The Aligarh Bulletin of Mathematics
Volume 44, Number 1 (2025), 85 - 103

ISSN: 0304-9787

Triple fixed point results for new classes of functions on cone metric spaces

Md W. Rahaman[1], L. Shambhu Singh[1] and Th. Chhatrajit Singh[2]*

[1]Department of Mathematics
Dhanamanjuri University, Manipur - 795001, India
[2]Manipur Technical University Imphal
Manipur - 795004, India
Email: mdrahman866@gmail.com, lshambu1162@gmail.com,
chhatrajit@gmail.com

(Received: March 18, 2025 Accepted: May 29, 2025)

Abstract

The contraction requirement in the Banach contraction principle requires that a function be continuous. Numerous authors circumvent this obligation and attenuate the hypotheses using metric spaces equipped with a partial order. This work presents many tripled fixed point theorems for functions exhibiting mixed monotone features in cone metric spaces, which are broader than partially ordered metric spaces.

1 Introduction and preliminaries

Fixed point theorems are indeed foundational in ensuring the existence and uniqueness of solutions to many problems across scientific disciplines. Researchers have been expanding these theorems by considering metric spaces with partial orders, which allow a relaxation of contraction conditions. The work by Ran and Reuring [27] pioneered this approach, particularly for applications in matrix equations.

Keywords and phrases: Fixed point, Riesz space, Archimedean Riesz space, cone metric space, tripled fixed point.

2020 AMS Subject Classification: 47H10, 54H25.

*Corresponding Author

Building on this, Nieto and Lopez [23] demonstrated the existence of unique fixed points for order- preserving contractions, adding novel findings in partially ordered metric spaces. Later, they broadened their results by examining order-reversing and non-monotone operators [22]. Lakshmikantham and Bhaskar [9] further advanced the field by introducing the concept of mixed monotone properties and coupled fixed points, which expanded the scope and application of fixed point theorems. They extended the scope of fixed point problems by utilizing mixed monotone operators, which led to further generalizations in ordered metric spaces and cone metric spaces [1, 3, 5, 7, 8, 10–12, 17, 20, 29–31]. Cevik and Altun contributed to this progress by presenting vector metric spaces, where the distance function is defined in a Riesz space and satisfies standard metric conditions. Also, they applied the BCP within this newly established framework [16]. This approach has spurred additional research, with numerous studies expanding fixed point results in vector metric spaces [4, 15, 24–26, 28]. Recent work has also extended previous findings through applications in vector metric spaces, reflecting the broad and evolving applicability of fixed point theory in diverse mathematical spaces [14, 22, 23].

In this paper, we introduce several tripled fixed point theorems within the framework of ordered cone metric spaces, specifically targeting functions that exhibit the mixed monotone property. Our work aims to generalize existing results in this area, providing a broader understanding of fixed point theory. To illustrate the applicability and effectiveness of our findings, we include some examples that demonstrate the utility of these theorems in practical scenarios. Through these contributions, we hope to enhance the existing literature on fixed point theorems and their applications in ordered metric spaces. Let E be a real Banach Space. A subset P of E is called a cone if

1. P is closed, non-empty and $P \neq 0$,
2. $a, b \in \mathbb{R}$ $a, b \geq 0$ and $i, h \in P$ imply $ai + bh \in P$,
3. $P \cap (-P) = 0$.

Given a cone $P \subset E$ we define the partial ordering $\leq$ with respect to P by $i \leq h$ if and only if $h - i \in P$. We write $i < h$ to denote that $i \leq h$ but $i \neq h$, while $i \ll h$ will stand for $h - i \in intP$ (interior of P).

Consider a partially ordered set P. It is considered lattice for every pair elements in P has both a supremum and an infimum. The absolute of an element $a \in P$ is given by $|a| = a \vee (-a)$ where $\vee$ represents the supremum. Additionally, if for every nonempty subset of P that is bounded above possesses supremum, then P is called Dedekind $(\sigma-)$complete. Here $a_n \downarrow a$ indicates a is infimum of

$\{a_n : n \in \mathbb{N}\}$ and (a_n) is order-preserving. If P is both ordered vector space and lattice, it is referred to as a Riesz space or vector lattice. Now, consider P as a Riesz space. P is called Archimedean Riesz space if $\frac{1}{n} \downarrow 0$ holds for every $a \in P^+$. A sequence (a_n) in P is called order convergent to a, written as $a_n \xrightarrow{o} a$, if there exists a sequence (b_n) in P such that $b_n \downarrow 0$ and $|a_n - a| \leq b_n$ for all n. Furthermore, if there exists a sequence (b_n) in P with $b_n \downarrow 0$ and $|a_n - a_{n+p}| \leq b_n$ for all n and p, then the sequence (a_n) is referred to as order-Cauchy. Also, P is o-Cauchy complete for each o-Cauchy sequence is o-convergent. Further for notions on Riesz spaces, we direct the reader to [18], [2].

We now revisit few important notions from [19]. Let Y be a nonempty set. Suppose the mapping $d : Y \times Y \to P$ satisfies the following conditions:
$0 < d(e, f)$ for all $e, f \in Y$ and $d(e, f) = 0$ iff $e = f$; $d(e, f) = d(f, e)$ for all $e, f \in Y$ and $d(e, f) \leq d(e, g) + d(g, f)$ for all $e, f, g \in Y$. Thus d is said to be a cone metric on Y and (Y, d) is said to be a cone metric space.

