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Tripled fixed points in complete D^*-metric spaces via altering distance functions

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Abstract

The concept of D^*-metric spaces has been recently developed as a generalization of metric spaces. The primary objective of this paper is to present tripled fixed points in complete D^*-metric spaces by modifying distance functions. Furthermore, we demonstrate the presence and uniqueness of a common tripled fixed point in a D^*-metric space by employing altering distance functions and weak compatibility. Furthermore, we demonstrate the application of integral equations in entire D^*-metric spaces.

1 Introduction

Study of functional analysis depends on metric fixed point theory. The renowned Polish mathematician Banach [8] was the one who initially presented it. Many

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authors have made numerous generalizations in various directions throughout the years due to its significance and use in various scientific domains. Metric fixed point theory research plays crucial role in many fields, including mathematical economics, operation research, differential equations, and other fields (see [1, 2, 7, 12, 16, 22, 27–30]). Numerous mathematicians developed various generalizations of metric spaces such as D-metric spaces by Dhage [13], b-metric spaces by L. G. Huang at X. Zhang [25], G-metric spaces [23], etc.

Dhage [13] works in functional analysis, differential equations, and fixed point theory. The term "D-metric spaces" is often associated with generalized metric spaces introduced by him, expanding the idea of metric spaces. These spaces are a generalization of the classical metric space where the metric is defined by a function that takes three points, rather than two. In classical metric spaces, the distance between two points is defined as a function satisfying specific properties like non-negativity, symmetry, and the triangle inequality. In a D-metric space, the distance is instead defined by a function involving three points, which adds more flexibility and generality to the framework. Dhage's [13] contributions in this area often involve proving fixed point theorems in such generalized spaces, which have applications in solving differential and integral equations, as well as in optimization problems.

Sedghi et al. [29] define D^*-metric spaces and alter the D-metric space. Sedghi et al. [29] made major contributions to the research of D*-metric spaces. In particular, his work focuses on generalizing and applying fixed-point theorems, which are vital in mathematical analysis and applications, particularly in fields like functional analysis and topology. One of his notable papers discusses the introduction of new definitions for D*-metric spaces and establishes a common fixed-point theorem for six weakly compatible mappings within such spaces. This work is important because it generalizes earlier results on fixed-point theory in metric spaces and offers broader applications, especially in solving equations and modeling dynamic systems. Sedghi's [29] contributions often involve enhancing the conditions under which fixed-point theorems hold in these generalized spaces. Some common fixed point theorems for contraction and generalized contraction mapping in D^*-metric spaces were established by Veerapandi and Pillai [35] in 2011.

In abstract metric spaces, for ω-compatible mappings, the common tripled fixed point theorem was studied by Aydi et al. [5] for various spaces. Triple fixed point outcomes have been established for several spaces by numerous scholars (see [3, 6, 11, 15, 21, 31–33]).

The notion of tripled fixed points and some conclusions pertaining to them was introduced by Berinde and Borcut [9] for contractive type mappings in partially ordered metric space with mixed monotone properties. The concept of a pair of nonlinear contractive mappings with a triple coincidence point was presented by Borcut et al. [10]. The Berinde and Borcut [9] concept of tripled fixed points is a mathematical extension of fixed-point theory, which focuses on the existence of common fixed points, particularly in the context of tripled systems for mappings acting on ordered metric spaces or partially ordered sets. This concept is part of the broader field of nonlinear functional analysis and has applications in various fields like differential equations, optimization, and dynamic systems. The tripled fixed-point concept, developed by Berinde and Borcut [9], generalizes the idea of fixed points to triples of elements in a set. Specifically, it deals with finding a triple in a product space such that a mapping satisfies: $f(x, y, z) = (x, y, z)$. This is an extension of the idea of single and coupled fixed points (which deal with pairs). This concept is often considered in partially ordered metric spaces where the elements of the space have an order, and the mapping satisfies certain contraction conditions (usually derived from Banach contraction principle extensions). Berinde and Borcut's [9] approach involves defining specific contraction conditions that guarantee the existence of a tripled fixed point. These conditions often rely on: (i) Complete partially ordered metric spaces and (ii) Mixed monotone property. The mixed monotone property is essential for proving the existence of tripled fixed points, as it ensures that the iterations of the map converge towards a common trippled fixed point.

Altering distance functions can play a crucial role in fixed point theorems. The Banach fixed-point theorem, for instance, guarantees the existence of fixed points in complete metric spaces, and altering distance function can extend these results to more generalized spaces. Khan, Swaleh, and Sessa [24] might have introduced or studied an altered form of the distance function that preserves some of the essential properties needed for certain mathematical applications, such as in proving generalized fixed point theorems. Their work might involve relaxing the strict conditions of classical metric spaces and extending results to broader or more abstract mathematical structures. In 1984, Khan et al. [24] developed the concept of altering distance functions on a metric space for self mapping. Guttia and Kumssa [19] expanded on the idea of altering distance function and called it the control function on fixed point theory. Pupa and Mocanu [26] introduced common fixed points and altering distance function under implicit relation. By using control function, many researchers extended the Banach contraction principle (see [14, 18, 20, 34]).

In this paper, we introduced a tripled fixed points in complete D^* metric spaces using altering distance function.

2 Preleminaries

The concept of a D^*-metric space originates from efforts to generalize traditional metric spaces in order to capture broader notions of distance and convergence that arise in mathematical analysis and its applications, especially fixed point theory. Firstly, we have some definitions.

Definition 2.1. [4] Let $\wp \neq \phi$ and let $D^* : \wp^3 \to [0, \infty)$ be a function defined on $\wp$ is called D^*-metric on $\wp$ if it satisfies the following conditions, for all $\hbar, \vartheta, \varrho, c \in \wp$

(Δ_1) $D^*(\hbar, \vartheta, \varrho) \geq 0$.

(Δ_2) $D^*(\hbar, \vartheta, \varrho) = 0 \Leftrightarrow \hbar = \vartheta = \varrho$

(Δ_3) $D^*(\hbar, \vartheta, \varrho) = D^*(\rho\{\hbar, \vartheta, \varrho\})$, where ρ is permutation.

(Δ_4) $D^*(\hbar, \vartheta, \varrho) \leq D^*(\hbar, \vartheta, c) + D^*(c, \varrho, \varrho)$.

Then, the pair $(\wp, D^*)$ is said to be D^*-metric space.

Definition 2.2. [4] Let $(\wp, D^*)$ be a D^*-metric space and $\mathcal{A} \subset \wp$. Let $\{\hbar_r\}$ be a sequence in $\wp$ and $\hbar \in \wp$. Then

(i) A subset $\mathcal{A}$ of $\wp$ is said to be D^*-bounded if $\exists\, c > 0$, such that $D^*(\hbar, \vartheta, \vartheta) < c$, for all $\hbar, \vartheta \in \mathcal{A}$.

(ii) A sequence $\{\hbar_r\}$ in $\wp$ converges to $\hbar$ if and only if $D^*(\hbar_r, \hbar_r, \hbar) = D^*(\hbar, \hbar, \hbar_r) \to 0$ as $r \to \infty$. That is, for each $\varepsilon > 0 \;\exists\, a \in \mathbb{N}$ such that $D^*(\hbar, \hbar, \hbar_r) < \varepsilon$ for all $r \geq a$.

(iii) A sequence $\{\hbar_r\}$ in $\wp$ is called a Cauchy sequence if for each $\varepsilon > 0 \;\exists\, a \in \mathbb{N}$ such that $D^*(\hbar_r, \hbar_r, \hbar_s) < \varepsilon$ for all $r, s \geq a$.

(iv) $(\wp, D^*)$ is complete if every Cauchy sequence is convergent.

