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On derivations and endomorphism in prime rings

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Abstract

Let $\mathcal{R}$ be a ring. Our purpose in this paper is to compare some subsets of a prime ring $\mathcal{R}$, defined by a specific type of commutativity conditions that involves derivations or endomorphisms with the center of $\mathcal{R}$. As consequences, several known theorems can be either generalized or deduced (viz.; [1], [2] and [4]).

Introduction

Assume that $\mathcal{R}$ is a ring with $\mathcal{Z}(\mathcal{R})$ as its center. In simple terms, a center-like subset is one that, for certain classes of rings, coincides with $\mathcal{Z}(\mathcal{R})$ and is defined

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by a commutative condition involving a specific kind of map. The literature on this concept contains a number of theorems. In [3], Herstein introduced the set

$$\mathfrak{H}(\mathcal{R},d)=\{a\in\mathcal{R}\ \mid\ [a,d(r)]=0\ \text{ for all }\ r\in\mathcal{R}\},$$

and proved that $\mathfrak{H}(\mathcal{R},d)=\mathcal{Z}(\mathcal{R})$, if d is a nonzero derivation of a 2-torsion free prime ring $\mathcal{R}$. In [2], Bell and Daif introduced and investigat the new center-like subsets listed below:

$$\begin{aligned}\mathcal{Z}^{*}(\mathcal{R},f) &= \{a\in\mathcal{R}\ \mid\ [r,a]=[f(a),f(r)]\ \text{ for all }\ r\in\mathcal{R}\},\\ \mathcal{Z}^{**}(\mathcal{R},f) &= \{a\in\mathcal{R}\ \mid\ [r,a]=[f(r),f(a)]\ \text{ for all }\ r\in R\},\\ \mathcal{Z}_1(R,f) &= \{a\in R\ \mid\ [f(r),f(a)]=[f(r),a]+[r,f(a)]\ \text{ for all }\ r\in\mathcal{R}\},\end{aligned}$$

where f is either an epimorphism or a derivation. In [4], Idrissi et al. introduced and studied a more general concept that consists of the following subsets

$$\begin{aligned}\mathcal{Z}^{+}(\mathcal{R},f) &= \{a\in\mathcal{R}\ \mid\ [f(a),f(r)]+[r,a]\in\mathcal{Z}(\mathcal{R})\ \text{ for all }\ r\in\mathcal{R}\},\\ \mathcal{Z}^{-}(\mathcal{R},f) &= \{a\in\mathcal{R}\ \mid\ [f(a),f(r)]-[r,a]\in\mathcal{Z}(\mathcal{R})\ \text{ for all }\ r\in\mathcal{R}\},\\ \mathcal{Z}^{*-}(\mathcal{R},f) &= \{a\in\mathcal{R}\ \mid\ [f(r),f(a)]-[f(r),a]-[r,f(a)]\in\mathcal{Z}(\mathcal{R})\ \text{ for all }\ r\in\mathcal{R}\},\end{aligned}$$

where f is either an endomorphism or a derivation. With some additional assumptions, they were able to compared each of the above mentioned subsets with the center of $\mathcal{R}$ for the class of 2-torsion free prime rings.

Recently, for a ring $\mathcal{R}$ equipped with derivations d_1 and d_2, Zemzami et al. [1] defined the following subsets:

$$\begin{aligned}\mathcal{Z}^{*}(\mathcal{R},d_1,d_2) &= \{a\in\mathcal{R}|[d_1(a),d_2(r)]=[r,a]\text{for all}r\in\mathcal{R}\},\\ \mathcal{Z}^{**}(\mathcal{R},d_1,d_2) &= \{a\in\mathcal{R}|[d_1(r),d_2(a)]=[r,a]\text{for all}r\in\mathcal{R}\},\\ \mathcal{Z}_1(\mathcal{R},d_1,d_2) &= \{a\in\mathcal{R}|[d_1(r),d_2(a)]=[d_2(a),r]+[a,d_1(r)]\text{ for all }\ r\in\mathcal{R}\}.\end{aligned}$$