Sequence (h_n) in Y is called vectorially converges (or P-converges for short) to an element h in Y, denoted as $h_n \xrightarrow{d,P} h$, if $\exists$ (b_n) in P such that $b_n \downarrow 0$ and $d(h_n, h) \leq b_n$ for all n. Similarly, a sequence (h_n) is called P-Cauchy if $\exists$ a sequence (b_n) in P such that $b_n \downarrow 0$ and $d(h_{n+p}, h_n) \leq b_n$ for all p and n. Moreover, if every P-Cauchy sequence in Y P-converges to a limit within Y, then Y is said to be P-complete.

The following lemma establishes a connection between o-convergence and P-convergence:

Lemma 1.1. *Suppose Y be a cone metric space and $(h_n) \in Y$. Then, $h_n \xrightarrow{d,P} h$ iff $d(h_n, h) \xrightarrow{o} 0$. Moreover, (h_n) is P-Cauchy iff $d(h_{n+p}, h_n) \xrightarrow{o} 0$ for all n and p.*

Further, $F : Y^3 \to Y$ is vectorial continuous if
$d(F(i_n, h_n, g_n), F(i, h, g)) \xrightarrow{o} 0$ for $d((i_n, h_n, g_n), (i, h, g)) \xrightarrow{o} 0$.

Before proceeding to the main discussion, we first revisit the concepts of the mixed monotone property and tripled fixed point, as introduced in [6]. Given a partially ordered set $(Y, \leq)$ we can extend this order to Y^3 as

$$(t, k, r) \leq (i, h, g) \Leftrightarrow t \leq i, h \leq k \text{ and } r \leq g,$$

for all $i, h, g, t, k, r \in Y$. We say $F : Y^3 \to Y$ have mixed monotone property for all $t, k, r \in Y$

$$t_1, t_2 \in Y, t_1 \leq t_2 \Rightarrow F(t_1, k, r) \leq F(t_2, k, r),$$

$$k_1, k_2 \in Y, k_1 \leq k_2 \Rightarrow F(t, k_1, r) \geq F(t, k_2, r),$$

and

$$r_1, r_2 \in Y, r_1 \leq r_2 \Rightarrow F(t, k, r_1) \leq F(t, k, r_2).$$

An element $(i, h, g) \in Y^3$ is said to be tripled fixed point of $F : Y^3 \to Y$ if $F(i, h, g) = i, F(h, i, h) = h$ and $F(g, h, i) = g$.

2 Main results

Throughout this study, unless mentioned otherwise, the ordered set $(Y, \leq)$ is a P-complete cone metric space where P is an Archimedean Riesz space.

Theorem 2.1. *Let $F : Y^3 \to Y$ have mixed monotone property. If either F is vectorial continuous or Y have below three properties:*

1. *if $i_n \xrightarrow{d,P} i$ and (i_n) is order-preserving, then $i_n \leq i$ for all n,*
2. *if $h_n \xrightarrow{d,P} i$ and (h_n) is order-reversing, then $h \leq h_n$ for all n,*
3. *if $g_n \xrightarrow{d,P} g$ and (g_n) is order-preserving, then $g_n \leq g$ for all n.*

Suppose F satisfies the contractive condition

$$d(F(i, h, g), F(u, v, w)) \leq \frac{k}{2}[d(i, u) + d(h, v) + d(g, w)]$$

for all $i, h, g, u, v, w \in Y$ where $u \leq i$, $h \leq v$, $w \leq g$, k be a constant such that $k \in [0, 1)$. If $\exists\ t, k, r \in Y$ such that $t \leq F(t, k, r)$, $k \geq F(k, t, k)$ and $r \leq F(r, k, t)$, then F has a triple fixed point.

Proof. Let $t = i_0$, $k = h_0$ and $r = g_0$. For the sequences (i_n), (h_n) and (g_n) defined by $i_n = T(i_{n-1}, h_{n-1}, g_{n-1})$, $h_n = T(h_{n-1}, i_{n-1}, h_{n-1})$ and $g_n = T(g_{n-1}, h_{n-1}, i_{n-1})$ for all n.