Some interesting lemmas of D^*-metric space are as follows:

Lemma 2.1. *[4] In D^*-metric space, $D^*(\hbar, \vartheta, \vartheta) = D^*(\hbar, \hbar, \vartheta)$.*

Lemma 2.2. *[4] Consider a D^*-metric space $(\wp, D^*)$, then $\lim_{r\to\infty} D^*(\hbar_r, \vartheta_r, \varrho_r) = D^*(\hbar, \vartheta, \varrho)$ whenever a sequence $\{(\hbar_r, \vartheta_r, \varrho_r)\}$ in $\wp^3$ converges to a point $(\hbar, \vartheta, \varrho)$ in $\wp^3$. That is $\lim_{r\to\infty} \hbar_r = \hbar$, $\lim_{r\to\infty} \vartheta_r = \vartheta$, $\lim_{r\to\infty} \varrho_r = \varrho$.*

Proof. Let $\{(\hbar_r, \vartheta_r, \varrho_r)\} \in \wp^3$, be such that $\lim\limits_{r\to\infty} \hbar_r = r$, $\lim\limits_{r\to\infty} \vartheta_r = \vartheta$, $\lim\limits_{r\to\infty} \varrho_r = \varrho$. Then for each $\varepsilon > 0$, $\exists\ a, b$ and $c \in \mathbb{N}$, such that $D^*(\hbar, \hbar, \hbar_r) < \frac{\varepsilon}{3}$, $\forall\ r \geq a, D^*(\vartheta, \vartheta, \vartheta_r) < \frac{\varepsilon}{3}$, $\forall\ r \geq b$ and $D^*(\varrho, \varrho, \varrho_r) < \frac{\varepsilon}{3}$, $\forall\ r \geq c$. Let $r' = \max\{a, b, c\}$, then by triangle inequality, for every $r \geq r'$, we obtain,

$$\begin{aligned} D^*(\hbar_r, \vartheta_r, \varrho_r) &\leq D^*(\hbar_r, \vartheta_r, \varrho) + D^*(\varrho, \varrho_r, \varrho_r) \\ &\leq D^*(\hbar, \vartheta, \varrho) + D^*(\hbar, \hbar_r, \hbar_r) + D^*(\vartheta, \vartheta_r, \vartheta_r) + D^*(\varrho, \varrho_r, \varrho_r) \\ &< D^*(\hbar, \vartheta, \varrho) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = D^*(\hbar, \vartheta, \varrho) + \varepsilon \end{aligned}$$

Then, $D^*(\hbar_r, \vartheta_r, \varrho_r) - D^*(\hbar, \vartheta, \varrho) < \varepsilon$. Moreover,

$$\begin{aligned} D^*(\hbar, \vartheta, \varrho) &\leq D^*(\hbar, \vartheta, \varrho_r) + D^*(\varrho_r, \varrho, \varrho) \\ &\leq D^*(\varrho_r, \vartheta_r, \varrho_r) + D^*(\hbar_r, \hbar, \hbar) + D^*(\vartheta_r, \vartheta, \vartheta) + D^*(\varrho_r, \varrho, \varrho) \\ &< D^*(\hbar_r, \vartheta_r, \varrho_r) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = D^*(\hbar_r, \vartheta_r, \varrho_r) + \varepsilon \end{aligned}$$

therefore, $D^*(\hbar, \vartheta, \varrho) - D^*(\hbar_r, \vartheta_r, \varrho_r) < \varepsilon \quad \Rightarrow |D^*(\hbar_r, \vartheta_r, \varrho_r) - D^*(\hbar, \vartheta, \varrho)| < \varepsilon$. Hence, $\lim\limits_{r\to\infty} D^*(\hbar_r, \vartheta_r, \varrho_r) = D^*(\hbar, \vartheta, \varrho)$ □

Lemma 2.3. *[4] Consider a D^*-metric space $(\wp, D^*)$, then a convergent sequence has a unique limit.*

Lemma 2.4. *[4] Consider a D^*-metric space $(\wp, D^*)$, then any convergent sequence in $(\wp, D^*)$ is a Cauchy sequence.*

Definition 2.3. [9] Let $\wp \neq \phi$. An element $(\hbar, \vartheta, \varrho) \in \wp$ is said to be a tripled fixed point of a given mapping $\chi : \wp^3 \to \wp$ if $\chi(\hbar, \vartheta, \varrho) = \hbar,\ \chi(\vartheta, \varrho, \hbar) = \vartheta$ and $\chi(\varrho, \hbar, \vartheta) = \varrho$.

Definition 2.4. [10] Let $\chi : \wp^3 \to \wp$ and $\xi : \wp \to \wp$ be two mappings. An element $(\hbar, \vartheta, \varrho)$ is said to be a tripled coincident point of χ and ξ if $\chi(\hbar, \vartheta, \varrho) = \xi\hbar,\ \chi(\vartheta, \varrho, \hbar) = \xi\vartheta$ and $\chi(\varrho, \hbar, \vartheta) = \xi\varrho$.

Definition 2.5. [10] Let $\chi : \wp^3 \to \wp$ and $\xi : \wp \to \wp$ be two mappings. An element $(\hbar, \vartheta, \varrho)$ is called a tripled common point of χ and ξ if $\chi(\hbar, \vartheta, \varrho) = \xi\hbar = \hbar,\ \chi(\vartheta, \varrho, \hbar) = \xi\vartheta = \vartheta$ and $\chi(\varrho, \hbar, \vartheta) = \xi\varrho = \varrho$.

Definition 2.6. [5] Consider a D^*-metric space $(\wp, D^*)$. A Pair (χ, ξ) is said to be weakly compatible if for all $\hbar, \vartheta, \varrho \in \wp, \xi\big(\chi(\hbar, \vartheta, \varrho)\big) = \chi(\xi\hbar, \xi\vartheta, \xi\varrho)$, whenever $\chi(\hbar, \vartheta, \varrho) = \xi\hbar,\ \chi(\vartheta, \varrho, \hbar) = \xi\vartheta$ and $\chi(\varrho, \hbar, \vartheta) = \xi\varrho$.

Definition 2.7. [24] A function $\psi : [0, \infty) \to [0, \infty)$ such that the following properties are met:

(i) ψ is a continuous and non decreasing,

(ii) $\psi(\imath) = 0 \ \Leftrightarrow\ \imath = 0$,

(iii) $\psi(\imath + \jmath) \leq \psi(\imath) + \psi(\jmath),\ \forall\, \imath, \jmath \in [0, \infty)$.

Then, the function ψ is called an altering distance function.

3 Main result

Now, we shall prove tripled fixed points in complete D^*-metric spaces by modifying distance functions.

Theorem 3.1. *Consider a D^*-metric space $(\wp, D^*)$. Consider a lower semi continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(\imath) = 0 \iff \imath = 0$ and let $\psi : [0, \infty) \to [0, \infty)$ be an altering distance function. Let $\mathcal{S} : \wp^3 \to \wp$ and $\mathcal{G} : \wp \to \wp$ are two mappings such that:*

$$\begin{aligned}\psi\bigg(D^*\Big(\mathcal{S}(\delta, \sigma, \kappa), &\mathcal{S}(\delta', \sigma', \kappa'), \mathcal{S}(\delta^*, \sigma^*, \kappa^*)\Big)\bigg)\\ &\leq \psi\Big(\mu\mathcal{M}(\delta, \sigma, \kappa, \delta', \sigma', \kappa', \delta^*, \sigma^*, \kappa^*)\Big)\\ &- \varphi\Big(\mu\mathcal{M}(\delta, \sigma, \kappa, \delta', \sigma', \kappa', \delta^*, \sigma^*, \kappa^*)\Big)\\ &+ \lambda\mathcal{N}(\delta, \sigma, \kappa, \delta', \sigma', \kappa', \delta^*, \sigma^*, \kappa^*). \qquad (3.1)\end{aligned}$$