Moreover, for a ring $\mathcal{R}$ equipped with endomorphisms T_1 and T_2, they introduced the following subsets:

$$\begin{aligned}\mathcal{Z}^{\wedge}(\mathcal{R},T_1,T_2) &= \{a\in\mathcal{R}\ \mid\ [r,a]=[T_1(a),T_2(r)]=[T_2(a),T_1(r)]\ \text{ for all }\ r\in\mathcal{R}\},\\ \mathcal{Z}^{\wedge\wedge}(\mathcal{R},T_1,T_2) &= \{a\in\mathcal{R}\ \mid\ [r,a]=[T_1(r),T_2(a)]=[T_2(r),T_1(a)]\ \text{ for all }\ r\in\mathcal{R}\}.\end{aligned}$$

In fact, under certain additional assumptions, the center $\mathcal{Z}(\mathcal{R})$ for the class of prime (semiprime) rings was compared to each of the aforementioned subsets. Further, they compared each of the aforementioned subsets with the center $\mathcal{Z}(\mathcal{R})$ for the class of prime (semiprime) rings.

This paper continues the work on the study of center-like subsets involving derivations and endomorphims in prime rings using more general identities. In fact, motivated by the results proved in [1], [2] and [4], for a ring $\mathcal{R}$ equipped with maps f and g, we introduce and study the following new center-like subsets:

$$\begin{aligned} C^*(\mathcal{R}, f, g) &= \{a \in \mathcal{R} \mid A_{(f,g,a)}(r) \in \mathcal{Z}(\mathcal{R}) \ \& \ A_{(g,f,a)}(r) \in \mathcal{Z}(\mathcal{R}) \ \text{ for all } \ r \in \mathcal{R}\}, \\ C^{**}(\mathcal{R}, f, g) &= \{a \in \mathcal{R} \mid A_{(f,-g,a)}(r) \in \mathcal{Z}(\mathcal{R}) \ \& \ A_{(-g,f,a)}(r) \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}\}, \\ C^{***}(\mathcal{R}, f) &= \{a \in \mathcal{R} \mid B_{(f,g,a)}(r) \in \mathcal{Z}(\mathcal{R}) \quad \text{for all } \ r \in \mathcal{R}\}, \\ C^{****}(\mathcal{R}, f) &= \{a \in \mathcal{R} \mid B_{(f,-g,a)}(r) \in \mathcal{Z}(\mathcal{R}) \quad \text{for all } \ r \in \mathcal{R}\}, \end{aligned}$$

where

$$A_{(f,g,a)}(r) = [f(r), g(a)] + [r, a] \quad \text{for all } \ a, r \in \mathcal{R},$$

and

$$B_{(f,g,a)}(r) = [f(r), g(a)] - [g(a), r] - [a, f(r)] \quad \text{for all } \ a, r \in \mathcal{R}.$$

We shall be concern with these sets when f and g are derivations or endomorphisms of prime rings.

1 Center-like subsets involving derivations

We begin with the following results:

Lemma 1.1. [[6], Lemma 4] Let $\mathcal{R}$ be a prime ring. If $b \in \mathcal{Z}(\mathcal{R}) \setminus \{0\}$ and $ab \in \mathcal{Z}(\mathcal{R})$ or $ba \in \mathcal{Z}(\mathcal{R})$, then $a \in \mathcal{Z}(\mathcal{R})$.

Lemma 1.2. [[5],Theorem 1] Let $\mathcal{R}$ be a prime ring with $char(\mathcal{R}) \neq 2$. If $d \neq 0$ is a derivation of $\mathcal{R}$ and $a \in \mathcal{R}$ such that $[a, d(r)] \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$, then $a \in \mathcal{Z}(\mathcal{R})$.

The results that follow are a generalization of [[4], Theorem 1], [[4], Theorem 2], [[4], Corollary 1] and [[1], Corollary 2].