Since the function F have mixed monotone property, then

$$i_1 = F(i_0, h_0, g_0) \leq F(i_1, h_1, g_1) = i_2,$$

$$h_1 = F(h_0, i_0, h_0) \geq F(h_1, i_1, h_1) = h_2,$$

$$g_1 = F(g_0, h_0, i_0) \leq F(g_1, h_1, i_1) = g_2,$$

which further implies

$$i_0 \leq i_1 \leq \cdots \leq i_n \leq \cdots,$$

$$h_0 \geq h_1 \geq \cdots \geq h_n \geq \cdots,$$

$$g_0 \leq g_1 \leq \cdots \leq g_n \leq \cdots.$$

Hence,

$$\begin{aligned}
d(i_{n+1}, i_n) =& \ d(F(i_n, h_n, g_n), F(i_{n-1}, h_{n-1}, g_{n-1})) \\
\leq& \ \frac{k}{2}[d(i_n, i_{n-1}) + d(h_n, h_{n-1}) + d(g_n, g_{n-1})] \\
=& \ \frac{k}{2}[d(F(i_{n-2}, h_{n-2}, g_{n-2}), F(i_{n-1}, h_{n-1}, g_{n-1})) \\
&+ d(F(h_{n-2}, i_{n-2}, h_{n-2}), F(h_{n-1}, i_{n-1}, h_{n-1})) \\
&+ d(F(g_{n-2}, h_{n-2}, i_{n-2}), F(g_{n-1}, h_{n-1}, i_{n-1}))] \\
\leq& \ \frac{k}{2}\Big[\frac{k}{2}[d(i_{n-2}, i_{n-1}) + d(h_{n-2}, h_{n-1}) + d(g_{n-2}, g_{n-1})] \\
&+ \frac{k}{2}[d(h_{n-2}, h_{n-1}) + d(i_{n-2}, i_{n-1}) + d(h_{n-2}, h_{n-1})] \\
&+ \frac{k}{2}[d(g_{n-2}, g_{n-1}) + d(h_{n-2}, h_{n-1}) + d(i_{n-2}, i_{n-1})]\Big] \\
=& \ \frac{k^2}{2}[d(i_{n-2}, i_{n-1}) + d(h_{n-2}, h_{n-1}) + d(g_{n-2}, g_{n-1})]
\end{aligned}$$

$$\frac{k^3}{2}[d(i_{n-2}, i_{n-3}) + d(h_{n-2}, h_{n-3})] \leq \cdots \leq \frac{k^n}{2}\Big[d(i_1, i_0) + d(h_1, h_0) + d(g_1, g_0)\Big].$$

Similarly, $d(h_n, h_{n+1}) \leq \frac{k^n}{2}[d(h_0, h_1) + d(i_0, i_1) + d(h_0, h_1)]$ and

$d(g_n, g_{n+1}) \leq \frac{k^n}{2}[d(g_0, g_1) + d(h_0, h_1) + d(i_0, i_1)]$ for all n. Thus,

$$\begin{aligned} d(i_n, i_{n+p}) \leq& d(i_n, i_{n+1}) + d(i_{n+1}, i_{n+2}) + \cdots + d(i_{n+p-1}, i_{n+p}) \\ \leq& \left(\frac{k^n}{2} + \frac{k^{n+1}}{2} + \cdots + \frac{k^{n+p-1}}{2}\right)[d(i_1, i_0) + d(h_1, h_0) \\ &+ d(g_1, g_0)] \\ \leq& \frac{k^n}{2(1-k)}[d(i_1, i_0) + d(h_1, h_0) + d(g_1, g_0)]. \end{aligned}$$

Similarly, $d(h_n, h_{n+p}) \leq \frac{k^n}{2(1-k)}[d(h_0, h_1) + d(i_0, i_1) + d(h_0, h_1)]$ and $d(g_n, g_{n+p}) \leq \frac{k^n}{2(1-k)}[d(g_0, g_1) + d(h_0, h_1) + d(i_0, i_1)]$ $\forall$ n and p. As, P is Archimedean Riesz space, sequences (g_n), (h_n), (i_n) are P-Cauchy sequences. Thus, $\exists$ g, h, $i \in h$ such that $g_n \xrightarrow{d,P} g$, $h_n \xrightarrow{d,P} h$, $i_n \xrightarrow{d,P} i$ because Y is P-complete. Thus, $\exists$ (c_n), (b_n), (a_n) such that $c_n \downarrow 0$, $b_n \downarrow 0$, $a_n \downarrow 0$ and $d(g_n, g) \leq c_n$, $d(h_n, h) \leq b_n$ and $d(i_n, i) \leq a_n$ for all n.

Case 1: Let F be a vectorially continuous function, $\exists$ (c_n) satisfying c_n decreases to 0 and

$$d(F(i_n, h_n, g_n), F(i, h, g)) \leq c_n \text{ for all } n.$$

As

$$\begin{aligned} d(F(i, h, g), i) \leq& d(F(i_n, h_n, g_n), F(i, h, g)) + d(F(i_n, h_n, g_n), i) \\ \leq& d(i_{n+1}, i) + c_n \\ \leq& a_{n+1} + c_n \\ \leq& a_n + c_n \qquad \text{for all } n, \end{aligned}$$

for all n, then $0 = d(i, F(i, h, g))$, that is, $i = F(i, h, g)$. Likewise, $h = F(h, i, h)$ and $g = F(g, h, i)$.

Case 2: $i_n \leq i$, $h \leq h_n$ and $g_n \leq g$ for all n. As,

$$\begin{aligned} d(F(i,h,g),i) \leq & d(F(i_n,h_n,g_n),F(i,h,g)) + d(F(i_n,h_n,g_n),i) \\ \leq & \frac{k}{2}[d(i_n,i)+d(h_n,h)+d(g_n,g)] + d(i_{n+1},i) \\ \leq & \frac{k}{2}[a_n+b_n+c_n] + a_{n+1} \\ \leq & \frac{k}{2}[b_n+c_n] + \frac{3k}{2}a_n \end{aligned}$$

for all n, thus $d(F(i,h,g),i) = 0$ implies $F(i,h,g) = i$. Similarly, $F(h,i,h) = h$ and $F(g,h,i) = g$. □

The above result confirms existence of a tripled fixed point and indicates the function does not have to be vectorially continuous under certain conditions. But, this does not suffice to ensure uniqueness of fixed point. For instance, suppose $Y = 2,3 \subseteq \mathbb{N}^+$ and Y^3 with an order relation defined as

$$(i,h,g) \leq (t,k,r) \Leftrightarrow \exists\, p,q,r \in \mathbb{N}^+ : t = px, h = qk, p = rz,$$

and $P = \mathbb{R}$ with usual order relation. Thus, Y is a P-complete cone metric space. As, $F : Y^3 \to Y$ defined as

$$F(i,h,g) = \begin{cases} 2, & \text{if } i=3, h=3, g=3 \\ 3, & \text{otherwise.} \end{cases}$$

All the conditions of Theorem 2.1 holds. Also, F has a tripled fixed point but not unique.