$(\mathcal{D}_1)$ *$\mathcal{S}(\wp^3) \subseteq \mathcal{G}(\wp)$,*

$(\mathcal{D}_2)$ *$(\mathcal{S}, \mathcal{G})$ are ω-compatible,*

$(\mathcal{D}_3)$ *$\mathcal{G}(\wp)$ is complete.*

where

$$
\begin{aligned}
&\mathcal{M}(\delta,\sigma,\kappa,\delta',\sigma',\kappa',\delta^*,\sigma^*,\kappa^*)\\
&=\max\left\{\begin{array}{c}
\max\left\{\begin{array}{c}
D^*(\mathcal{G}\delta,\mathcal{G}\delta',\mathcal{G}\delta^*), D^*\Big(\mathcal{S}(\delta,\sigma,\kappa),\mathcal{S}(\delta^*,\sigma^*,\kappa^*),\mathcal{G}\delta^*\Big),\\
D^*\Big(\mathcal{S}(\delta,\sigma,\kappa),\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{G}\delta^*\Big),\\
D^*\Big(\mathcal{S}(\delta,\sigma,\kappa)\mathcal{G}\delta,\mathcal{G}\delta^*\Big),\\
D^*\Big(\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{G}\delta^*,\mathcal{S}(\delta^*,\sigma^*,\kappa^*)\Big),\\
D^*\Big(\mathcal{G}\delta^*,\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{G}\delta'\Big),\\
D^*\Big(\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{S}(\delta,\sigma,\kappa),\mathcal{G}\delta'\Big),\\
D^*\Big(\mathcal{S}(\delta^*,\sigma^*,\kappa^*),\mathcal{S}(\delta^*,\sigma^*,\kappa^*),\mathcal{G}\delta^*\Big)
\end{array}\right\},\\
\max\left\{\begin{array}{c}
D^*(\mathcal{G}\sigma,\mathcal{G}\sigma',\mathcal{G}\sigma^*), D^*\Big(\mathcal{S}(\sigma,\kappa,\delta),\mathcal{S}(\sigma^*,\kappa^*,\delta^*),\mathcal{G}\sigma^*\Big),\\
D^*\Big(\mathcal{S}(\sigma,\kappa,\delta),\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{G}\sigma^*\Big),\\
D^*\Big(\mathcal{S}(\sigma,\kappa,\delta),\mathcal{G}\sigma,\mathcal{G}\sigma^*\Big),\\
D^*\Big(\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{G}\sigma^*,\mathcal{S}(\sigma^*,\kappa^*,\delta^*)\Big),\\
D^*\Big(\mathcal{G}\sigma^*,\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{G}\sigma'\Big),\\
D^*\Big(\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{S}(\sigma,\kappa,\delta),\mathcal{G}\sigma'\Big),\\
D^*\Big(\mathcal{S}(\sigma^*,\kappa^*,\delta^*),\mathcal{S}(\sigma^*,\kappa^*,\delta^*),\mathcal{G}\sigma^*\Big)
\end{array}\right\},\\
\max\left\{\begin{array}{c}
D^*(\mathcal{G}\kappa,\mathcal{G}\kappa',\mathcal{G}\kappa^*), D^*\Big(\mathcal{S}(\kappa,\delta,\sigma),\mathcal{S}(\kappa^*,\delta^*,\sigma^*),\mathcal{G}\kappa^*\Big),\\
D^*\Big(\mathcal{S}(\kappa,\delta,\sigma)\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{G}\kappa^*\Big),\\
D^*\Big(\mathcal{S}(\kappa,\delta,\sigma),\mathcal{G}\kappa,\mathcal{G}\kappa^*\Big),\\
D^*\Big(\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{G}\kappa^*,\mathcal{S}(\kappa^*,\delta^*,\sigma^*)\Big),\\
D^*\Big(\mathcal{G}\kappa^*,\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{G}\kappa'\Big),\\
D^*\Big(\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{S}(\kappa,\delta,\sigma),\mathcal{G}\kappa'\Big),\\
D^*\Big(\mathcal{S}(\kappa^*,\delta^*,\sigma^*),\mathcal{S}(\kappa^*,\delta^*,\sigma^*),\mathcal{G}\kappa^*\Big)
\end{array}\right\}
\end{array}\right\}
\end{aligned}
$$

and

$$\mathcal{N}(\delta,\sigma,\kappa,\delta',\sigma',\kappa',\delta^*,\sigma^*,\kappa^*)$$
$$= \min\left\{\begin{array}{c} \min\left\{\begin{array}{c} D^*(\mathcal{G}\delta,\mathcal{G}\delta',\mathcal{G}\delta^*), D^*\Big(\mathcal{S}(\delta,\sigma,\kappa),\mathcal{S}(\delta^*,\sigma^*,\kappa^*),\mathcal{G}\delta^*\Big),\\ D^*\Big(\mathcal{S}(\delta,\sigma,\kappa),\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{G}\delta^*\Big), D^*\Big(\mathcal{S}(\delta,\sigma,\kappa)\mathcal{G}\delta,\mathcal{G}\delta^*\Big),\\ D^*\Big(\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{G}\delta^*,\mathcal{S}(\delta^*,\sigma^*,\kappa^*)\Big),\\ D^*\Big(\mathcal{G}\delta^*,\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{G}\delta'\Big), D^*\Big(\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{S}(\delta,\sigma,\kappa),\mathcal{G}\delta'\Big),\\ D^*\Big(\mathcal{S}(\delta^*,\sigma^*,\kappa^*),\mathcal{S}(\delta^*,\sigma^*,\kappa^*),\mathcal{G}\delta^*\Big) \end{array}\right\},\\ \min\left\{\begin{array}{c} D^*(\mathcal{G}\sigma,\mathcal{G}\sigma',\mathcal{G}\sigma^*), D^*\Big(\mathcal{S}(\sigma,\kappa,\delta),\mathcal{S}(\sigma^*,\kappa^*,\delta^*),\mathcal{G}\sigma^*\Big),\\ D^*\Big(\mathcal{S}(\sigma,\kappa,\delta),\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{G}\sigma^*\Big), D^*\Big(\mathcal{S}(\sigma,\kappa,\delta),\mathcal{G}\sigma,\mathcal{G}\sigma^*\Big),\\ D^*\Big(\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{G}\sigma^*,\mathcal{S}(\sigma^*,\kappa^*,\delta^*)\Big),\\ D^*\Big(\mathcal{G}\sigma^*,\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{G}\sigma'\Big), D^*\Big(\mathcal{S}(\sigma',\kappa',\delta'),\mathcal{S}(\sigma,\kappa,\delta),\mathcal{G}\sigma'\Big),\\ D^*\Big(\mathcal{S}(\sigma^*,\kappa^*,\delta^*),\mathcal{S}(\sigma^*,\kappa^*,\delta^*),\mathcal{G}\sigma^*\Big) \end{array}\right\},\\ \min\left\{\begin{array}{c} D^*(\mathcal{G}\kappa,\mathcal{G}\kappa',\mathcal{G}\kappa^*), D^*\Big(\mathcal{S}(\kappa,\delta,\sigma),\mathcal{S}(\kappa^*,\delta^*,\sigma^*),\mathcal{G}\kappa^*\Big),\\ D^*\Big(\mathcal{S}(\kappa,\delta,\sigma)\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{G}\kappa^*\Big), D^*\Big(\mathcal{S}(\kappa,\delta,\sigma),\mathcal{G}\kappa,\mathcal{G}\kappa^*\Big),\\ D^*\Big(\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{G}\kappa^*,\mathcal{S}(\kappa^*,\delta^*,\sigma^*)\Big),\\ D^*\Big(\mathcal{G}\kappa^*,\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{G}\kappa'\Big),\\ D^*\Big(\mathcal{S}(\kappa',\delta',\sigma'),\mathcal{S}(\kappa,\delta,\sigma),\mathcal{G}\kappa'\Big),\\ D^*\Big(\mathcal{S}(\kappa^*,\delta^*,\sigma^*),\mathcal{S}(\kappa^*,\delta^*,\sigma^*),\mathcal{G}\kappa^*\Big) \end{array}\right\} \end{array}\right\}$$

with $0<\mu<1$ and $\lambda\geq 0$. Then $\mathcal{S},\mathcal{G}$ has a unique common tripled fixed point in $\wp$.

Proof. Consider an arbitrary $\delta,\sigma,\kappa\in\wp$ and from $(\mathcal{D}_1)$, the sequences $\{\delta_r\},\{\sigma_r\},\{\kappa_r\},\{\delta'_r\},\{\sigma'_r\},\{\kappa'_r\}$ in $\wp$ such that

$$\begin{aligned} \mathcal{S}(\delta_r,\sigma_r,\kappa_r) &= \mathcal{G}\delta_{r+1}=\delta'_r,\\ \mathcal{S}(\sigma_r,\kappa_r,\delta_r) &= \mathcal{G}\sigma_{r+1}=\sigma'_r,\\ \mathcal{S}(\kappa_r,\delta_r,\sigma_r) &= \mathcal{G}\kappa_{r+1}=\kappa'_r,\ \forall\, r=0,1,2,... \end{aligned}$$

We shall prove that $\mathcal{S}$ and $\mathcal{G}$ have unique common tripled fixed point in $\wp$. Consider $D^*(\delta'_r,\delta'_r,\delta'_{r+1})>0, D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1})>0$ and $D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})>0\ \forall\, r$. If not, $\exists$ a positive integer r so that $\delta'_r=\delta'_{r+1},\sigma'_r=\sigma'_{r+1}$ and $\kappa'_r=\kappa'_{r+1}$ and so $(\delta'_r,\sigma'_r,\kappa'_r)$ is a tripled fixed point of $\mathcal{S},\mathcal{G}$.