Theorem 1.1. *If d and δ are two derivations of a 2-torsion free prime ring $\mathcal{R}$, then $C^*(\mathcal{R}, d, \delta) = \mathcal{Z}(\mathcal{R})$.*

Proof. (i) Since $d(\mathcal{Z}(\mathcal{R})) \subseteq \mathcal{Z}(\mathcal{R})$ and $\delta(\mathcal{Z}(\mathcal{R})) \subseteq \mathcal{Z}(\mathcal{R})$, so $\mathcal{Z}(\mathcal{R}) \subseteq C^*(\mathcal{R}, d, \delta)$.
(ii) Taking $a \in C^*(\mathcal{R}, d, \delta)$, that is,

$$A_{(d,\delta,a)}(r) \in \mathcal{Z}(\mathcal{R}) \ \text{ for all } \ r \in \mathcal{R}, \tag{1.1}$$

and

$$A_{(\delta,d,a)}(r) \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.2}$$

Putting $\delta(a)$ instead of r in (1.1), we get

$$[\delta(a), a] + [d(\delta(a)), \delta(a)] \in \mathcal{Z}(\mathcal{R}). \tag{1.3}$$

Replacing r by $(\delta(a))^2$ in (1.1), we arrive at

$$([\delta(a), a] + [d(\delta(a)), \delta(a)])\,\delta(a) \in \mathcal{Z}(\mathcal{R}).$$

Using Lemma 1.1, we obtain

$$[\delta(a), a] + [d(\delta(a)), \delta(a)] = 0 \text{ or } \delta(a) \in \mathcal{Z}(\mathcal{R}). \tag{1.4}$$

Suppose that $\delta(a) \notin \mathcal{Z}(\mathcal{R})$, so $[\delta(a), a] + [d(\delta(a)), \delta(a)] = 0$. Substituting $r\delta(a)$ for r in (1.1), we get

$$A_{(d,\delta,a)}(r)\delta(a) + [x, \delta(a)]d(\delta(a)) \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.5}$$

Now, taking $\delta(a)r$ instead of r in (1.1), one can easily prove that

$$A_{(d,\delta,a)}(r)\delta(a) + d(\delta(a))[r, \delta(a)] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.6}$$

Comparing (1.5) with (1.6), we obtain

$$[d(\delta(a)), [r, \delta(a)]] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}.$$

Define the map $\phi : \mathcal{R} \to \mathcal{R}$ as follows: $\phi(r) = [r, \delta(a)]$ for all $r \in \mathcal{R}$. Then, the last relation becomes $[d(\delta(a)), \phi(\mathcal{R})] \subseteq \mathcal{Z}(\mathcal{R})$. Since ϕ is a nonzero derivation of $\mathcal{R}$, by using Lemma 1.2, we find $d(\delta(a)) \in \mathcal{Z}(\mathcal{R})$, which means that $[\delta(a), a] = 0$. Taking ra instead of r in (1.1), we obviously get

$$A_{(d,\delta,a)}(r)a + [r, \delta(a)]d(a) \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.7}$$

On the other hand, substituting ar for r in (1.1), we obtain

$$A_{(d,\delta,a)}(r)a + d(a)[r, \delta(a)] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.8}$$

Subtracting (1.7) from (1.8), we get $[d(a), [r, \delta(a)]] \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$. That implies, $[d(a), \phi(\mathcal{R})] \subseteq \mathcal{Z}(\mathcal{R})$. Thus, by applying Lemma 1.2, we obtain $d(a) \in \mathcal{Z}(\mathcal{R})$.

Now, if $\delta(a) \in \mathcal{Z}(\mathcal{R})$, so Equation (1.1) gives $[r,a] \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$, which implies that $a \in \mathcal{Z}(\mathcal{R})$. Then, both cases of relation (1.4) lead to $d(a) \in \mathcal{Z}(\mathcal{R})$. Hence, (1.2) reduces to $[r,a] \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$, thus $a \in \mathcal{Z}(\mathcal{R})$. Hence, $C^*(\mathcal{R}, d, \delta) = \mathcal{Z}(\mathcal{R})$. $\square$

The following result is obtain by making a small modification in the proof of Theorem 1.1.

Theorem 1.2. *If d and δ are two derivations of a 2-torsion free prime ring $\mathcal{R}$, then $C^{**}(\mathcal{R}, d, \delta) = \mathcal{Z}(\mathcal{R})$.*

Corollary 1.1. *If d and δ are two derivations of a 2-torsion free prime ring $\mathcal{R}$, then the following assertions are equivalent:*

(i) $[r,a] - [d(a), \delta(x)] \in \mathcal{Z}(\mathcal{R})$ *for all* $r, a \in \mathcal{R}$.

