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Analysis of cosmological parameters in $f(T, B)$ gravity

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Abstract

In this paper, cosmological solutions for Fridmann-Lemaitre Robertson Walker (FLRW) universe in $f(T, B)$ gravity has been obtained. The particular form of $f(T, B) = f_0 TB$ is taken with hybrid expansion of scale factor. The analysis of stability condition, energy condition of this model has been studied. For this model, to test the stability of this model sound speed is also calculated.

1 Introduction:

General theory of Relativity (GR) explains gravity is not a force but space time curvature which gives the geometry of our universe. Einstein field equations give the explanation how space time behaves on macroscopic scales. But there are limitations to explain some phenomenon like inflation, dark matters, dark energy and accelerated expansion of our universe. Modification of GR has been essential according to the cosmological observations of type-Ia supernova (SNe-Ia) and other astronomical evidences [1, 17, 19]. So many cosmologies have been interested in modified gravity because it provides qualitative solutions to a variety of fundamental issues about the accelerated expansion of the universe with time. Thus the study of modified gravity theories has been necessary in recent years.

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Einstein added an extra term in his field equations called cosmological constant to attain a static universe in 1917. But after the discovery of the expansion of the universe by Hubble in 1929, Einstein realized that his idea of cosmological constant is wrong and rejected it. But in modern cosmology, concept of cosmological constant is used to interpret for the explanation for dark energy, the cause of the acceleration of the expansion of the universe. For this purpose, general relativity theory (GR) has been modified to many cosmological models such as ΛCDM cosmological model [16] which describes the evolution of the universe. It incorporates three components namely cosmological constant representing dark energy, cold dark matter, and ordinary matter. Although this model explains the observed expansion of the universe it has limitation like 'Hubble tension' arising from discrepancies in the measured Hubble constant from different methods.

Many cosmologist has been interested in modified gravity theories because they can be explain a systematic and geometric explanation of the accelerated expansion of universe. One of the modified gravity theories is $f(R)$ gravity which is constraucted by replacing the Ricci-scalar R in the Einstein-Hilbert action with a function of R [**?**, 24] and some researcher [4, 12, 13, 21, 23] have been analyzed the behaviour of the differnet models universe in modified theory of gravity. Teleparallel equivalent of general relativity (TEGR) firstly proposed by Albert Einstein is the modification in GR by replacing the general metric coefficient by tetrad fields. In GR the Ricci-scalar R is used to represent gravitational interaction while in Teleparallel gravity torsion scalar T is used to represent gravitational interaction and Weitzenbock connection is used in Teleparallel gravity [25]. This Teleparallel gravity theory which is called $f(T)$ gravity has been studied by many researchers [9–11, 15, 26, 27]. Another generalization of TEGR is that $f(T, B)$ gravity having an additional attracting property where the Ricci-scalar R is equivalent to the addition of torsion scalar T and Boundary term B. It has been a prominent gravity theory since it satisfies observational data to explain the accelerated expansion of the universe [22]. Many researchers have been studied $f(T, B)$ gravity theory [2, 3, 5, 7, 8, 18, 20, 28].

In this work, cosmological solutions are obtained by taking scale factor [13] $a(t) = \beta t^{\frac{2}{3}} e^{\alpha t}$ in $f(T, B)$ gravity in the framework of FLRW metric.

2 Field equation in $\mathrm{f(T, B)}$ gravity

The action used in this paper for $f(T, B)$gravity [6] is as follows:

$$S_{f(T,B)} = \frac{1}{k^2}\int d^4x e f(T, B) + \int e d^4x L_m, \tag{2.1}$$

where $k^2 = 8\pi G$ and L_m is the matter Lagrangian in the Jordan form. $f(T, B)$ is a function of the torsion scalar T and the boundary term $B = \frac{2}{e}\partial_\mu (eT^\mu)$ in which $T_\mu = T^\mu_{v\mu} \cdot L_m$ and determinant of tetrad component is given by $e = det\left(e^i_\mu\right)$. The field equations obtain by using equation (2.1) are as follows:

$$2e\delta^\lambda_v \nabla^\mu \nabla_\mu \partial_B f - 2e\nabla^\lambda \nabla_v \partial_B f + eB\partial_B f \delta^\lambda_v + 4e\left(\partial_\mu \partial_B f + \partial_\mu \partial_T f\right) S_v^{\mu\lambda}$$
$$+ 4e^\mu_v \partial_\mu \left(eS_a^{\mu\lambda}\right) \partial_r f - 4e\partial_{Tf} T^\sigma{}_{\mu v} S_a^{\mu\lambda} - ef\delta^\lambda_v = 16\pi e T^\lambda_v. \tag{2.2}$$

Stress energy momentum tensor for perfect fluid is written as

$$T_v = (\rho + p)u_v u^\lambda - p\delta^\lambda_v, \tag{2.3}$$

where ρ is the proper energy density and p the proper pressure of the perfect fluid inside the universe and u is the four velocity vector such that $u^\lambda u_\lambda = 1$.