Using (3.1), for each $r \in \mathbb{N}$, we get

$$\begin{aligned}
&\psi\Big(D^*(\delta'_r, \delta'_r, \delta'_{r+1})\Big) \\
&\qquad = \psi\Big[D^*\Big(\mathcal{S}(\delta_r, \sigma_r, \kappa_r), \mathcal{S}(\delta_r, \sigma_r, \kappa_r), \mathcal{S}(\delta_{r+1}, \sigma_{r+1}, \kappa_{r+1})\Big)\Big] \\
&\qquad \le \psi\Big(\mu\mathcal{M}(\delta_r, \sigma_r, \kappa_r, \delta_r, \sigma_r, \kappa_r, \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1})\Big) \\
&\qquad\quad - \varphi\Big(\mu\mathcal{M}(\delta_r, \sigma_r, \kappa_r, \delta_r, \sigma_r, \kappa_r, \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1})\Big) \\
&\qquad\quad + \lambda\mathcal{N}(\delta_r, \sigma_r, \kappa_r, \delta_r, \sigma_r, \kappa_r, \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1}). \qquad (3.2)
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{M}(\delta_r, \sigma_r, \kappa_r, \delta_r, \sigma_r, \kappa_r, \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1}) \\
&= \max\left\{\begin{array}{c}
\max\left\{\begin{array}{c} D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r), \\ D^*(\delta'_r, \delta'_{r+1}, \delta'_r), \\ D^*(\delta'_r, \delta'_r, \delta'_r), D^*(\delta'_r, \delta'_{r-1}, \delta'_r), \\ D^*(\delta'_r, \delta'_r, \delta'_{r+1}), D^*(\delta'_r, \delta'_r, \delta'_{r-1}, \\ D^*(\delta'_r, \delta'_r, \delta'_{r-1}), D^*(\delta'_{r+1}, \delta'_{r+1}, \delta'_r) \end{array}\right\}, \\
\max\left\{\begin{array}{c} D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r), \\ D^*(\sigma'_r, \sigma'_{r+1}, \sigma'_r), \\ D^*(\sigma'_r, \sigma'_r, \sigma'_r), D^*(\sigma'_r, \sigma'_{r-1}, \sigma'_r), \\ D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), D^*(\sigma'_r, \sigma'_r, \sigma'_{r-1}, \\ D^*(\sigma'_r, \sigma'_r, \sigma'_{r-1}), D^*(\sigma'_{r+1}, \sigma'_{r+1}, \sigma'_r) \end{array}\right\}, \\
\max\left\{\begin{array}{c} D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r), \\ D^*(\kappa'_r, \kappa'_{r+1}, \kappa'_r), \\ D^*(\kappa'_r, \kappa'_r, \kappa'_r), D^*(\kappa'_r, \kappa'_{r-1}, \kappa'_r), \\ D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}), D^*(\kappa'_r, \kappa'_r, \kappa'_{r-1}, \\ D^*(\kappa'_r, \kappa'_r, \kappa'_{r-1}), D^*(\kappa'_{r+1}, \kappa'_{r+1}, \kappa'_r) \end{array}\right\}
\end{array}\right\}
\end{aligned}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ \begin{array}{c} D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r), D^*(\delta'_r, \delta'_r, \delta'_{r+1}), \\ D^*(\delta'_r, \delta'_r, \delta'_r), D^*(\delta'_r, \delta'_{r-1}, \delta'_{r-1}), \\ D^*(\delta'_r, \delta'_r, \delta'_{r+1}), D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r) \end{array} \right\} \\ \max \left\{ \begin{array}{c} D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r), D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), \\ D^*(\sigma'_r, \sigma'_r, \sigma'_r), D^*(\sigma'_r, \sigma'_{r-1}, \sigma'_{r-1}), \\ D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r) \end{array} \right\} \\ \max \left\{ \begin{array}{c} D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r), D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}), \\ D^*(\kappa'_r, \kappa'_r, \kappa'_r), D^*(\kappa'_r, \kappa'_{r-1}, \kappa'_{r-1}), \\ D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}), D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r) \end{array} \right\} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r), 0, D^*(\delta'_r, \delta'_r, \delta'_{r+1}) \right\} \\ \max \left\{ D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r), 0, D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}) \right\} \\ \max \left\{ D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r), 0, D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \right\} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r), D^*(\delta'_r, \delta'_r, \delta'_{r+1}) \right\} \\ \max \left\{ D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r), D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}) \right\} \\ \max \left\{ D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r), D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \right\} \end{array} \right\}.$$

Similarly

$$\begin{aligned} &\mathcal{N}(\delta_r, \sigma_r, \kappa_r, \delta_r, \sigma_r, \kappa_r, \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1}) \\ &\quad = \min \left\{ \begin{array}{c} \min \left\{ D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r), 0, D^*(\delta'_r, \delta'_r, \delta'_{r+1}) \right\} \\ \min \left\{ D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r), 0, D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}) \right\} \\ \min \left\{ D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r), 0, D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \right\} \end{array} \right\} \\ &\quad = \min\{0, 0, 0\} \\ &\quad = 0. \end{aligned}$$

We prove that

$$D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r) \geq D^*(\delta'_r, \delta'_r, \delta'_{r+1}), \quad D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r) \geq D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1})$$

and

$$D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r) \geq D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}), \ \forall\, r \in \mathbb{N}.$$

Consider

$$D^*(\delta'_{r-1},\delta'_{r-1},\delta'_r) < D^*(\delta'_r,\delta'_r,\delta'_{r+1}),\ \ D^*(\sigma'_{r-1},\sigma'_{r-1},\sigma'_r) < D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1})$$

and

$$D^*(\kappa'_{r-1},\kappa'_{r-1},\kappa'_r) < D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1}),\ \forall\, r\in\mathbb{N}.$$

Then we get,

$$\begin{aligned}&\mathcal{M}(\delta_r,\sigma_r,\kappa_r,\delta_r,\sigma_r,\kappa_r,\delta_{r+1},\sigma_{r+1},\kappa_{r+1})\\ &\qquad\qquad = \max\Big\{D^*(\delta'_r,\delta'_r,\delta'_{r+1}),D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}),D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})\Big\}.\end{aligned}$$

Then, from (3.2) we obtain

$$\begin{aligned}&\psi\Big(D^*(\delta'_r,\delta'_r,\delta'_{r+1})\Big)\\ &\quad\le \psi\left(\mu\max\left\{\begin{array}{c}D^*(\delta'_r,\delta'_r,\delta'_{r+1}),\\ D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}),\\ D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})\end{array}\right\}\right)-\varphi\left(\mu\max\left\{\begin{array}{c}D^*(\delta'_r,\delta'_r,\delta'_{r+1}),\\ D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}),\\ D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})\end{array}\right\}\right)\\ &\quad\le \psi\Big(\mu\max\Big\{D^*(\delta'_r,\delta'_r,\delta'_{r+1}),D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}),D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})\Big\}\Big)\end{aligned}$$

Since, ψ is increasing, we obtain

$$D^*(\delta'_r,\delta'_r,\delta'_{r+1}) \le \mu\max\left\{\begin{array}{c}D^*(\delta'_r,\delta'_r,\delta'_{r+1}),\\ D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}),\\ D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})\end{array}\right\}. \tag{3.3}$$

Similarly,

$$D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}) \le \mu\max\left\{\begin{array}{c}D^*(\delta'_r,\delta'_r,\delta'_{r+1}),\\ D^*(\sigma'_r,\sigma'_r,\sigma'_{r+1}),\\ D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1})\end{array}\right\}. \tag{3.4}$$

And

$$D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \leq \mu \max \left\{ \begin{array}{c} D^*(\delta'_r, \delta'_r, \delta'_{r+1}), \\ D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), \\ D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \end{array} \right\}. \tag{3.5}$$

Combining (3.3),(3.4) and (3.5), we obtain

$$\begin{aligned} &\max \left\{ D^*(\delta'_r, \delta'_r, \delta'_{r+1}), D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \right\} \\ &\quad \leq \mu \max \left\{ D^*(\delta'_r, \delta'_r, \delta'_{r+1}), D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \right\} \end{aligned}$$

which is a contradiction, since $0 < \mu < 1$.
Thus,

$$D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r) \geq D^*(\delta'_r, \delta'_r, \delta'_{r+1}), D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r) \geq D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1})$$

and

$$D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r) \geq D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}).$$

Thus by above inequality, we get

$$\begin{aligned} &\max \left\{ D^*(\delta'_r, \delta'_r, \delta'_{r+1}), D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}), D^*(\kappa'_r, \kappa'_r, \kappa'_{r+1}) \right\} \\ &\quad \leq \mu \max \left\{ D^*(\delta'_{r-1}, \delta'_{r-1}, \delta'_r), D^*(\sigma'_{r-1}, \sigma'_{r-1}, \sigma'_r), D^*(\kappa'_{r-1}, \kappa'_{r-1}, \kappa'_r) \right\} \\ &\quad \leq \mu^2 \max \left\{ \begin{array}{c} D^*(\delta'_{r-2}, \delta'_{r-2}, \delta'_{r-1}), \\ D^*(\sigma'_{r-2}, \sigma'_{r-2}, \sigma'_{r-1}), D^*(\kappa'_{r-2}, \kappa'_{r-2}, \kappa'_{r-1}) \end{array} \right\} \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad \leq \mu^r \max \left\{ \begin{array}{c} D^*(\delta'_0, \delta'_0, \delta'_1), \\ D^*(\sigma'_0, \sigma'_0, \sigma'_1), D^*(\kappa'_0, \kappa'_0, \kappa'_1) \end{array} \right\}. \end{aligned}$$