(ii) $[r,a] + [d(a), \delta(x)] \in \mathcal{Z}(\mathcal{R})$ *for all* $r, a \in \mathcal{R}$.

(iii) $\mathcal{R}$ *is a commutative integral domain.*

The following results extend [[1], Theorem 2], [[4], Theorem 3], [[4], Theorem 4] and [[4], Corollary 2].

Theorem 1.3. *If d and δ are two derivations of a 2-torsion free prime ring $\mathcal{R}$, then $C^{***}(\mathcal{R}, d, \delta) = \mathcal{Z}(\mathcal{R})$.*

Proof. All we have to do is demonstrate $C^{***}(\mathcal{R}, d) \subseteq \mathcal{Z}(\mathcal{R})$.
Let $a \in C^{***}(\mathcal{R}, d, \delta)$. Then

$$B_{(d,\delta,a)}(r) \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.9}$$

Taking ra instead of r in (1.9), we get

$$B_{(d,\delta,a)}(r)a + rB_{(d,\delta,a)}(a) + [r, \delta(a) + a]d(a) + d(r)[a, \delta(a)] \in \mathcal{Z}(\mathcal{R}) \tag{1.10}$$

for all $r \in \mathcal{R}$. Replacing r by ar in (1.10), it follows that

$$aB_{(d,\delta,a)}(r) + B_{(d,\delta,a)}(a)r + d(a)[r, \delta(a) + a] + [a, \delta(a)]d(r) \in \mathcal{Z}(\mathcal{R}) \tag{1.11}$$

for all $r \in \mathcal{R}$. Using (1.10) and (1.11), we conclude that

$$[[r, \delta(a) + a], d(a)] + [d(r), [a, \delta(a)]] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{1.12}$$

Now taking $r\delta(a)$ in the place of r in Equation (1.9), we obtain

$$B_{(d,\delta,a)}(r)\delta(a) + [r, \delta(a) + a]d(\delta(a)) + r[d(\delta(a)), \delta(a) + a] - d(r)[a, \delta(a)] \in \mathcal{Z}(\mathcal{R}) \tag{1.13}$$

for all $r \in \mathcal{R}$. Replacing r by $\delta(a)r$ in (1.9), it is clear that

$$B_{(d,\delta,a)}(r)\delta(a)+d(\delta(a))[r,\delta(a)+a]+[d(\delta(a)),\delta(a)+a]r-[a,\delta(a)]d(r) \in \mathcal{Z}(\mathcal{R}) \quad (1.14)$$

for all $r \in \mathcal{R}$. Substituting r by $\delta(a)$ in (1.9), we arrive at $[d(\delta(a)),\delta(a)+a] \in \mathcal{Z}(\mathcal{R})$. Using (1.13) together with (1.14), we conclude that

$$[[r,\delta(a)+a],d(\delta(a))]-[d(r),[a,\delta(a)]] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \quad (1.15)$$

Applying (1.12), the above relation forces

$$[[r,\delta(a)+a],d(\delta(a))+\delta(a)] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \quad (1.16)$$

If $\delta(a)+a \in \mathcal{Z}(\mathcal{R})$, then (1.9) gives $[\delta(a),r] \in \mathcal{Z}(\mathcal{R})$, that is, $\delta(a) \in \mathcal{Z}(\mathcal{R})$. Hence, $a \in \mathcal{Z}(\mathcal{R})$. Suppose that $\delta(a)+a \notin \mathcal{Z}(\mathcal{R})$. Let us define the map $\phi_1 : \mathcal{R} \to \mathcal{R}$ by $\phi_1(r) = [r,\delta(a)+a]$ for all $r \in \mathcal{R}$. It is clear that ϕ_1 is a nonzero derivation of $\mathcal{R}$, thus, (1.16) reduces to $[d(\delta(a))+\delta(a),\phi_1(\mathcal{R})] \subseteq \mathcal{Z}(\mathcal{R})$. Using Lemma 1.2, we find $d(\delta(a))+\delta(a) \in \mathcal{Z}(\mathcal{R})$. Let $z = \delta(a)+a$. Then, $d(z) \in \mathcal{Z}(\mathcal{R})$ and (1.9) gives $[\delta(a),a] \in \mathcal{Z}(\mathcal{R})$. From (1.15), it follows that $[d(\delta(a)),\phi_1(\mathcal{R})] \subseteq \mathcal{Z}(\mathcal{R})$, so by the Lemma 1.2, we deduce that $d(\delta(a)) \in \mathcal{Z}(\mathcal{R})$, hence $\delta(a) \in \mathcal{Z}(\mathcal{R})$, therefore, as above we get $a \in \mathcal{Z}(\mathcal{R})$. That gives $\delta(a)+a \in \mathcal{Z}(\mathcal{R})$; a contradiction. Thus, $\delta(a)+a \in \mathcal{Z}(\mathcal{R})$, then (1.9) forces that $\delta(a) \in \mathcal{Z}(\mathcal{R})$. Again by (1.9), we conclude that $a \in \mathcal{Z}(\mathcal{R})$. □