To establish uniqueness, an additional condition must be imposed to assumptions of Theorem 2.1. If Y^3 is lattice or Dedekind ($\sigma-$) complete, then the tripled fixed point is unique. Two elements (t,k,r) and (i,h,g) in Y^3 are comparable or at least one of the supremum $(t,k,r) \vee (i,h,g) = (i \vee t, h \wedge k, g \vee r)$ and the infimum $(t,k,r) \wedge (i,h,g) = (i \wedge t, h \vee k, g \wedge r)$ are in Y^3, then the tripled fixed point is unique. For any two elements (t,k,r) and (i,h,g) in Y^3, the conditions below are weaker and equivalent than the previous ones:

1. Set $\{(t,k,r),(i,h,g)\}$ have upper bound or lower bound in Y^3,

2. $\exists$ an element (u,v,w) in Y^3 comparable to (i,h,g), (t,k,r).

We now replace the previous conditions with one of the following two alternative conditions.

Theorem 2.2. *If condition (1) is incorporated into the assumptions of Theorem 2.1, F possesses a unique tripled fixed point.*

Proof. Let (t,k,r) be another tripled fixed point of F. Our claim is that $d((t,k,r),(i,h,g)) = 0$. Case I: If (t,k,r) and (i,h,g) are comparable, we get

$$\begin{aligned}
d((i,h,g),(t,k,r)) &= d(i,t) + d(h,k) + d(g,r) \\
&= d(F(i,h,g),F(t,k,r)) + d(F(h,i,h),F(k,t,k)) \\
&\quad + d(F(g,h,i),F(r,k,t)) \\
&\le k[d(i,t) + d(h,k) + d(g,r)] \\
&= kd((i,h,g),(t,k,r)).
\end{aligned}$$

As $k \in [0,1)$ implies $d((i,h,g),(t,k,r)) = 0$.

Case II: If (t,k,r) and (i,h,g) are incomparable, $\exists$ upper or lower bound $(a_1,a_2,a_3) \in h^3$ of $\{(i,h,g),(t,k,r))\}$. Since, (a_1,a_2,a_3) is comparable to (i,h,g), (t,k,r), we have $(i,h,g) = (F^n(i,h,g),F^n(h,i,h),F^n(g,h,i))$ and $(t,k,r) = (F^n(t,k,r),F^n(k,t,k),F^n(r,k,t))$ are comparable with $(F^n(a_1,a_2,a_3),F^n(a_2,a_1,a_2),F^n(a_3,a_2,a_1))$ for all n. So, we have

$$\begin{aligned}
d((i,h,g),(t,k,r)) &= d\Big((F^n(i,h,g),F^n(h,i,h),F^n(g,h,i)), \\
&\quad (F^n(t,k,r),F^n(k,t,k),F^n(r,k,t))\Big) \\
&\le d((F^n(i,h,g),F^n(h,i,h),F^n(g,h,i)), \\
&\quad (F^n(a_1,a_2,a_3),F^n(a_2,a_1,a_2),F^n(a_3,a_2,a_1))) \\
&\quad + d((F^n(a_1,a_2,a_3),F^n(a_2,a_1,a_2),F^n(a_3,a_2,a_1)), \\
&\quad (F^n(t,k,r),F^n(k,t,k),F^n(r,k,t))) \\
&\le k^n([d(a_1,i) + d(a_2,h) + d(a_3,g)] \\
&\quad + [d(a_1,t) + d(a_2,k) + d(a_3,r)])
\end{aligned}$$

for all n. Since P is Archimedean Riesz space, $d((t,k,r),(i,h,g)) = 0$. □

Theorem 2.3. *Suppose (i,h,g) be tripled fixed point F. Along with the conditions of first Theorem, for each triple of elements of Y has lower or upper bound in Y implies $i = h = g$.*

Proof. Suppose (i,h,g) be tripled fixed point of F.

Case I: If i, h, g are comparable elements, we have

$$d(i,h) = d(F(i,h,g), F(h,i,h)) \leq kd(i,h).$$

Thus, $i = h$ as $k \in [0,1)$. Similarly, $i = g$. So, $i = h = g$.