Thus, we have

$$D^*(\delta'_r, \delta'_r, \delta'_{r+1}) \leq \mu^r \max \left\{ D^*(\delta'_0, \delta'_0, \delta'_1), D^*(\sigma'_0, \sigma'_0, \sigma'_1), D^*(\kappa'_0, \kappa'_0, \kappa'_1) \right\}$$

$$D^*(\sigma'_r, \sigma'_r, \sigma'_{r+1}) \leq \mu^r \max \left\{ D^*(\delta'_0, \delta'_0, \delta'_1), D^*(\sigma'_0, \sigma'_0, \sigma'_1), D^*(\kappa'_0, \kappa'_0, \kappa'_1) \right\}$$

and

$$D^*(\kappa'_r,\kappa'_r,\kappa'_{r+1}) \le \mu^r \max\Big\{D^*(\delta'_0,\delta'_0,\delta'_1), D^*(\sigma'_0,\sigma'_0,\sigma'_1), D^*(\kappa'_0,\kappa'_0,\kappa'_1)\Big\}.$$

Using rectangle inequality, for $r > s$, we obtain

$$\begin{aligned}
D^*(\delta'_r,\delta'_r,\delta'_s) &\le D^*(\delta'_s,\delta'_{s+1},\delta'_{s+1}) + D^*(\delta'_{s+1},\delta'_r,\delta'_r)\\
&\le D^*(\delta'_s,\delta'_{s+1},\delta'_{s+1}) + D^*(\delta'_{s+2},\delta'_{s+2},\delta'_{s+1}) + D^*(\delta'_{s+2},\delta'_r,\delta'_r)\\
&\le D^*(\delta'_s,\delta'_{s+1},\delta'_{s+1}) + D^*(\delta'_{s+2},\delta'_{s+2},\delta'_{s+1}) + \cdots\\
&\quad + D^*(\delta'_{r-1},\delta'_r,\delta'_r)\\
&\le \Big(\mu^s+\mu^{s+1}+\cdots+\mu^{r-1}\Big)\max\left\{\begin{array}{c} D^*(\delta'_0,\delta'_0,\delta'_1),\\ D^*(\sigma'_0,\sigma'_0,\sigma'_1),\\ D^*(\kappa'_0,\kappa'_0,\kappa'_1)\end{array}\right\}\\
&\le (\mu^s+\mu^{s+1}+\cdots)\max\left\{\begin{array}{c} D^*(\delta'_0,\delta'_0,\delta'_1),\\ D^*(\sigma'_0,\sigma'_0,\sigma'_1),\\ D^*(\kappa'_0,\kappa'_0,\kappa'_1)\end{array}\right\}\\
&\le \frac{\mu^s}{1-\mu}\max\left\{\begin{array}{c} D^*(\delta'_0,\delta'_0,\delta'_1),\\ D^*(\sigma'_0,\sigma'_0,\sigma'_1),\\ D^*(\kappa'_0,\kappa'_0,\kappa'_1)\end{array}\right\} \to 0 \text{ as } s\to\infty.
\end{aligned}$$

Similarly, $D^*(\sigma'_r,\sigma'_r,\sigma'_s) \to 0$ as $r,s\to\infty$ and $D^*(\kappa'_r,\kappa'_r,\kappa'_s) \to 0$ as $r,s\to\infty$. This proves that $\{\delta'_r\},\{\sigma'_r\}$ and $\{\kappa'_r\}$ are Cauchy sequences in the D^*-metric space. Consider $\mathcal{G}(\wp)$ is complete subspace of $(\wp, D^*)$, then $\{\delta'_r\},\{\sigma'_r\}$ and $\{\kappa'_r\}$ converge to $\hbar,\vartheta,\varrho$ respectively in $\mathcal{G}(\wp)$. Thus, $\exists\,\hbar',\vartheta',\varrho' \in \mathcal{G}(\wp)$ such that

$$\lim_{r\to\infty}\delta'_r = \hbar = \mathcal{G}\hbar', \qquad \lim_{r\to\infty}\sigma'_r = \vartheta = \mathcal{G}\vartheta' \text{ and } \lim_{r\to\infty}\kappa'_r = \varrho = \mathcal{G}\varrho' \tag{3.6}$$

We claim that $\mathcal{S}(\hbar',\vartheta',\varrho') = \hbar, \quad \mathcal{S}(\vartheta',\varrho',\hbar') = \vartheta$ and $\mathcal{S}(\varrho',\hbar',\vartheta') = \varrho$

By using (3.1), we obtain

$$\begin{aligned}
&\psi\Big(D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{S}(\hbar',\vartheta',\varrho'),\delta'_{r+1}\Big)\Big)\\
&\qquad\qquad= \psi\Big(D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{S}(\delta_{r+1},\sigma_{r+1},\kappa_{r+1})\Big)\Big)\\
&\qquad\qquad\le \psi\Big(\mu\mathcal{M}(\hbar',\vartheta',\varrho',\hbar',\vartheta',\varrho',\delta_{r+1},\sigma_{r+1},\kappa_{r+1}\Big)\\
&\qquad\qquad\quad - \varphi\Big(\mu\mathcal{M}(\hbar',\vartheta',\varrho',\hbar',\vartheta',\varrho',\delta_{r+1},\sigma_{r+1},\kappa_{r+1}\Big)\\
&\qquad\qquad\quad + \lambda\mathcal{N}(\hbar',\vartheta',\varrho',\hbar',\vartheta',\varrho',\delta_{r+1},\sigma_{r+1},\kappa_{r+1}) \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
&\lim_{r\to\infty}\mathcal{M}(\hbar',\vartheta',\varrho',\hbar',\vartheta',\varrho',\delta_{r+1},\sigma_{r+1},\kappa_{r+1})\\
&= \lim_{r\to\infty}\max\left\{\begin{array}{c}
\max\left\{\begin{array}{c}
D^*(\mathcal{G}\hbar',\mathcal{G}\hbar',\delta'_r), D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\delta'_{r+1},\delta'_r\Big),\\
D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{S}(\hbar',\vartheta',\varrho'),\delta'_r\Big), D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{G}\hbar',\delta'_r\Big),\\
D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\delta'_r,\delta'_{r+1}\Big), D^*\Big(\delta'_r,\mathcal{S}(\hbar',\vartheta',\varrho'),\delta'_r\Big),\\
D^*\Big(\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{S}(\hbar',\vartheta',\varrho'),\mathcal{G}\hbar'\Big),\\
D^*(\delta'_{r+1},\delta'_{r+1},\delta'_r)
\end{array}\right\},\\
\max\left\{\begin{array}{c}
D^*(\mathcal{G}\vartheta',\mathcal{G}\vartheta',\sigma'_r), D^*\Big(\mathcal{S}(\vartheta',\varrho',\hbar'),\sigma'_{r+1},\sigma'_r\Big),\\
D^*\Big(\mathcal{S}(\vartheta',\varrho',\hbar'),\mathcal{S}(\vartheta',\varrho',\hbar'\sigma'_r\Big), D^*\Big(\mathcal{S}(\vartheta',\varrho',\hbar'),\mathcal{G}\vartheta',\sigma'_r\Big),\\
D^*\Big(\mathcal{S}(\vartheta',\varrho',\hbar'),\sigma'_r,\sigma'_{r+1}\Big), D^*\Big(\sigma'_r,\mathcal{S}(\vartheta',\varrho',\hbar'),\sigma'_r\Big),\\
D^*\Big(\mathcal{S}(\vartheta',\varrho',\hbar'),\mathcal{S}(\vartheta',\varrho',\hbar'),\mathcal{G}\vartheta'\Big),\\
D^*(\sigma'_{r+1},\sigma'_{r+1},\sigma'_r)
\end{array}\right\},\\
\max\left\{\begin{array}{c}
D^*(\mathcal{G}\vartheta',\mathcal{G}\vartheta',\kappa'_r), D^*\Big(\mathcal{S}(\varrho',\hbar',\vartheta'),\kappa'_{r+1},\kappa'_r\Big),\\
D^*\Big(\mathcal{S}(\varrho',\hbar',\vartheta'),\mathcal{S}(\varrho',\hbar',\vartheta')\kappa'_r\Big), D^*\Big(\mathcal{S}(\varrho',\hbar',\vartheta'),\mathcal{G}\varrho',\kappa'_r\Big),\\
D^*\Big(\mathcal{S}(\varrho',\hbar',\vartheta'),\kappa'_r,\kappa'_{r+1}\Big), D^*\Big(\kappa'_r,\mathcal{S}(\varrho',\hbar',\vartheta'),\kappa'_r\Big),\\
D^*\Big(\mathcal{S}(\varrho',\hbar',\vartheta'),\mathcal{S}(\varrho',\hbar',\vartheta'),\mathcal{G}\varrho'\Big),\\
D^*(\kappa'_{r+1},\kappa'_{r+1},\kappa'_r)
\end{array}\right\}
\end{array}\right\}
\end{aligned}$$

$$= \max \left\{ \begin{array}{c} \max\Big\{0, D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big)\Big\}, \\ \max\Big\{0, D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big)\Big\}, \\ \max\Big\{0, D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big)\Big\} \end{array} \right\}$$

$$= \max\Big\{D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big)\Big\}$$

and

$$\lim_{r\to\infty} \mathcal{N}(\hbar', \vartheta', \varrho', \hbar', \vartheta', \varrho', \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1}) = \min \left\{ \begin{array}{c} \min\Big\{0, D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big)\Big\}, \\ \min\Big\{0, D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big)\Big\}, \\ \min\Big\{0, D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big)\Big\} \end{array} \right\} = 0.$$