The following results is obtained by applying Theorem 1.3 to the derivation "$-d$".

Theorem 1.4. *If d and δ are two derivations of a 2-torsion free prime ring $\mathcal{R}$, then $C^{****}(\mathcal{R},d,\delta) = \mathcal{Z}(\mathcal{R})$.*

Corollary 1.2. *If d and δ are two derivations of a 2-torsion free prime ring $\mathcal{R}$, then the following assertions are equivalent:*

(i) $[d(r),\delta(a)]-[d(r),a]-[\delta(a),r] \in \mathcal{Z}(\mathcal{R})$ *for all* $r,a \in \mathcal{R}$.

(ii) $[d(r),\delta(a)]+[d(r),a]+[\delta(a),r] \in \mathcal{Z}(\mathcal{R})$ *for all* $r,a \in \mathcal{R}$.

(iii) $\mathcal{R}$ *is a commutative integral domain.*

Theorem 1.3 cannot be applied to semiprime rings, which is demonstrated by the following example.

Example 1.1. *Let* $\mathcal{R} = \mathbb{Z}[X] \times M_2(\mathbb{Z})$ *and* $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Define

$$\begin{array}{rcl} d: \mathcal{R} & \to & \mathcal{R} \\ (P, N) & \mapsto & (P', 0) \end{array}$$

and

$$\begin{array}{rcl} \delta: \mathcal{R} & \to & \mathcal{R} \\ (P, N) & \mapsto & (0, AN - NA). \end{array}$$

Clearly, $\mathcal{R}$ is semiprime, d and δ are two derivations of $\mathcal{R}$. Moreover, $(0, A) \in C^{***}(\mathcal{R}, d, \delta)$ and $(0, A) \in C^{****}(\mathcal{R}, d, \delta)$, but $(0, A) \notin \mathcal{Z}(\mathcal{R})$.

2 Center-like subsets involving endomorphisms

Throughout this section, if an endomorphism T of $\mathcal{R}$ is not the identity map, then T is said to be nontrivial.

The result that follows is a generalization of [[1], Theorem 5].

Theorem 2.1. *Let $\mathcal{R}$ be a 2-torsion free prime ring. If T and H are two endomorphism of $\mathcal{R}$ such that at least one of them is non trivial, then $C^*(\mathcal{R}, T, H) \subseteq \mathcal{Z}(\mathcal{R})$.*

Proof. **Case (1):** Assume that $T = H \neq I_{\mathcal{R}}$. Then by (Theorem 6 [4]) we have $C^*(\mathcal{R}, T, H) = \mathcal{Z}(\mathcal{R})$.

Case (2): Assume that $T \neq I_{\mathcal{R}} \neq H$ and $T \neq H$. Let $a \in C^*(\mathcal{R}, T, H)$, that is,

$$[T(r), H(a)] + [r, a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}, \tag{2.1}$$

and

$$[H(r), T(a)] + [r, a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.2}$$

Taking a instead of r in (2.1), we get $[T(a), H(a)] \in \mathcal{Z}(\mathcal{R})$. Putting a^2 in place of r in (2.9), we find $2[T(a), H(a)]T(a) \in \mathcal{Z}(\mathcal{R})$. Consequently, by Lemma 1.1, we obtain $[T(a), H(a)] = 0$. Knowing that $[[T(r), H(a)], T(a)] = -[[r, a], T(a)]$ for all $r \in \mathcal{R}$, and substituting $[r, a]$ for r in (2.1), we find

$$[[r, a], T(a) + a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}.$$

Using Lemma 1.2, we arrive at $T(a)+a \in \mathcal{Z}(\mathcal{R})$ or $a \in \mathcal{Z}(\mathcal{R})$.
Assume that $T(a)+a \in \mathcal{Z}(\mathcal{R})$. Taking ra in place of r in (2.1), we arrive at

$$[T(r), H(a)]T(a) + [r,a]a \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}.$$

