(t)\\
&= E_\alpha(At^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1}(T-s)^{1-\alpha} E_{\alpha,\alpha}(A(t-s)^\alpha) BB^* E_{\alpha,\alpha}(A^*(T-s)^\alpha)\\
&W^{-1}[u_1 - p(T,u)]ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s,u(s),Ku(s))ds
\end{aligned} \tag{5.7}$$

Substituting (5.6) in to (5.7) one can easily see that $\Psi u(T) = u_1$ which means that the control $v(t)$ steers the system (5.4) from u_0 to u_1. This can be achieved by proving that the integral equation (5.7) has a solution, that is the operator Ψ has a fixed point. Observe that Ψ maps B_r into itself. From the assumptions

$$\|\Psi u(t)\| \leq 2M_1 + 2TM_2(L_1 + L_1M)r + 2TM_2L_1(M' + N) + \|u_1\| \leq r$$

and

$$\|\Psi u_1 - \Psi u_2\| \leq 2TM_2L_1(1+M)\|u_1 - u_2\|.$$

By $(A3)$, Ψ is a contraction. Hence there exists a unique fixed point $u \in B_r$ such that $\Psi u(t) = u(t)$ and $u(T) = u_1$. Thus the system (5.4) is controllable on J.

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