Case II: If i, h, g are incomparable and b be upper bound of $\{i,h,g\}$, then

$$F(b,h,g) \geq F(i,h,g), F(h,b,h) \leq F(h,i,h) \text{ and } F(g,h,b) \geq F(g,h,i),$$

$$F(b,h,b) \geq F(b,h,g), F(b,h,b) \geq F(i,h,i) \text{ and } F(b,h,b) \geq F(g,h,b),$$

$$F(h,b,h) \leq F(b,h,b).$$

Also,

$$\begin{aligned}
F^{m+1}(i,h,g) &= F^m(T(i,h,g), F(h,i,h), F(g,h,i)) \\
&\leq F^m(F(b,h,g), F(h,b,h), F(g,h,b)) \\
&= F^{m+1}(b,h,g), \\
F^{m+1}(h,i,h) &= F^m(F(h,i,h), F(i,h,i), F(h,i,h)) \\
&\geq F^m(F(h,b,h), F(b,h,b), F(h,b,h)) \\
&= F^{m+1}(h,b,h), \\
F^{m+1}(g,h,i) &= F^m(T(g,h,i), F(h,g,h), F(i,h,g)) \\
&\leq F^m(F(g,h,b), F(h,g,h), F(b,h,g)) \\
&= F^{m+1}(g,h,b),
\end{aligned}$$

$$\begin{aligned} F^{m+1}(b,h,g) &= F^m(F(b,h,g),F(h,b,h),F(g,h,b)) \\ &\le F^m(F(b,h,b),F(h,b,h),F(b,h,b)) \\ &= F^{m+1}(b,h,b), \end{aligned}$$

$$\begin{aligned} F^{m+1}(g,h,b) &= F^m(F(g,h,b),F(h,g,h),F(b,h,g)) \\ &\le F^m(F(b,h,b),F(h,b,h),F(b,h,b)) \\ &= F^{m+1}(b,h,b), \end{aligned}$$

$$\begin{aligned} F^{m+1}(i,h,i) &= F^m(F(i,h,i),F(h,i,h),F(i,h,i)) \\ &\le F^m(F(b,h,b),F(h,b,h),F(b,h,b)) \\ &= F^{m+1}(b,h,b). \end{aligned}$$

Hence,

$$\begin{aligned} d(i,h) = &\ d(F^{m+1}(i,h,g),F^{m+1}(h,i,h)) \\ \le &\ d\Big(F(F^m(i,h,g),F^m(h,i,h),F^m(g,h,i)), \\ &\qquad F(F^m(b,h,g),F^m(h,b,h),F^m(g,h,b))\Big) \\ &+ d\Big(F(F^m(b,h,g),F^m(h,b,h),F^m(g,h,b)), \\ &F(F^m(b,h,b),F^m(h,b,h),F^m(b,h,b))\Big) \\ &+ d\Big(F(F^m(h,i,h),F^m(i,h,i),F^m(h,i,h)), \\ &\qquad F(F^m(h,b,h),F^m(b,h,b),F^m(h,b,h))\Big) \\ &+ d\Big(F(F^m(h,b,h),F^m(b,h,b),F^m(h,b,h)), \\ &\qquad F(F^m(b,h,b),F^m(h,b,h),F^m(b,h,b))\Big). \end{aligned}$$

implies

$$d(i,h) \le \frac{k}{2}\Big[d(F^m(b,h,g),F^m(i,h,g)) + d(F^m(h,b,h),$$
$$F^m(h,i,h)) + d(F^m(g,h,b),F^m(g,h,i))$$
$$+ \cdots + d(F^m(b,h,b),F^m(h,b,h))\Big].$$

Thus, $d(i,h) \le k^{m+1}[d(i,b)+d(h,b)+d(g,b)]$ and hence $d(i,h)=0$ as P is Archimedean Riesz space. □

If every triplet of elements in Y^3 has an upper or lower bound, then Y forms a lattice. By incorporating condition (1) into the assumptions of Theorem 2.1, we can ensure both the uniqueness of the tripled fixed point and the equality of its components. However, the presence of an upper or lower bound for each triplet in Y does not necessarily imply that every triplet in Y^3 also has at least one upper or lower bound. Additionally, it can be proven that the components of a fixed point are identical if t,k and r in first Theorem are comparable.

Theorem 2.4. *With the conditions of Theorem 2.1 is incorporated for any tripled fixed point (i,h,g) of F, if t,k and r are comparable in Y, then $i=h=g$.*

Proof. We have $t \le F(t,k,r)$ by Theorem 2.1. Using mathematical induction, we claim $i_m \le h_m$ for all m and $t \le k$. If $t \le k$, then $h_1 = F(k,t,k) \ge F(t,k,r) = i_1$ as F have mixed monotone property. As $h_m \ge i_m$ for some m, we have

$$\begin{aligned} i_{m+1} &= F^{m+1}(t,k,r) \\ &= F(F^m(t,k,r),F^m(k,t,k),F^m(r,k,t)) \\ &= F(i_m,h_m,g_m) \\ &\le F(h_m,i_m,h_m) \\ &= h_{m+1}, \end{aligned}$$

i.e., $h_m \ge i_m$ for all m. As $g_m \xrightarrow{d,P} g$, $h_m \xrightarrow{d,P} h$ and $i_m \xrightarrow{d,P} i$, there exist three sequences (a_n), (b_n) and (c_n) in P such that $a_n \downarrow 0$, $b_n \downarrow 0$, $c_n \downarrow 0$ and

$d(g_m, g) \leq c_m$, $d(h_m, h) \leq b_m$, $d(i_m, i) \leq a_m$. Thus

$$\begin{aligned} d(i,h) \leq\ & d(i, F^{m+1}(t,k,r)) + d(F^{m+1}(t,k,r), F^{m+1}(k,t,k)) \\ & + d(h, F^{m+1}(k,t,k)) \\ \leq\ & a_{m+1} + d(F(F^m(t,k,r), F^m(k,t,k), F^m(r,k,t)), \\ & F(F^m(k,t,k), F^m(t,k,t), F^m(k,t,k))) \\ & + b_{m+1} \\ \leq\ & a_m + kd(F^m(t,k,r), F^m(k,t,k) + b_m \\ \leq\ & a_m + k[d(F^m(t,k,r), i) + d(i,h) + d(h, F^m(k,t,k))] + b_m \\ \leq\ & (k+1)(a_m + b_m) + kd(i,h). \end{aligned}$$

