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A study of Galois group of covering spaces as a topological group

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Abstract

Let $Deck_X(\tilde{X})$ denotes the set of all covering transformations of a covering $p : (\tilde{X}, \tilde{x}) \to (X, x)$, where (X, x) is a path connected, locally path connected pointed topological spaces, then $Deck_X(\tilde{X})$ is a group [7]. If $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is a Galois covering, then we write $Gal_X(\tilde{X})$ for $Deck_X(\tilde{X})$ and call it the Galois group of $\tilde{X}$ over X. We have studied the Galois covering through the sheets in [8] , [13]. Also in [9], we have studied the Galois group of covering spaces. In this paper, first we will show that $Gal_X(\tilde{X})$ is a topological group if we consider $\tilde{X}$ is compact. Then we will study some properties of it.

1 Introduction

Throughout the paper, we assume that (X, x) is a path-connected, locally path-connected pointed topological space and maps are base point preserving continuous surjective maps. For simplicity we write X in place of (X, x). Let $\tilde{X}$ be a connected space , X be a space and $p : \tilde{X} \to X$ be a continuous map. If for every $x \in$ X has a path connected open neighborhood U such that $p^{-1}(U)$ is open in $\tilde{X}$

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and each component of $p^{-1}(U)$ is mapped homeomorphically onto U by p , then p is called a covering map and the pair $(\tilde{X}, p)$ is called a covering space of X. The fundamental group $\pi_1(X, x)$ is a group acting from right on the set $p^{-1}(x)$. Also $\pi_1(X, x)$ acts transitively on the set $p^{-1}(x)$. Thus the set $p^{-1}(x)$ is a homogeneous right $\pi_1(X, x)$-space. A covering transformation of a covering space $(\tilde{X}, p)$ of X is a homeomorphism $h : \tilde{X} \to \tilde{X}$ such that $ph = p$. Let, the set of all covering transformations of $(\tilde{X}, p)$ is denoted by $Deck_X(\tilde{X})$ and it forms a group under usual composition of mappings. If the covering space $(\tilde{X}, p)$ of X is a Galois covering space then we write $Gal_X(\tilde{X})$ for $Deck_X(\tilde{X})$ and call it the Galois group of $\tilde{X}$ over X. In this paper we will show that $Gal_X(\tilde{X})$ is a topological group if $\tilde{X}$ is compact and then we will study some properties of it. For further information see [11], [12] and [14].

2 Some useful definitions and results

Definition 2.1. *Let $(\tilde{X}, p)$ be a covering space of a locally path connected space X. If the group $Deck_x(\tilde{X})$ acts transitively on the set $p^{-1}(x)$, for every $x \in X$, then $(\tilde{X}, p)$ is called Galois.*

Definition 2.2. *Let $(\tilde{X}, p)$ be a Galois covering space of a locally path connected space X. Then, we write $Gal_X(\tilde{X})$ for $Dec_X(\tilde{X})$ and call this the Galois group of $\tilde{X}$ over X.*

Definition 2.3. *Let X and Y be two topological spaces. Let $C(X, Y)$ denotes the set of all continuous maps between X and Y. Given a compact subset K of X and an open subset U of Y, let $V(K, U)$ denotes the set of all functions $f \in C(X, Y)$ such that $f(K) \subseteq U$. Then, the collection of all such $V(K, U)$ is a subbase for the compact-open topology on $C(X, Y)$.*

Definition 2.4. *If X, Y, Z are topological spaces with Y locally compact Housdorff space, then the composition map $C(Y, Z) \times C(X, Y) \to C(X, Z)$, given by $(f, g) \mapsto f \circ g$ is continuous(here all the function spaces are given the compact-open topology and $C(Y, Z) \times C(X, Y)$ is given the product topology).*

Definition 2.5. *A topological group is a group G, which is also a topological space such that:*
(i) $\cdot : G \times G \to G$,(in case the product) given by $(x, y) \mapsto xy$, is continuous, where

$G \times G$ has the product topology;
(ii) The inversion map $^{-1} : G \to G$, given by $x \mapsto x^{-1}$ is continuous.

Definition 2.6. *A category $\mathcal{C}$ consists of*
(i) a class of objects A, B, C,.... denoted by $Ob(\mathcal{C})$;
(ii) for each (A, B) a set of morphisms from A to B denoted by $\mathcal{C}(X, Y)$;
(iii) for A, B and C and a pair of morphisms $f : A \to B$ and $g : B \to C$, their composition $g \circ f : A \to C$, satisfies :
(a) associativity: if $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f \in \mathcal{C}(A, D)$;
(b) identity: for each $B \in \mathcal{C}$ there is a morphism $I_B \in \mathcal{C}(B, B)$ such that if $f \in \mathcal{C}(A, B)$, then $I_B \circ f = f$ and if $h \in \mathcal{C}(B, C)$, then $h \circ I_B = h$.