Applying the upper limit as $r \to \infty$ in (3.7), we get

$$\begin{aligned} &\psi\Big(D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \mathcal{S}(\hbar', \vartheta', \varrho'), \hbar\Big)\Big) \\ &\quad = \lim_{r\to\infty} \psi\Big(D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \mathcal{S}(\hbar', \vartheta', \varrho'), \delta'_{r+1}\Big)\Big) \\ &\quad \le \psi\Big(\mu \lim_{r\to\infty} \mathcal{M}(\hbar', \vartheta', \varrho', \hbar', \vartheta', \varrho', \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1})\Big) \\ &\quad\quad - \varphi\Big(\mu \lim_{r\to\infty} \mathcal{M}(\hbar', \vartheta', \varrho', \hbar', \vartheta', \varrho', \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1})\Big) \\ &\quad\quad + \lambda \lim_{r\to\infty} \mathcal{N}(\hbar', \vartheta', \varrho', \hbar', \vartheta', \varrho', \delta_{r+1}, \sigma_{r+1}, \kappa_{r+1}) \\ &\quad \le \psi\left(\mu \max\left\{ \begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array} \right\}\right) \\ &\quad\quad - \varphi\left(\mu \max\left\{ \begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array} \right\}\right) \\ &\quad\quad + \lambda(0) \end{aligned}$$

$$\leq \psi\left(\mu \max\left\{\begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array}\right\}\right)$$

which implies that

$$D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big) \leq \psi\left(\mu \max\left\{\begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), \\ D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array}\right\}\right).$$

Similarly

$$D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big) \leq \psi\left(\mu \max\left\{\begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), \\ D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array}\right\}\right)$$

and

$$D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \leq \psi\left(\mu \max\left\{\begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), \\ D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array}\right\}\right).$$

Therefore, we have

$$\max\left\{\begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), \\ D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array}\right\} \leq \mu \max\left\{\begin{array}{c} D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big), \\ D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big), \\ D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) \end{array}\right\}$$

which is not possible. Hence,

$$D^*\Big(\mathcal{S}(\hbar', \vartheta', \varrho'), \hbar, \hbar\Big) = 0, \qquad D^*\Big(\mathcal{S}(\vartheta', \varrho', \hbar'), \vartheta, \vartheta\Big) = 0$$

$$\text{and} \qquad D^*\Big(\mathcal{S}(\varrho', \hbar', \vartheta'), \varrho, \varrho\Big) = 0$$

which implies that

$$\mathcal{S}(\hbar', \vartheta', \varrho') = \hbar, \qquad \mathcal{S}(\vartheta', \varrho', \hbar') = \vartheta \qquad \text{and} \qquad \mathcal{S}(\varrho', \hbar', \vartheta') = \varrho.$$

It follows that

$$\mathcal{S}(\hbar',\vartheta',\varrho')=\hbar=\mathcal{G}\hbar',\quad \mathcal{S}(\vartheta',\varrho',\hbar')=\vartheta=\mathcal{G}\vartheta' \quad\text{and}\quad \mathcal{S}(\varrho',\hbar',\vartheta')=\varrho=\mathcal{G}\varrho'.$$

Since $\{\mathcal{S},\mathcal{G}\}$ is weakly compatible pair, we have

$$\mathcal{S}(\hbar,\vartheta,\varrho)=\mathcal{G}\hbar, \qquad \mathcal{S}(\vartheta,\varrho,\hbar)=\mathcal{G}\vartheta \qquad\text{and}\qquad \mathcal{S}(\varrho,\hbar,\vartheta)=\mathcal{G}\varrho.$$

Now we prove that $\mathcal{G}\hbar=\hbar,\ \mathcal{G}\vartheta=\vartheta$ and $\mathcal{G}\varrho=\varrho$. Applying the upper limit as $r\to\infty$ and using (3.1), we obtain

$$\begin{aligned}
\psi\Big(D^*(\mathcal{G}\hbar,\mathcal{G}\hbar,\hbar)\Big) &= \lim_{r\to\infty}\psi\Big(D^*(\mathcal{G}\hbar,\mathcal{G}\hbar,\delta'_{r+1})\Big)\\
&= \lim_{r\to\infty}\psi\Big(D^*\Big(\mathcal{S}(\hbar,\vartheta,\varrho),\mathcal{S}(\hbar,\vartheta,\varrho),\mathcal{S}(\delta_{r+1},\sigma_{r+1},\kappa_{r+1})\Big)\Big)\\
&\le \psi\Big(\mu\lim_{r\to\infty}\mathcal{M}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,\delta_{r+1},\sigma_{r+1},\kappa_{r+1})\Big)\\
&\quad -\varphi\Big(\mu\lim_{r\to\infty}\mathcal{M}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,\delta_{r+1},\sigma_{r+1},\kappa_{r+1})\Big)\\
&\quad +\lambda\lim_{r\to\infty}\mathcal{N}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,\delta_{r+1},\sigma_{r+1},\kappa_{r+1}) \qquad (3.8)
\end{aligned}$$

where

$$\lim_{r\to\infty}\mathcal{M}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,\delta_{r+1},\sigma_{r+1},\kappa_{r+1})=\max\left\{\begin{array}{c} D^*(\mathcal{G}\hbar,\hbar,\hbar),\\ D^*(\mathcal{G}\vartheta,\vartheta,\vartheta),\\ D^*(\mathcal{G}\varrho,\varrho,\varrho)\end{array}\right\}$$

and

$$\begin{aligned}
\lim_{r\to\infty}\mathcal{N}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,\delta_{r+1},\sigma_{r+1},\kappa_{r+1}) &= \min\left\{\begin{array}{c}\min\Big\{0,D^*(\mathcal{G}\hbar,\hbar,\hbar)\Big\},\\ \min\Big\{0,D^*(\mathcal{G}\vartheta,\vartheta,\vartheta)\Big\},\\ \min\Big\{0,D^*(\mathcal{G}\varrho,\varrho,\varrho)\Big\}\end{array}\right\}\\
&=0.
\end{aligned}$$

From (3.8), we get

$$\begin{aligned}\psi\Big(D^*(\mathcal{G}\hbar,\hbar,\hbar)\Big) &\leq \psi\left(\mu\max\left\{\ D^*(\mathcal{G}\hbar,\hbar,\hbar), D^*(\mathcal{G}\vartheta,\vartheta,\vartheta), D^*(\mathcal{G}\varrho,\varrho,\varrho)\ \right\}\right)\\ &\quad -\varphi\left(\mu\max\left\{\begin{array}{c} D^*(\mathcal{G}\hbar,\hbar,\hbar),\\ D^*(\mathcal{G}\vartheta,\vartheta,\vartheta), D^*(\mathcal{G}\varrho,\varrho,\varrho)\end{array}\right\}\right)+\lambda(0)\\ &\leq \psi\Big(\mu\max\Big\{D^*(\mathcal{G}\hbar,\hbar,\hbar), D^*(\mathcal{G}\vartheta,\vartheta,\vartheta), D^*(\mathcal{G}\varrho,\varrho,\varrho)\Big\}\Big)\end{aligned}$$

which implies that

$$D^*(\mathcal{G}\hbar,\hbar,\hbar)\leq \mu\max\Big\{D^*(\mathcal{G}\hbar,\hbar,\hbar), D^*(\mathcal{G}\vartheta,\vartheta,\vartheta), D^*(\mathcal{G}\varrho,\varrho,\varrho)\Big\}.$$

Similarly

$$D^*(\mathcal{G}\vartheta,\vartheta,\vartheta)\leq \mu\max\Big\{D^*(\mathcal{G}\hbar,\hbar,\hbar), D^*(\mathcal{G}\vartheta,\vartheta,\vartheta), D^*(\mathcal{G}\varrho,\varrho,\varrho)\Big\}$$

and

$$D^*(\mathcal{G}\varrho,\varrho,\varrho)\leq \mu\max\Big\{D^*(\mathcal{G}\hbar,\hbar,\hbar), D^*(\mathcal{G}\vartheta,\vartheta,\vartheta), D^*(\mathcal{G}\varrho,\varrho,\varrho)\Big\}.$$