Therefore, $(1-k)d(i,h) \leq (k+1)(a_m + b_m)$. Thus, $0 = d(i,h)$. Likewise, $0 = d(i,g)$ and $0 = d(h,g)$. □

The next result shows that we get a unique tripled fixed point using condition $F(t,k,r) \leq t$, $F(k,t,k) \geq k$ and $F(r,k,t) \leq r$ instead of the condition $t \leq F(t,k,r)$, $k \geq F(k,t,k)$ and $r \leq F(r,k,t)$ in Theorem 2.1.

Theorem 2.5. *Suppose Y hold condition (1) and $F : Y^3 \to Y$ have mixed monotone property on Y and either F is vectorial continuous or Y have the properties below:*

1. *if $i_n \xrightarrow{d,P} i$ and (i_n) is order-preserving, then $i \leq i_n$ for all n,*
2. *if $h_n \xrightarrow{d,P} h$ and (h_n) is order-reversing, then $h_n \leq h$ for all n,*
3. *if $g_n \xrightarrow{d,P} g$ and (g_n) is order-preserving, then $g \leq g_n$ for all n.*

Also, F satisfies

$$d(F(i,h,g), F(u,v,w)) \leq \frac{k}{2}[d(i,u) + d(h,v) + d(g,w)]$$

for all i, h, g, u, v, w in h where $u \leq i$, $h \leq v$, $w \leq g$, k be a constant such that $k \in [0,1)$. If $\exists\ t, k, r \in h$ satisfying $F(t,k,r) \leq t$, $F(k,t,k) \geq k$ and $F(r,k,t) \leq r$ implies F has a unique tripled fixed point in Y^3.

Proof. Assume $t = i_0$, $k = h_0$, $r = g_0$, $F^n(i_0, h_0, g_0) = i_n$, $F^n(h_0, i_0, h_0) = h_n$ and $F^n(g_0, h_0, i_0) = g_n$. Since, $F(i_0, h_0, g_0) \leq i_0$, $T(h_0, i_0, h_0) \geq h_0$, $T(g_0, h_0, i_0) \leq g_0$, thus

$$\cdots \leq i_n \leq \cdots \leq i_2 \leq i_1 \leq i_0,$$

$$h_0 \leq h_1 \leq h_2 \leq \cdots \leq h_n \leq \cdots,$$

$$\cdots \leq g_n \leq \cdots \leq g_2 \leq g_1 \leq g_0.$$

As in the proof of Theorem 2.1, we get

$$\begin{aligned} d(i_{n+1}, i_n) \leq & \frac{k^n}{2}[d(i_1, i_0) + d(h_1, h_0) + d(g_1, g_0)] \\ d(h_{n+1}, h_n) \leq & \frac{k^n}{2}[d(i_1, i_0) + d(h_1, h_0) + d(g_1, g_0)] \\ d(g_{n+1}, g_n) \leq & \frac{k^n}{2}[d(i_1, i_0) + d(h_1, h_0) + d(g_1, g_0)], \end{aligned}$$

implies (g_n), (h_n), (i_n) are E-Cauchy sequences. As h is E-complete, $\exists\, i, h$ and g in h where $i_n \xrightarrow{d,P} i$, $h_n \xrightarrow{d,P} h$ and $g_n \xrightarrow{d,P} g$. So, there exist three sequences in P, say, (a_n), (b_n) and (c_n) satsifying $c_n \downarrow 0$, $b_n \downarrow 0$, $a_n \downarrow 0$ and $d(g_n, g) \leq c_n$, $d(h_n, h) \leq b_n$, $d(i_n, i) \leq a_n$. Therefore, (i, h, g) is a tripled fixed point of F where F is vectorially continuous. As, $i_n \xrightarrow{d,P} i$ and $i_n \leq i\ \forall\, n$, we get

$$\begin{aligned} d(F(i, h, g), i) \leq & \; d(F(i, h, g), F(i_n, h_n, g_n)) + d(F(i_n, h_n, g_n), i) \\ \leq & \; \frac{k}{2}[d(i, i_n) + d(h, h_n) + d(g, g_n)] + d(i, i_{n+1}) \\ \leq & \; \frac{k}{2}[c_n + b_n + a_n] + a_{n+1} \\ \leq & \; \frac{k}{2}[c_n + b_n] + \frac{3k}{2}a_n. \end{aligned}$$

Hence, $d(F(i, h, g), i) = 0$, that is, $F(i, h, g) = i$. Similarly, we get $F(h, i, h) = h$ and $F(g, h, i) = g$. □

Presented below are the corollaries:

A function F may still have a tripled fixed point if a vectorially convergent sequence, with comparable consecutive terms, contains a subsequence where all terms remain comparable to the limit, i.e., Y fulfills the following condition:

(A) if (i_n) converges vectorially to i and holds $i_n \gtrless i_{n+1}\ \forall\ n$, then there exist a subsequence (i_{n_m}) of (i_n) such that $i_{n_m} \gtrless i\ \forall\ n_m$. Here, the symbol $\gtrless$ is used for any two comparable elements.