Thus, $[[T(r),H(a)]T(a) + [r,a]a, a] = 0$ for all $r \in \mathcal{R}$. Since $[[T(r),H(a)],a] = -[[r,a]a,a]$ for all $r \in \mathcal{R}$, we obtain $[[r,a],a](T(a)+a) \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$. Consequently, according to Lemma 1.2, either $T(a) = -a$ or $[[r,a],a] \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$ in which case $a \in \mathcal{Z}(\mathcal{R})$. If $T(a) = -a$, arguing as above we get $T(a) = -a = H(a)$ or $a \in \mathcal{Z}(\mathcal{R})$.
Suppose that $T(a) = -a = H(a)$, then (2.10) implies that

$$[H(r) - r, a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.3}$$

Substituting ra for r in (2.3), we obtain

$$[H(r) - r, a]a \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.4}$$

By Lemma 1.2, either $[H(r)-r,a] = 0$ for all $r \in \mathcal{R}$ or $a \in \mathcal{Z}(\mathcal{R})$. That is, $[H(r)-r,a] = 0$ for all $r \in \mathcal{R}$. Now, taking $[t,a]r$ instead of r in the last expression, we get

$$(H(t)-t)[r,a] + [t,a](H(r)-r) = 0 \text{ for all } t, r \in \mathcal{R}. \tag{2.5}$$

Replacing t by $[t,a]$ in (2.5), it follows that $[[t,a],a](H(r)-r) = 0$ for all $t, r \in \mathcal{R}$, which gives $[[t,a],a]\mathcal{R}(H(r)-r) = 0$ for all $t, r \in \mathcal{R}$. By using the primeness of $\mathcal{R}$ and the fact that $H \neq I_{\mathcal{R}}$, we arrive at $[[t,a],a] = 0$ for all $t \in \mathcal{R}$, which implies $a \in \mathcal{Z}(\mathcal{R})$.

Case (3) Assume that $T \neq I_{\mathcal{R}}$ and $H = I_{\mathcal{R}}$. Let $a \in C^*(\mathcal{R}, T, I_{\mathcal{R}})$. Then

$$[T(r) + r, a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}, \tag{2.6}$$

and

$$[r, T(a) + a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.7}$$

Taking a instead of r in (2.6), we arrive at $[T(a),a] \in \mathcal{Z}(\mathcal{R})$. Putting a^2 instead of r in (2.7), one can verify that $[T(a),a] = 0$. Now, replacing r by ra in (2.7), we arrive at $[r, T(a)+a]a \in \mathcal{Z}(\mathcal{R})$ for all $r \in \mathcal{R}$. Using Lemma 1.1, we find that $a \in \mathcal{Z}(\mathcal{R})$ or $T(a)+a \in \mathcal{Z}(\mathcal{R})$. Taking ra instead of r in (2.6), we obtain

$$[r,a](T(a)+a) \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.8}$$

Using Lemma 1.1, we get $a \in \mathcal{Z}(\mathcal{R})$ or $T(a) = -a$.
Assume that $T(a) = -a$. Replacing r by ra in (2.6), we find that

$$[T(r) - r, a]a \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}.$$

Again by Lemma 1.1, we get $[T(r) - r, a] = 0$ for all $r \in \mathcal{R}$. Arguing as above we obtain $[[t, a], a](T(r) - r) = 0$ for all $t, r \in \mathcal{R}$ which gives $[[t, a], a]\mathcal{R}(T(r) - r) = 0$ for all $t, r \in \mathcal{R}$. Using the primeness of $\mathcal{R}$ together with the fact that $T \neq I_R$, we conclude that $[[t, a], a] = 0$ for all $t \in \mathcal{R}$ leading to $a \in \mathcal{Z}(\mathcal{R})$.