Therefore, we obtain

$$\max\left\{\begin{array}{c} D^*(\mathcal{G}\hbar,\hbar,\hbar),\\ D^*(\mathcal{G}\vartheta,\vartheta,\vartheta),\\ D^*(\mathcal{G}\varrho,\varrho,\varrho)\end{array}\right\}\leq \mu\max\left\{\begin{array}{c} D^*(\mathcal{G}\hbar,\hbar,\hbar),\\ D^*(\mathcal{G}\vartheta,\vartheta,\vartheta),\\ D^*(\mathcal{G}\varrho,\varrho,\varrho)\end{array}\right\}.$$

Hence,

$$D^*(\mathcal{G}\hbar,\hbar,\hbar)=0,\qquad D^*(\mathcal{G}\vartheta,\vartheta,\vartheta)=0\qquad\text{and}\qquad D^*(\mathcal{G}\varrho,\varrho,\varrho)=0$$

which implies that

$$\mathcal{G}\hbar=\hbar,\qquad \mathcal{G}\vartheta=\vartheta\qquad\text{and}\qquad \mathcal{G}\varrho=\varrho.$$

It follows that

$$\mathcal{S}(\hbar,\vartheta,\varrho)=\hbar=\mathcal{G}\hbar,\quad \mathcal{S}(\vartheta,\varrho,\hbar)=\vartheta=\mathcal{G}\vartheta\quad\text{and}\quad \mathcal{S}(\varrho,\hbar,\vartheta)=\varrho=\mathcal{G}\varrho.$$

Thus, $(\hbar,\vartheta,\varrho)$ is the common tripled fixed point of $\mathcal{S}$ and $\mathcal{G}$.

The uniqueness of the common tripled fixed point in $\wp$ will be demonstrated in the next section. Consider there is an another tripled fixed point of (x, y, z), Then, we obtain

$$\begin{aligned}\psi\Big(D^*(\hbar,\hbar,x)\Big) &= \psi\bigg(D^*\Big(\mathcal{S}(\hbar,\vartheta,\varrho),\mathcal{S}(\hbar,\varrho,\vartheta),\mathcal{S}(x,y,z)\Big)\bigg)\\ &\leq \psi\Big(\mu\mathcal{M}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,x,y,z)\Big)\\ &\quad-\varphi\Big(\mu\mathcal{M}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,x,y,z)\Big)\\ &\quad+\lambda\mathcal{N}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,x,y,z) \qquad (3.9)\end{aligned}$$

where

$$\mathcal{M}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,x,y,z)=\max\Big\{D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\Big\}$$

and

$$\mathcal{N}(\hbar,\vartheta,\varrho,\hbar,\vartheta,\varrho,x,y,z)=0.$$

From (3.9), we obtain

$$\begin{aligned}\psi\Big(D^*(\hbar,\hbar,x)\Big) &\leq \psi\left(\mu\max\left\{\ D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\ \right\}\right)\\ &\quad-\varphi\left(\mu\max\left\{\ D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\ \right\}\right)\\ &\quad+\lambda(0)\\ &\leq \bigg(\mu\max\Big\{D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\Big\}\bigg)\end{aligned}$$

which implies that

$$D^*(\hbar,\hbar,x)\leq\bigg(\mu\max\Big\{D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\Big\}\bigg).$$

Similarly

$$D^*(\vartheta,\vartheta,y)\leq\bigg(\mu\max\Big\{D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\Big\}\bigg)$$

and

$$D^*(\varrho,\varrho,z)\leq\bigg(\mu\max\Big\{D^*(\hbar,\hbar,x),D^*(\vartheta,\vartheta,y),D^*(\varrho,\varrho,z)\Big\}\bigg).$$

Therefore, we obtain

$$\max\left\{\begin{array}{c} D^*(\hbar,\hbar,x), \\ D^*(\vartheta,\vartheta,y), \\ D^*(\varrho,\varrho,z) \end{array}\right\} \le \mu \max\left\{\begin{array}{c} D^*(\hbar,\hbar,x), \\ D^*(\vartheta,\vartheta,y), \\ D^*(\varrho,\varrho,z) \end{array}\right\}$$

which is not possible. Hence

$$D^*(\hbar,\hbar,x)=0, \qquad D^*(\vartheta,\vartheta,y)=0 \qquad \text{and} \qquad D^*(\varrho,\varrho,z)=0$$

$$\Rightarrow \hbar = x, \quad \vartheta = y \quad \text{and} \quad \varrho = z.$$

Thus, the uniqueness of the common tripled fixed point of $\mathcal{S}$ and $\mathcal{G}$ is represented by $(\hbar,\vartheta,\varrho)$. Now, we will prove the uniqueness of fixed point in $\wp$.

$$\begin{aligned} \psi\Big(D^*(\hbar,\vartheta,\varrho)\Big) &= \psi\bigg(D^*\Big(\mathcal{S}(\hbar,\vartheta,\varrho),\mathcal{S}(\vartheta,\varrho,\hbar),\mathcal{S}(\varrho,\hbar,\vartheta)\Big)\bigg) \\ &\le \psi\Big(\mu\mathcal{M}(\hbar,\vartheta,\varrho,\vartheta,\varrho,\hbar,\varrho,\hbar,\vartheta)\Big) \\ &\quad - \varphi\Big(\mu\mathcal{M}(\hbar,\vartheta,\varrho,\vartheta,\varrho,\hbar,\varrho,\hbar,\vartheta)\Big) \\ &\quad + \lambda\mathcal{N}(\hbar,\vartheta,\varrho,\vartheta,\varrho,\hbar,\varrho,\hbar,\vartheta) \end{aligned} \tag{3.10}$$

where

$$\mathcal{M}(\hbar,\vartheta,\varrho,\vartheta,\varrho,\hbar,\varrho,\hbar,\vartheta) = \max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\}$$

and

$$\mathcal{N}(\hbar,\vartheta,\varrho,\vartheta,\varrho,\hbar,\varrho,\hbar,\vartheta) = 0.$$

From (3.10), we get

$$\begin{aligned} \psi\Big(D^*(\hbar,\vartheta,\varrho)\Big) &\le \psi\left(\mu\max\left\{ D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta) \right\}\right) \\ &\quad -\varphi\left(\mu\max\left\{ D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta) \right\}\right) \\ &\quad +\lambda(0) \\ &\le \psi\bigg(\mu\max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\}\bigg) \end{aligned}$$

which implies that

$$D^*(\hbar,\vartheta,\varrho) \le \mu\max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\}.$$

Similarly

$$D^*(\vartheta,\varrho,\hbar)\le \mu\max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\}$$

and

$$D^*(\varrho,\hbar,\vartheta)\le \mu\max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\}.$$

Therefore, we get

$$\max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\} \le \mu\max\Big\{D^*(\hbar,\vartheta,\varrho),D^*(\vartheta,\varrho,\hbar),D^*(\varrho,\hbar,\vartheta)\Big\}$$

which is not possible. Hence

$$D^*(\hbar,\vartheta,\varrho)=0,\qquad D^*(\vartheta,\varrho,\hbar)=0\qquad\text{and}\qquad D^*(\varrho,\hbar,\vartheta)=0,$$

and hence, we get $\hbar=\vartheta=\varrho$. Which means that $\mathcal{S}$ and $\mathcal{G}$ have a unique common fixed point. □

Corollary 3.1. *Under the hypothesis of Theorem 3.1, by setting $\lambda=0$, we conclude that $\mathcal{S}$ and $\mathcal{G}$ have a common tripled fixed point in $\wp$.*

Corollary 3.2. *Consider a complete D^*-metric space $(\wp,D^*)$. Let $\mathcal{S}:\wp^3\to\wp$ be a mapping such that*

$$D^*\Big(\mathcal{S}(\delta,\sigma,\kappa),\mathcal{S}(\delta',\sigma',\kappa'),\mathcal{S}(\delta^*,\sigma^*,\kappa^*)\Big) \le \mu\max\Big\{D^*(\delta,\delta',\delta^*),D^*(\sigma,\sigma',\sigma^*),D^*(,\kappa,\kappa',\kappa^*)\Big\}$$

for all $\delta,\sigma,\kappa,\delta',\sigma',\kappa',\delta^*,\sigma^*,\kappa^*\in\wp$ with $0<\mu<1$, then there is a unique tripled fixed point of $\mathcal{S}\in\wp$.