Corollary 2.1. *Suppose a function $F : Y^3 \to Y$ has mixed monotone property and let Y satisfy condition (1), if either Y has property (A) or F is vectorial continuous. Also, let F hold*

$$d(F(u,v,w),T(i,h,g)) \leq \frac{k}{2}[d(u,i)+d(v,h)+d(w,g)]$$

for all i,h,g,u,v,w in h where $u \leq i$, $h \leq v$, $w \leq g$, k be a constant such that $k \in [0,1)$. If $\exists\ t,k,r \in h$ satisfying either $F(t,k,r) \leq t$ and $F(k,t,k) \geq k$ or $t \leq F(t,k,r)$ and $k \geq F(k,t,k)$, then F has a unique tripled fixed point in Y^3.

Proof. Assume $t = i_0$, $k = h_0$, $r = g_0$, $F^n(i_0,h_0,g_0) = i_n$, $F^n(h_0,i_0,h_0) = h_n$ and $F^n(g_0,h_0,i_0) = g_n$ for all n. Suppose $i_0 \leq F(i_0,h_0,g_0) = i_1$, $h_0 \geq F(h_0,i_0,h_0) = h_1$, $g_0 \leq F(g_0,h_0,i_0) = g_1$ (or $i_0 \geq F(i_0,h_0,g_0) = i_1$, $h_0 \leq F(h_0,i_0,h_0) = h_1$, $g_0 \geq F(g_0,h_0,i_0) = g_1$). Then,

$$d(i_n,i_{n+1}) \leq \frac{k^n}{2}[d(i_0,i_1)+d(h_0,h_1)+d(g_0,g_1)],$$

$$d(h_n,h_{n+1}) \leq \frac{k^n}{2}[d(i_0,i_1)+d(h_0,h_1)+d(g_0,g_1)],$$

$$d(g_n,g_{n+1}) \leq \frac{k^n}{2}[d(i_0,i_1)+d(h_0,h_1)+d(g_0,g_1)].$$

because $i_{n+1} \gtrless i_n$, $h_{n+1} \gtrless h_n$ and $g_{n+1} \gtrless g_n$ for all n. Hence, (i_n), (h_n) and (g_n) are E-Cauchy sequences. E-Completeness of h implies $\exists\ g$, h and i in h such that $g_n \xrightarrow{d,P} g$, $h_n \xrightarrow{d,P} h$ and $i_n \xrightarrow{d,P} i$. From Theorem 2.1, $\exists\ (i_{n_m})$, (h_{n_m}),

(g_{n_m}) where the terms are comparable to limit i, h, g respectively. Thus,

$$
\begin{aligned}
d(F(i,h,g),i) \leq & d(F(i,h,g),F(i_{n_m},h_{n_m},g_{n_m})) + d(F(i_{n_m},h_{n_m},g_{n_m}),i) \\
\leq & \frac{k}{2}[d(i,i_{n_m}) + d(h,h_{n_m}) + d(g,g_{n_m})] + d(i_{n_m+1},i) \\
\leq & \frac{k}{2}[a_{n_m} + b_{n_m} + c_{n_m}] + a_{n_m+1} \\
\leq & \frac{k}{2}[b_{n_m} + c_{n_m}] + \frac{3k}{2}a_{n_m}.
\end{aligned}
$$

As $c_n \downarrow 0$, $b_n \downarrow 0$, $a_n \downarrow 0$, we get $c_{n_m} \downarrow 0$, $b_{n_m} \downarrow 0$, $a_{n_m} \downarrow 0$ and $F(i,h,g) = i$. Likewise, $F(h,i,h) = h$ and $F(g,h,i) = i$. □

We derive similar results by considering functions that are order-reversing with respect to the first and third variables and order-preserving with respect to the second variable. For convenience, we refer to such functions as having the second type mixed monotone property. Specifically, a function $F : Y^3 \to Y$ is said to have the second type mixed monotone property if, for any $t, k, r \in Y$, the following conditions hold:

$$t_1, t_2 \in Y, t_1 \leq t_2 \Rightarrow F(t_2,k,r) \leq F(t_1,k,r),$$

$$k_1, k_2 \in Y, k_1 \leq k_2 \Rightarrow F(t,k_2,r) \geq F(t,k_1,r),$$

and

$$r_1, r_2 \in Y, r_1 \leq r_2 \Rightarrow F(t,k,r_2) \leq F(t,k,r_1).$$

Corollary 2.2. *Suppose Y hold condition (1) and $F : Y^3 \to Y$ have either second or first type mixed monotone property on Y. Suppose either Y has property (A) or F is vectorial continuous. Also, F satisfy*

$$d(F(i,h,g),F(u,v,w)) \leq \frac{k}{2}[d(i,u) + d(h,v) + d(g,w)]$$

$\forall\ i, h, g, u, v, w$ in Y where $u \leq i$, $h \leq v$, $w \leq g$, k be a constant such that $k \in [0, 1)$. If $\exists\ t, k, r \in Y$ satisfying either $F(t, k, r) \leq t$ and $F(k, t, k) \geq k$ or $t \leq F(t, k, r)$ and $k \geq F(k, t, k)$, then F has a unique tripled fixed point in Y^3.