Using comoving coordinates $u^\lambda = (1, 0, 0, 0)$, in equation (2.3), the non-zero component of the stress energy momentum energy is given by

$$T^0_0 = \rho,\ T^1_1 = T^2_2 = T^3_3 = -p. \tag{2.4}$$

In this work, the flat space time described by the isotropic homogeneous FLRW line element is given by

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right], \tag{2.5}$$

where $a(t)$ is the scale factor of the universe and (t, r, θ, φ) are the co-moving co-ordinates. The choice of tetrad is given by

$$e^i_\mu = diag\ (1,\ a(t),\ a(t),\ a(t)).$$

The field equation (2.2) with the line element (2.5) yields two independent equations

$$-3H^2(3f_B + 2f_T) + 3H\dot{f_B} - 3\dot{H}f_B + \frac{1}{2}f = k^2\rho, \tag{2.6}$$

$$-\left(3H^2 + \dot{H}\right)(3f_B + 2f_T) - 2H\dot{f_T} + \ddot{f_B} + \frac{1}{2}f = -k^2 p, \tag{2.7}$$

where $f_B = \frac{\partial f}{\partial B}$ and $f_T = \frac{\partial f}{\partial T}$.

Torsion scalar can be obtain as

$$T = -6H^2. \tag{2.8}$$

Ricci scalar are can be obtained as

$$R = -T + B. \tag{2.9}$$

Boundary term B can be obtained as

$$B = -6\left(3H^2 + \dot{H}\right). \tag{2.10}$$

The Ricci scalar R can be obtained by using equation (2.9) as

$$R = -6\left(2H^2 + \dot{H}\right). \tag{2.11}$$

The equations (2.6) and (2.7) can be rewrite in standard form as

$$3H^2 = k^2\rho_{tot}, \tag{2.12}$$

$$2\dot{H} + 3H^2 = -k^2 p_{tot}. \tag{2.13}$$

The parameters ρ_{tot} and p_{tot} can be written as:

$$\rho_{tot} = \rho + \rho_d, \tag{2.14}$$

$$p_{tot} = p + p_d. \tag{2.15}$$

where ρ_d and p_d are the parts of the energy density and pressure respectively. Using equations (2.6), (2.7), (2.12) – (2.15), we find ρ_d and p_d as

$$k^2\rho_d = 3H^2\left(1 + 3f_B + 2f_T\right) - 3H\dot{f}_B + 3\dot{H}f_B - \frac{1}{2}f, \tag{2.16}$$

$$k^2 p_d = -3H^2\left(1 + 3f_B + 2f_T\right) - \dot{H}\left(2 + 3f_B - 2f_T\right) - 2H\dot{f}_T + \ddot{f}_B + \frac{1}{2}f, \tag{2.17}$$

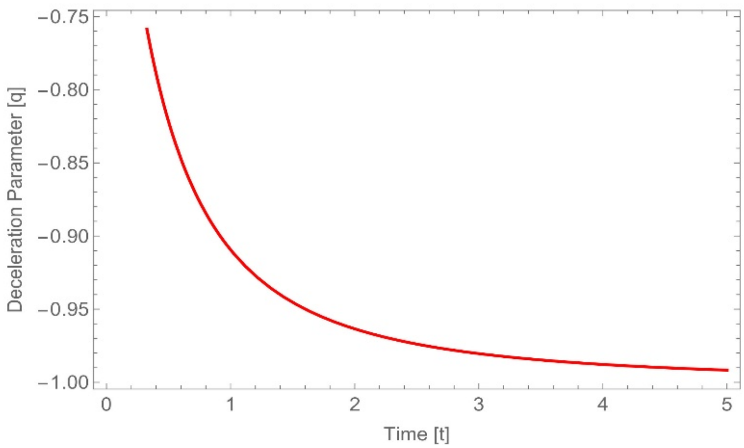

Figure 1: Graph of deceleration parameter with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

3 Dynamical parameters and their physical discussions

In this work, scale factor $a(t)$ is taken as

$$a(t) = \beta t^{\frac{2}{3}} e^{\alpha t}, \tag{3.1}$$

where α and β are integrating constants.