Definition 2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravarient functor F from $\mathcal{C}$ to $\mathcal{D}$ consists of*
(i) an object map that maps every A of $\mathcal{C}$ to $F(A)$ of $\mathcal{D}$; and
(ii) a morphism map which maps every morphism $f : A \to B$ in $\mathcal{C}$ to a morphism $F(f) : F(B) \to F(A)$ in $\mathcal{D}$ such that:
(a) $F(I_A) = I_{F(A)}$;
(b) $F(g \circ f) = F(f) \circ F(g)$, for $g : B \to C$.

Definition 2.8. *If $f \in C(X, Y)$ be base point preserving then its homotopy class is the equivalence class $[f] = \{g \in C(X, Y) : f \simeq g\}$, where $f \simeq g$ is read as f is homotopic to g. The family of all such homotopy class is denoted by $[X, Y]$.*

Lemma 2.1. *Let $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ are continuous. If $f_1 \simeq f_2$ and $g_1 \simeq g_2$ then $g_1 \circ f_1 \simeq g_2 \circ f_2$; that is $[g_1 \circ f_1] = [g_2 \circ f_2]$.*

3 Galois group of covering spaces as a topological group

In this section, first we will show that $Gal_X(\tilde{X})$ is a topological group if we consider $\tilde{X}$ is compact and then we will study some properties of it. $Gal_X(\tilde{X})$ is mainly the collection of all covering transformations $h : \tilde{X} \to \tilde{X}$ where $\tilde{X}$ is a Galois covering of X.

Theorem 3.1. *$Gal_X(\tilde{X})$ is a topological group with respect to compact-open topology and composition of mappings, where $\tilde{X}$ is compact Hausdorff space.*

Proof: $Gal_X(\tilde{X})$ is already a group under composition '$\circ$' of mappings. Also it is a topological space with respect to compact-open topology. So, to show $Gal_X(\tilde{X})$ is a topological group, we have to show:
(i) $\circ : Gal_X(\tilde{X}) \times Gal_X(\tilde{X}) \to Gal_X(\tilde{X})$, given by $(h_1, h_2) \mapsto h_1 \circ h_2$, is continuous and
(ii) $^{-1} : Gal_X(\tilde{X}) \to Gal_X(\tilde{X})$, given by $h \mapsto h^{-1}$, is continuous.
Let $h_1, h_2 \in Gal_X(\tilde{X})$ and (K, U) be a neighborhood of $h_1 \circ h_2$. As $\tilde{X}$ is compact Hausdorff space, it is normal. So, we can find an open V such that $h_2(K) \subseteq V \subset \bar{V} \subseteq h_1^{-1}(U)$. Therefore, $(\bar{V}, U)$ and (K, V) are neighborhoods of h_1 and h_2 respectively. So, the neighborhood $(\bar{V}, U) \times (K, V)$ of (h_1, h_2) is taken into (K, U) by the composition map '$\circ$'. So, '$\circ$' is continuous.
Again, Let (K_1, U_1) be a neighborhood of $h \in Gal_X(\tilde{X})$. Therefore, $h(K_1) \subseteq U_1 \implies h(\tilde{X} - K_1) \supseteq \tilde{X} - U_1 \implies h^{-1}(\tilde{X} - U_1) \subseteq \tilde{X} - K_1$. As $\tilde{X}$ is compact, $\tilde{X} - U_1$ is also compact. Hence, $(\tilde{X} - U_1, \tilde{X} - K_1)$ is a neighborhood of h^{-1}. Hence, '$^{-1}$' is continuous.
Hence, $Gal_X(\tilde{X})$ is a topological group with respect to compact-open topology and composition of mappings.

4 Some properties

$Gal_X(\tilde{X})$ is a topological group with respect to compact-open topology and composition of mappings with the identity element $e_{\tilde{X}}$.
Therefore $Gal_X(\tilde{X})$ is a topological space. For simplicity we can write it as (G^T, e^T).

Definition 4.1. *If $f : Y \to Z$ is a base point preserving continuous map, its homotopy class is the equivalence class $[f] = \{g \in C(Y, Z) : f \simeq g\}$. The family of all such homotopy class is denoted by $[Y; Z]$.*

Now, Consider a pointed topological space (W, w_0). Consider S= set of all base point preserving continuous maps from (W, w_0) to (G^T, e^T).
Under the composition '$\lhd$' defined by $(f_1 \lhd f_2)(w) = (f_1 \circ f_2)(w)$ for all $f_1, f_2 \in S$ and $w \in W$, S forms a group.