3 Conclusion

Our findings highlight conditions under which triple fixed points exist for certain classes of functions, providing a significant generalization of classical fixed point theorems. These results demonstrate the versatility of cone metric spaces in accommodating complex function types and mappings that may not fit within the constraints of traditional metric spaces. The established theorems not only generalize existing results but also introduce new avenues for studying coupled and multi-point fixed point problems. This work paves the way for the application of triple fixed point results in domains like differential equations, dynamic systems, and mathematical modeling, which frequently require simultaneous resolution of relationships between multiple variables. Future studies may explore further generalizations, including higher-order fixed points, and investigate practical applications in optimization, computational science, and real-world systems modeled by cone metric spaces. The development of such tools enriches the mathematical toolkit for addressing complex problems across diverse disciplines.

Acknowledgment:

The authors like to express sincere gratitude to the referees for their helpful comments and suggestion to improve the quality of the manuscript.

References

[1] M. Abbas, M. Ali Khan and S. Radenovic, *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput., **217**(1)(2010), 195–202.

[2] C. D. Aliprantis , K. C. Border, *Infinite Dimensional Analysis*, Springer-Verlag (Berlin), **1999**.

[3] I. Altun, B. Damjanović and D. Djorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett., **23** (3)(2010), 310–316.

[4] I. Altun and C. Cevik, *Some common fixed point theorems in vector metric spaces*, Filomat, **25**(2011), 105–113.

[5] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal. Theory, Methods & Applications, **74**(18)(2011), 7347–7355.

[6] V. Berinde, M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74**(15)(2011), 4889-4897.

[7] V. Berinde, *Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal. Theory, Methods & Applications, **75** (6)(2012), 3218–3228.

[8] V Berinde and M. Păcurar, *Coupled fixed point theorems for generalized symmetric Meir–Keeler contractions in ordered metric spaces*, Fixed Point Theory Appl., **115**(2012), https://doi.org/10.1186/1687-1812-2012-115.

[9] T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. Theory, Methods & Applications, **65**(7)(2006), 1379–1393.

[10] M. Borcut, *Tripled fixed point theorems for monotone mappings in partially ordered metric spaces*, Carpathian J. Math., **28** (2)(2012), 215–222.

[11] M. Borcut, *Tripled coincidence theorems for monotone mappings in partially ordered metric spaces*, Creat. Math. Inform., **21**(2)(2012), 135–142.

[12] N. V. Can, V. Berinde, N. V. Luong, N. h. Thuan, *A coupled coincidence point theorem in partially ordered metric spaces*, Kragujevac J. Math., **37**(1)(2013), 103–119.

[13] C. Cevik, *On continuity of functions between vector metric spaces*, J. Funct. Spaces, 753969 (2014), http://dx.doi.org/10.1155/2014/753969.

[14] C. Çevik, I. Altun, H. Şahin and Ç. C. Özeken, *Some fixed point theorems for contractive mapping in ordered vector metric spaces*, J. Nonlinear Sci. Appl., **10**(4)(2017), 1424–1432.

[15] C. Cevik, C. C. Ozeken, *Coupled fixed point results for new classes of functions on ordered vector metric space*, Acta Math. Hungar, **172**(1)(2024), 1-18.

[16] C. Cevik and I. Altun, *Vector metric spaces and some properties*, Topol. Methods Nonlinear Anal., **34**(2)(2009), 375–382.

[17] E. Karapınar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl., **59** (12)(2010), 3656–3668.

[18] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, I, North-Holland (Amsterdam), **1971**.

[19] H. Long-Guang, Z. Xian, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2)(2007), 1468-1476.

[20] A. Naziri-Kordkandi, *A Class of Common Fixed Point Results in Cone Metric Spaces*, Caspian J. Math. Sci. (CJMS), **13**(1)(2024), 143-154.

[21] V. Lakshmikantham and L. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(12)(2009), 4341-4349.

[22] J. J. Nieto and R. Rodríguez-López, *Existence and uniqueness of fixed point in par tially ordered sets and applications to ordinary differential equations*, Acta Math. Sin., **23**(2007), 2205–2212.

[23] J. J. Nieto and R. Rodrıguez-Lopez, *Contractive mapping theorems in partially or dered sets and applications to ordinary differential equations*, Order, **22**(2005), 223–239.

[24] Z. Páles and I. R. Petre, *Iterative fixed point theorems in E-metric spaces*, Acta Math. Hungar., **140** (2013), 134–144.

[25] I. R. Petre, *Fixed point theorems in vector metric spaces for single-valued operators*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convex., **9** (2011), 59-80.

[26] I. R. Petre, *Fixed points for ϕ-contractions in E-Banach spaces*, Fixed Point Theory, **13**(2)(2012), 623–640.

[27] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(5)(2003), 1435–1443.

[28] H. Sahin, *Best proximity point theory on vector metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **70**(1)(2021), 130–142.

[29] T. C. Singh, *Some Theorems on Fixed Points in N-Cone Metric Spaces with Certain Contractive Conditions*, Eng. Lett., **32**(9)(2024), 1833-1839.

[30] T. C. Singh and L. S. Singh, *Tripled fixed point theorems in N-cone metric space under F-invariant set*, South East Asian J. of Math. Math. Sci., **20**(1)(2024), 155-168.

[31] F. Sabetghadam, H. P. Masiha and A. H. Sanatpour, *Some coupled fixed point theo rems in cone metric spaces*, Fixed Point Theory Appl., 125426(2009), https://doi.org/10.1155/2009/125426.