Case (4) Assume that $T = I_{\mathcal{R}}$ and $H \neq I_{\mathcal{R}}$. By using similar arguments as in **Case (3)**, we get the proof. □

Corollary 2.1. *Let $\mathcal{R}$ be a 2-torsion free prime ring and T a nontrivial endomorphism of $\mathcal{R}$. Then $Z^+(\mathcal{R}, T) = \mathcal{Z}(\mathcal{R})$.*

We now give a generalization of [[4], Theorem 5] and [[1], Theorem 5].

Theorem 2.2. *Let $\mathcal{R}$ be a 2-torsion free prime ring. If T and H are two endomorphism of $\mathcal{R}$ such that at least one of them is non trivial, then $C^{**}(\mathcal{R}, T, H) \subseteq \mathcal{Z}(\mathcal{R})$.*

Proof. **Case (1):** Assume that $T = H \neq I_{\mathcal{R}}$. Then, by [[4], Theorem 5], we have $C^{**}(\mathcal{R}, T, H) = \mathcal{Z}(\mathcal{R})$.

Case (2): Assume that T and H are two non-trivial endomorphisms such that $T \neq H$. Let $a \in C^{**}(\mathcal{R}, T, H)$, that is,

$$[T(r), H(a)] - [r, a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}, \tag{2.9}$$

and

$$[H(r), T(a)] - [r, a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.10}$$

Taking a instead of r in (2.9), we get $[T(a), H(a)] \in \mathcal{Z}(\mathcal{R})$. Replacing r by a^2 in (2.9), we obtain $2[T(a), H(a)]T(a) \in \mathcal{Z}(\mathcal{R})$. In light of this, by Lemma 1.1, we can deduce that $[T(a), H(a)] = 0$.
Putting $[r, a]$ in place of r in (2.9) and using the fact that $[[T(r), H(a)], T(a)] = [[r, a], T(a)]$ for all $r \in \mathcal{R}$, we get

$$[[r, a], T(a) - a] \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}. \tag{2.11}$$

Applying Lemma 1.2, we get $a \in \mathcal{Z}(\mathcal{R})$ or $T(a) - a \in \mathcal{Z}(\mathcal{R})$.

Suppose that $T(a)-a\in\mathcal{Z}(\mathcal{R})$. Replacing r by ra in (2.9), we arrive at

$$[T(r),H(a)]T(a)-[r,a]a\in\mathcal{Z}(\mathcal{R}) \text{ for all } r\in\mathcal{R}.$$

Thus, $[[T(r),H(a)]T(a)-[r,a]a,a]=0$ for all $r\in\mathcal{R}$. Since $[[T(r),H(a)],a]=[[r,a]a,a]$ for all $r\in\mathcal{R}$, it follows that $[[r,a],a](T(a)-a)\in\mathcal{Z}(\mathcal{R})$ for all $r\in\mathcal{R}$. Consequently, by Lemma 1.2, either $T(a)=a$ or $[[r,a],a]\in\mathcal{Z}(\mathcal{R})$ for all $r\in\mathcal{R}$ in which case $a\in\mathcal{Z}(\mathcal{R})$.
If $T(a)=a$, arguing as above we get $T(a)=a=H(a)$ or $a\in\mathcal{Z}(\mathcal{R})$.
Suppose that $T(a)=a=H(a)$, then (2.10) implies that

$$[H(r)-r,a]\in\mathcal{Z}(\mathcal{R}) \text{ for all } r\in\mathcal{R}. \tag{2.12}$$

Taking ra instead of r in (2.12), we find that

$$[H(r)-r,a]a\in\mathcal{Z}(\mathcal{R}) \text{ for all } r\in\mathcal{R}. \tag{2.13}$$

Using Lemma 1.2, leads to $[H(r)-r,a]=0$ for all $r\in\mathcal{R}$. Now, replacing r by $[t,a]r$ in the last equation, we get

$$(H(t)-t)[r,a]+[t,a](H(r)-r)=0 \text{ for all } t,r\in\mathcal{R}. \tag{2.14}$$

Putting $[t,a]$ for r in (2.14), it follows that $[[t,a],a](H(r)-r)=0$ for all $t,r\in\mathcal{R}$ which implies $[[t,a],a]\mathcal{R}(H(r)-r)=0$ for all $t,r\in\mathcal{R}$. Using the fact that $\mathcal{R}$ is a prime and the presumption that $H\neq I_{\mathcal{R}}$, we conclude that $[[t,a],a]=0$ for all $t\in\mathcal{R}$. Hence, $a\in\mathcal{Z}(\mathcal{R})$.