Example 3.1. *Let $X=[0,\infty)$ and $D^*(x,y,z)=\max\big\{|x-y|,|y-z|,|z-x|\big\}$. In this case (X,D^*) is a complete D^*-metric space. Let $\mathcal{S}:X^3\to X$ and $\mathcal{G}:X\to X$ be given by $\mathcal{G}(x)=\frac{x}{3}$ and $\mathcal{S}(x,y,z)=\frac{x-y+z}{24}$, also $\psi(t)=\frac{2t}{7}$ and $\Phi(t)=\frac{t}{7}$ for all $t\in[0,\infty)$. Then, obviously, $\mathcal{S}(X^3)\subseteq\mathcal{G}(X)$, and the pair $(\mathcal{S},\mathcal{G})$*

is ω-compatible. Now we have

$$
\begin{aligned}
&\psi\Big(D^*\big(\mathcal{S}(a,b,c),\mathcal{S}(a,b,c),\mathcal{S}(x,y,z)\big)\Big)\\
&\qquad=\frac{2}{7}D^*\Big(\mathcal{S}(a,b,c),\mathcal{S}(a,b,c),\mathcal{S}(x,y,z)\Big)\\
&\qquad\leq\frac{1}{2}\max\Big\{|\mathcal{S}(a,b,c)-\mathcal{S}(x,y,z)|\Big\}\\
&\qquad=\frac{1}{2}\max\Big\{|\frac{a-b+c}{24}-\frac{x-y+z}{24}|\Big\}\\
&\qquad=\frac{1}{48}\max\Big\{|a-b+c-x+y-z|\Big\}\\
&\qquad\leq\frac{1}{14}\max\Big\{\ \max\big\{|\tfrac{a}{3}-\tfrac{x}{3}|,0\big\},\max\big\{|\tfrac{b}{3}-\tfrac{y}{3}|,0\big\},\max\big\{|\tfrac{c}{3}-\tfrac{z}{3}|,0\big\}\ \Big\}\\
&\qquad\leq\frac{1}{7}\left(\frac{1}{2}\max\left\{\begin{array}{c}\max\Big\{D^*(\mathcal{G}a,\mathcal{G}a,\mathcal{G}x),0\Big\},\\ \max\Big\{D^*(\mathcal{G}b,\mathcal{G}b,\mathcal{G}y),0\Big\},\\ \max\Big\{D^*(\mathcal{G}c,\mathcal{G}c,\mathcal{G}z),0\Big\}\end{array}\right\}\right)\\
&\qquad\leq\psi\Big(\mu\mathcal{M}(a,b,c,a,b,c,x,y,z)\Big)-\Phi\Big(\mu\mathcal{M}(a,b,c,a,b,c,x,y,z)\Big)\\
&\qquad\quad+\lambda\mathcal{N}(a,b,c,a,b,c,x,y,z).
\end{aligned}
$$

Thus all the condition of the above theorem are satisfied and (0,0,0) is unique common tripled fixed point of $\mathcal{S}$ and $\mathcal{G}$.

4 Application to integral equations

This section applies Corollary 3.1 to study the existence of a unique solution to an initial value problem.

Theorem 4.1. *Let us take the initial value problem*

$$\frac{dl}{d\imath}=\mathcal{T}\Big(\imath,\delta'(\imath),\delta'(\imath),\delta'(\imath)\Big),\ \imath\in I=[0,1],\ \delta'(0)=\delta'_0 \tag{4.1}$$

where $\mathcal{T}:I\times\mathcal{R}^2\to\mathcal{R}$ and $\delta'_0\in\mathcal{R}$ with

$$\int_0^{\imath} \mathcal{T}\Big(\jmath, \delta'(\jmath), \sigma'(\jmath), \kappa'(\jmath)\Big) d\jmath$$

$$= \max \left\{ \begin{array}{c} \int_0^{\imath} \mathcal{T}\Big(\jmath, \delta'(\jmath), \delta'(\jmath), \delta'(\jmath)\Big) d\jmath, \\ \int_0^{\imath} \mathcal{T}\Big(\jmath, \sigma'(\jmath), \sigma'(\jmath), \sigma'(\jmath)\Big) d\jmath, \\ \int_0^{\imath} \mathcal{T}\Big(\jmath, \kappa'(\jmath), \kappa'(\jmath), \kappa'(\jmath)\Big) d\jmath \end{array} \right\}.$$

Then, the initial value problem (4.1) has a unique solution in $C(I, \mathcal{R})$.

Proof. Initial Value Problem (4.1)'s equivalent integral equation is

$$\delta'(\imath) = \delta'_0 + \int_0^{\imath} \mathcal{T}\Big(\jmath, \delta'(\jmath), \delta'(\jmath), \delta'(\jmath)\Big) d\jmath.$$

Let $\wp = C(I, \mathcal{R})$ and $D^*(\delta', \sigma', \kappa') = |\delta' - \sigma'| + |\sigma' - \kappa'| + |\kappa' - \delta'| \;\forall\, \delta', \sigma', \kappa' \in \wp$. Define $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(\imath) = \frac{2\imath}{5},\ \varphi(\imath) = \frac{\imath}{6}$. Define $\mathcal{S} : \wp^3 \to \wp$ and $\mathcal{G} : \wp \to \wp$ by

$$\begin{aligned} \mathcal{S}(\delta', \sigma', \kappa')(\imath) &= \frac{\delta'_0}{16} + \int_0^{\imath} \mathcal{T}\Big(\imath, \delta'(\imath), \sigma'(\imath), \kappa'(\imath)\Big) d\imath, \\ \mathcal{G}(\delta')(\imath) &= \delta'_0 + 16 \int_0^{\imath} \mathcal{T}\Big(\jmath, \delta'(\jmath), \delta'(\jmath), \delta'(\jmath)\Big) d\jmath. \end{aligned}$$

Now

$$\begin{aligned} &\psi\bigg(D^*\Big(\mathcal{S}(\delta', \sigma', \kappa')(\imath), \mathcal{S}(\delta', \sigma', \kappa')(\imath), \mathcal{S}(a, b, c)(\imath)\Big)\bigg) \\ &\quad = \frac{2}{5} D^*\Big(\mathcal{S}(\delta', \sigma', \kappa')(\imath), \mathcal{S}(\delta', \sigma', \kappa')(\imath), \mathcal{S}(a, b, c)(\imath)\Big) \\ &\quad = \frac{4}{5} |R(\delta', \sigma', \kappa')(\imath) - R(a, b, c)(\imath)| \end{aligned}$$

$$= \frac{2}{40} \max \Big\{ |\mathcal{G}(\delta')(\imath) - \mathcal{G}(a)(\imath)|, |\mathcal{G}(\sigma')(\imath) - \mathcal{G}(b)(\imath)|, |\mathcal{G}(\kappa')(\imath) - \mathcal{G}(c)(\imath)| \Big\}$$
$$= \frac{1}{40} \max \Big\{ D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c) \Big\}$$

$$= \frac{1}{20} \bigg(\frac{1}{2} \max \Big\{ D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c) \Big\} \bigg)$$
$$\leq \frac{7}{30} \bigg(\frac{1}{2} \max \Big\{ D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c) \Big\} \bigg)$$

$$\leq \frac{2}{5} \left(\frac{1}{2} \max \Big\{ \; D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c) \; \Big\} \right)$$
$$- \frac{1}{6} \left(\frac{1}{2} \max \Big\{ \; D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c) \; \Big\} \right)$$

$$\leq \frac{2}{5} \left(\frac{1}{2} \max \left\{ \begin{array}{c} \max \Big\{ D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), 0 \Big\}, \\ \max \Big\{ D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), 0 \Big\}, \\ \max \Big\{ D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c), 0 \Big\} \end{array} \right\} \right)$$
$$- \frac{1}{6} \left(\frac{1}{2} \max \left\{ \begin{array}{c} \max \Big\{ D^*(\mathcal{G}\delta', \mathcal{G}\delta', \mathcal{G}a), 0 \Big\}, \\ \max \Big\{ D^*(\mathcal{G}\sigma', \mathcal{G}\sigma', \mathcal{G}b), 0 \Big\}, \\ \max \Big\{ D^*(\mathcal{G}\kappa', \mathcal{G}\kappa', \mathcal{G}c), 0 \Big\} \end{array} \right\} \right)$$
$$\leq \psi\Big(\mu\mathcal{M}(\delta', \sigma', \kappa', \delta', \sigma', \kappa', a, b, c) \Big) - \varphi\Big(\mu\mathcal{M}(\delta', \sigma', \kappa', \delta', \sigma', \kappa', a, b, c) \Big).$$

The equation (4.1) has a unique solution in $C(I, \mathcal{R})$, as deduced by Corollary 3.1. □

5 Conclusion

This paper can successfully modify distance functions and represent tripled fixed point in complete D^*-metric spaces. After employing altering distance function and weak compatibility, the presence and uniqueness of a common tripled fixed point in a D^*-metric space are successfully represented. The application of integral equation in entire D^*-metric are demonstrated.

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