The Hubble parameter $H(t) = \frac{\dot{a}}{a}$ and deceleration parameter $q = \frac{-\ddot{a}a}{\dot{a}^2}$ can be calculated as

$$H(t) = \alpha + \frac{2}{3t}, \tag{3.2}$$

$$q(t) = \frac{-3\alpha t(4 + 3\alpha t) + 2t}{(2 + 3\alpha t)^2}. \tag{3.3}$$

From the Figure 1, it can be seen that deceleration parameter $q = 0$ at $t = 0$ and it decreases with time t and at late times $q \to -1$ when $t \to \infty$. It shows that the expansion of the universe is accelerating. From the Figure 2, Hubble parameter is decreases with time which is also the condition for the accelerating expansion of universe.

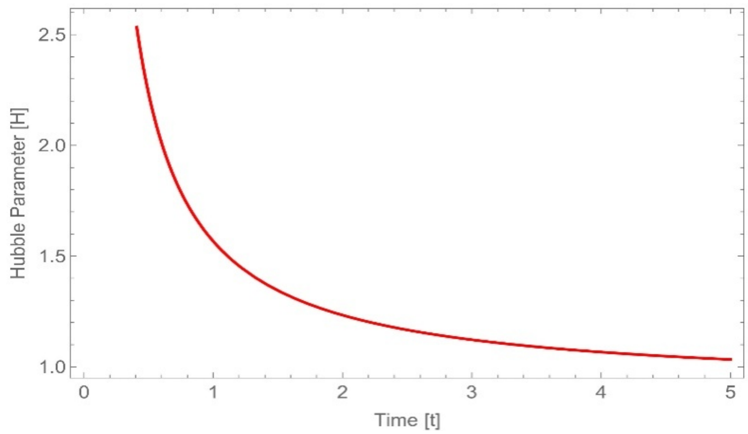

Figure 2: Graph of Hubble parameter with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

4 Model 1

The model with $f(T, B)$proportional to the product of T and B, i.e., $f(T, B) = f_0 TB$ where f_0 is an arbitrary constant. In this case we have,

$$k^2\rho_d = 3\alpha^2 + 4\frac{\alpha}{t} + \frac{4}{3}\frac{1}{t^2} + f_0 216\alpha^4 + f_0 576\frac{\alpha^3}{t} + f_0 576\frac{\alpha^2}{t^2} + f_0\frac{32}{27}\frac{\alpha}{t^3} + f_0\frac{16}{81}\frac{1}{t^4} \tag{4.1}$$

$$k^2 p_d = -3\alpha^2 - 4\frac{\alpha}{t} - f_0 216\alpha^4 - f_0 576\frac{\alpha^3}{t} - f_0 408\frac{\alpha^2}{t^2} - f_0 32\frac{\alpha}{t^3} + f_0 96\frac{1}{t^4}. \tag{4.2}$$

From equation (4.1) and (4.2), the equation of state parameter ω can be obtained as

$$\omega = \frac{-3\alpha^2 - 4\frac{\alpha}{t} - f_0 216\alpha^4 - f_0 576\frac{\alpha^3}{t} - f_0 408\frac{\alpha^2}{t^2} - f_0 32\frac{\alpha}{t^3} + f_0 96\frac{1}{t^4}}{3\alpha^2 + 4\frac{\alpha}{t} + \frac{4}{3}\frac{1}{t^2} + f_0 216\alpha^4 + f_0 576\frac{\alpha^3}{t} + f_0 576\frac{\alpha^2}{t^2} + f_0\frac{32}{27}\frac{\alpha}{t^3} + f_0\frac{16}{81}\frac{1}{t^4}}. \tag{4.3}$$

From Figure 3, it is observed that energy density is a decreasing function of

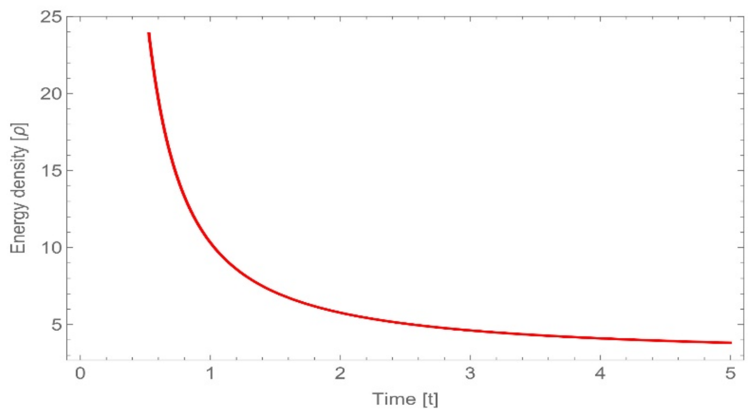

Figure 3: Graph of energy density with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