Case (3) Assume that $T\neq I_{\mathcal{R}}$ and $H=I_{\mathcal{R}}$. Let $a\in C^{**}(\mathcal{R},T,I_{\mathcal{R}})$. Then,

$$[T(r)-r,a]\in\mathcal{Z}(\mathcal{R}) \text{ for all } r\in\mathcal{R}, \tag{2.15}$$

and

$$[r,T(a)-a]\in\mathcal{Z}(\mathcal{R}) \text{ for all } r\in\mathcal{R}. \tag{2.16}$$

Replacing r by a in (2.15), we arrive at $[T(a),a]\in\mathcal{Z}(\mathcal{R})$. Taking a^2 instead of r in (2.16), one can verify that $[T(a),a]=0$. Now replacing r by ra in (2.16), we find that $[r,T(a)-a]a\in\mathcal{Z}(\mathcal{R})$ for all $r\in\mathcal{R}$. So, by Lemma 1.1, we get $a\in\mathcal{Z}(\mathcal{R})$ or $T(a)-a\in\mathcal{Z}(\mathcal{R})$. Replacing r by ra in (2.15), we find that

$$[r,a](T(a)-a)\in\mathcal{Z}(\mathcal{R}) \text{ for all } r\in\mathcal{R}.$$

Using Lemma 1.1, leads to $a\in\mathcal{Z}(\mathcal{R})$ or $T(a)=a$. Assume that $T(a)=a$. Replacing r

by ra in (2.15), we find that

$$[T(r)-r,a]a \in \mathcal{Z}(\mathcal{R}) \text{ for all } r \in \mathcal{R}.$$

Again by Lemma 1.1, we find $[T(r)-r,a]=0$ for all $r \in \mathcal{R}$. Arguing as above we obtain $[[t,a],a](T(r)-r)=0$ for all $t,r \in \mathcal{R}$, which implies $[[t,a],a]\mathcal{R}(T(r)-r)=0$ for all $t,r \in \mathcal{R}$. Using the fact that $T \neq I_{\mathcal{R}}$ and the primeness of $\mathcal{R}$, we obtain $[[t,a],a]=0$ for all $t \in \mathcal{R}$ leading to $a \in \mathcal{Z}(\mathcal{R})$.

Case (4) Assume that $T = I_{\mathcal{R}}$ and $H \neq I_{\mathcal{R}}$. By using similar arguments as in **Case (3)**, we get the proof. □

Theorem 2.2 cannot be applied to semiprime rings, which is demonstrated by the following example.

Example 2.1. *Consider* $\mathcal{R} = \mathbb{C} \times M_2(\mathbb{Z})$ *and* $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. *The fact that* $\mathcal{R}$ *is a semi-prime ring can be proven without a doubt. Define*

$$\begin{aligned} T:\mathcal{R} &\rightarrow \mathcal{R} \\ (z,N) &\mapsto (z,0) \end{aligned}$$

and

$$\begin{aligned} H:\mathcal{R} &\rightarrow \mathcal{R} \\ (z,N) &\mapsto (\bar{z},N). \end{aligned}$$

Then, T *and* H *are two endomorphisms of* $\mathcal{R}$. *Moreover,* $(A,0) \in C^{**}(\mathcal{R},T,H)$ *but* $(A,0) \notin \mathcal{Z}(\mathcal{R})$.

3 Conclusions

In this study it is shown that certain subsets, defined by commutativity conditions involving derivations and endomorphisms, coincide within the center of prime rings. As the applications of the main results, we obtained improved versions of the results proved in [1] and [4], respectively. These results are open problems for rings with involution.

4 Declarations

Authors Contributions
All authors made equal contributions.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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