Theorem 4.1. *$[W; G^T]$ is a group.*

Proof: $[W; G^T]$= set of all homotopy class of base point preserving continuous maps from W to G^T, i.e, $[W; G^T] = \{[f] | f : W \to G^T$ is a continuous map :

$f(w_0) = e^T\}$. Now, we define a composition '$*$' on $[W, G^T]$ by the rule $[f] * [g] = [f \triangleleft g]$ for all $f, g \in S$. Now, $f_1 \in [f]$ and $g_1 \in [g] \implies f_1 \simeq f$ and $g_1 \simeq g$ respectively.
$\implies f_1 \triangleleft g_1 \simeq f \triangleleft g \implies [f_1 \triangleleft g_1] = [f \triangleleft g] \implies [f_1] * [g_1] = [f] * [g]$
$\implies$ '$*$' is well defined.
Now as $(S, \triangleleft)$ is a group $\implies [W; G^T]$ is a group under '$*$'.

Theorem 4.2. *If $g : Y \to Z$ is a base point preserving continuous map, then g induces a homomorphism $g^* : [Z; G^T] \to [Y; G^T]$.*

Proof: Define $g^* : [Z; G^T] \to [Y; G^T]$ by $g^*([h]) = [h \circ g]$ for all $[h] \in [Z; G^T]$. $h_1, h_2 : Z \to G^T$ and $h_1 \simeq h_2 \implies h_1 \circ g \simeq h_2 \circ g \implies [h_1 \circ g] = [h_2 \circ g]$ i.e $[h_1] = [h_2] \implies g^*([h_1]) = g^*([h_2]) \implies$ This map is well defined.
Now, $g^*([h_1] * [h_2]) = g^*([h_1 \triangleleft h_2]) = [(h_1 \triangleleft h_2) \circ g]$, by definition.
Thus, for all $y \in Y, [((h_1 \triangleleft h_2) \circ g)(y)] = [(h_1 \triangleleft h_2)(g(y))] = [(h_1 \circ h_2)(g(y))] = [((h_1 \circ g) \triangleleft (h_2 \circ g))(y)] \implies [(h_1 \triangleleft h_2) \circ g)] = [(h_1 \circ g) \triangleleft (h_2 \circ g)] = [h_1 \circ g] * [h_2 \circ g] = g^*([h_1]) * g^*([h_2])$. Therefore, $g^*([h_1] * [h_2]) = g^*([h_1]) * g^*([h_2]) \implies g^*$ is a group homomorphism.

Let 'Htp' denotes the category of pointed topological spaces and homotopy classes of theire base point preserving continuous maps and 'Grp' be the category of groups and homomorphisms. Then we have the following result.

Theorem 4.3. *For the Galois topological group G^T, there exists a contravarient functor $F(G^T) : Htp \to Grp$.*

Proof: Using previous two theorems, define $F(G^T)(Y) = [Y; G^T]$ which is a group. Also, for $g : Y \to Z$ in 'Htp', $g^* = F(G^T)(g) : [Z; G^T] \to [Y; G^T]$ by $g^*([h]) = [h \circ g]$ for all $[h] \in [Z; G^T]$.
Let $f_1 : Y \to Z$ and $f_2 : Z \to W$ be base point preserving continuous maps. Then, $f_2 \circ f_1 : Y \to W$ is also a base point preserving continuous map.
Then $(f_2 \circ f_1)^* = F(G^T)(f_2 \circ f_1) : [W; G^T] \to [Y; G^T]$ by $(f_2 \circ f_1)^*([f]) = [f \circ (f_2 \circ f_1)]$ for all $f \in [W; G^T]$.
Thus, for all $y \in Y, [(f \circ (f_2 \circ f_1))(y)] = [f((f_2 \circ f_1)(y))] = [f(f_2(f_1(y)))] = [(f \circ f_2)(f_1(y))] = [((f \circ f_2) \circ f_1)(y)]$
$\implies [f \circ (f_2 \circ f_1)] = [(f \circ f_2) \circ f_1] = f_1^*([f \circ f_2)]) = f_1^*(f_2^*([f])) = (f_1^* \circ f_2^*)([f])$. Therefore, $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.
Now, for the identity map $I_Y : Y \to Y, I_Y^* = F(G^T)(I_Y) : [Y; G^T] \to [Y; G^T]$ defined by $I_Y^*([h]) = [h \circ I_Y] = [h]$. Hence, $F(G^T)$ is a contravariant functor.

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