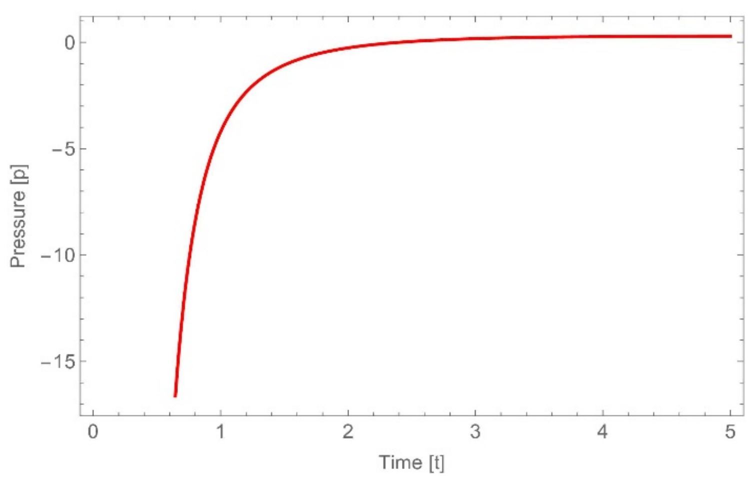

Figure 4: Graph of pressure with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

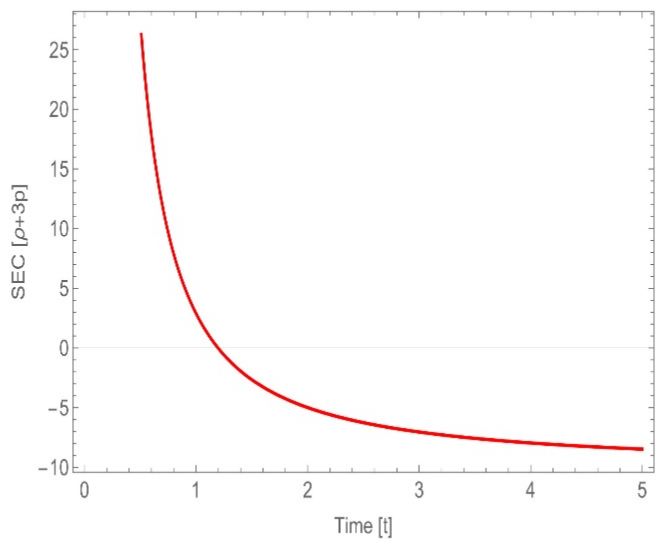

Figure 5: Graph of SEC with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

time t. From Figure 4 it is observed that pressure lies in the negative region which supports the observed phenomena of the expansion of the universe.

5 Energy condition

This section analyse the different types of energy conditions for the model chosen in $f(T,\ B)$ gravity [4, 12, 13, 21, 23]. The energy conditions are listed below as

WEC : $\rho_d \geq 0$, $\rho_d + p_d \geq 0$,

SEC : $\rho_d + 3p_d \geq 0$,

NEC : $\rho_d + p_d \geq 0$,

DEC : $\rho_d - p_d \geq 0$.

Weak Energy Condition (WEC) and Null Energy Condition (NEC) are satisfied for the model in $f(T,\ B)$ gravity which is shown in Figure 6. In Figure 5, it can be seen that Strong Energy Condition (SEC) is validated at the initial stage but it is violated with the increase of tine t. Thus, the results show that WEC, NEC and DEC are all fulfilled for the model. The violation of SEC supports the acceleration of the expansion of universe.

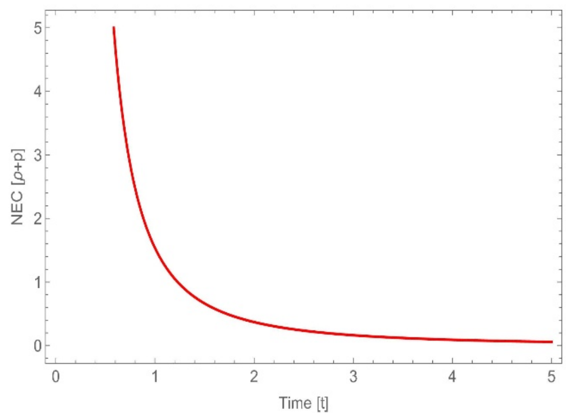

Figure 6: Graph of NEC with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

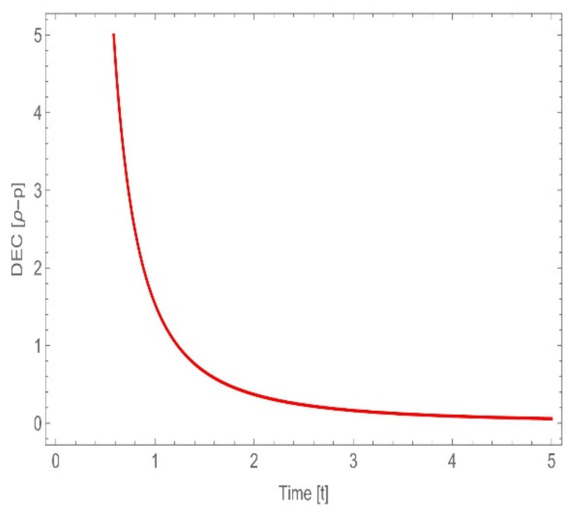

Figure 7: Graph of DEC with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

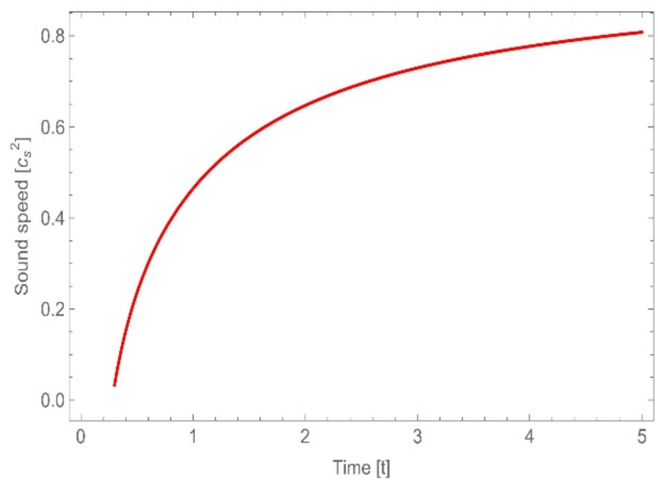

Figure 8: Graph of sound speed with time t for $\alpha = 0.8$ and $f_0 = 0.001$.

6 Stability of the model

The adiabatic speed of sound through the cosmic fluid is defined as

$$C_s^2 = \frac{dp}{d\rho} = \frac{\frac{dp}{dt}}{\frac{d\rho}{dt}}. \tag{6.1}$$

The equation for $\frac{dp}{d\rho}$ can be calculate as

$$\frac{dp}{d\rho} = \frac{4\frac{\alpha}{t^2} + f_0 576\frac{\alpha^3}{t^2} + f_0 816\frac{\alpha^2}{t^2} + f_0 96\frac{\alpha}{t^4} - f_0 384\frac{1}{t^4}}{-4\frac{\alpha}{t^2} - \frac{8}{3}\frac{1}{t^3} - f_0 576\frac{\alpha^3}{t^2} - 1152\frac{\alpha^2}{t^3} - f_0\frac{32}{9}\frac{\alpha}{t^4} - f_0\frac{64}{81}\frac{1}{t^5}}\frac{d\rho}{dt}. \tag{6.2}$$

It shows that $\frac{dp}{d\rho} > 0$ showing the stable behavior of the model.

7 Conclusion

This paper studied the exact solutions of dynamical parameters with hybrid law scale factor with the discussion of energy condition and stability condition in the context of $f(T, B) = f_0TB$, model of $f(T, B)$ gravity.

In this model, expansion of universe is accelerated with time. The Equation of State Parameter (EOS) is also calculated and it is seen that equation of State Parameter ω is leading to -1 as $t \to \infty$. Hence the model coincides with the ΛCDM model. Deceleration parameter decreases with time t and its value is negative which confirms accelerating behavior. Hubble parameter is a decreasing function of time t which is shown in Figure 2. It supports the accelerated expansion of universe. For the energy conditions, WEC, NEC and DEC are fulfilled for the model while the SEC is validated at the initial stage but it is violated with the increase of tine t which supports accelerated expansion of the universe. From the result of the speed of sound calculation, it is seen that the model in this paper is stable.

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