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Fixed points results for $(\theta - \phi)-$ rational type contractive condition in b-metric spaces

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Abstract

This paper investigates fixed point results for mappings that satisfying (θ-ϕ)-rational type contractive conditions in partial b-metric spaces. These spaces are more general than metric spaces because they let relaxed triangle inequalities work. We establish the existence and uniqueness of fixed points for such mappings, utilizing novel rational contractive conditions involving control functions. These results extend classical fixed-point theorems and are applicable to a broader range of spaces and mappings. The results are especially useful for mathematical analysis, optimization, and computational methods that use iterative steps. They give us new information about how dynamical systems behave in generalized metric frameworks.

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1 Introduction and definition

One of the field that many researchers are actively studying is fixed point theory, particularly in analysis. Fixed point theory can be applied in a variety of fields, including biology, computer science, engineering, economics, etc. The Banach Contraction Principle is known as one of the most important findings in fixed point theory. Over the past few years, the result is expanded in various ways. Bakhtin [3] first introduced the concept behind b-metric and broadly used by Czerwik [5, 19]. The map $b-metric$ is itself a generalization of map metric. Each and every metric is always a b-metric even though the converse is not true. Also every metric is always a partial metric even though the opposite might not be hold. S.G. Mathews [23] first defined the partial metric space. In the partial metric space, the ordinary metric is replaced by a partial metric with the characteristic that any point in this space may have a non zero self distance. Every metric space is always a partial metric space, but the opposite may not hold true. Shukla [27] first defined the partial b-metric space. Numbers of publications and references dealing with partial, rectangular b-metric space began to surface (refer to [1, 6, 9–11, 14, 15, 30]). Zead Mustafa et al. [24] altered the triangle property of partial b-metric and established a condition for convergence and some rules for working in partial b-metric space. Our findings have several implications, one of which is the generalization of similar findings in previous research (see [4, 7, 8, 16–18, 20–22, 25, 26, 28, 29]) and the citations therein.

In this paper, we establish some theorems for the existence and uniqueness of a fixed point in partial b-metric space along with the concept of $(\theta - \phi)-$ rational type Contraction mapping using auxiliary functions.

Definition 1.1. *Considering A as a set having at least one element and $d : A \times A \to [0, +\infty)$ satisfying:*

(a) $d(\hbar, \aleph) = 0 \iff \hbar = \aleph$ *for all* $\hbar, \aleph \in A$.

(b) $d(\hbar, \aleph) = d(\aleph, \hbar)$ *for all* $\hbar, \aleph \in A$.

(c) $d(\hbar, \aleph) \leq d(\hbar, \varrho) + d(\varrho, \aleph)$ *for all* $\hbar, \aleph, \varrho \in A$.

Then, the map d is called metric on A and (A, d) is a metric space.

Definition 1.2. *[3] Considering A as a set having at least one element and $d : A \times A \to [0, +\infty)$ satisfying:*

(a) $d(\hbar, \aleph) = 0 \iff \hbar = \aleph$ *for all* $\hbar, \aleph \in A$.

(b) $d(\hbar,\aleph)=d(\aleph,\hbar)$ *for all* $\hbar,\aleph\in A$.

(c) $d(\hbar,\aleph)\leq b[d(\hbar,\varrho)+d(\varrho,\aleph)]$ *for all* $\hbar,\aleph,\varrho\in A$.

Then, the map d *is called b-metric on* A *and* (A,d) *is b-metric space.*

Definition 1.3. *[23] Considering* A *as a set having at least one element and* $p:A\times A\rightarrow[0,+\infty)$ *satisfying:*

(a) $p(\hbar,\aleph)\geq 0\ for\ all\ \hbar,\aleph\in A$ *and* $p(\hbar,\aleph)=p(\hbar,\hbar)=p(\aleph,\aleph)\iff\hbar=\aleph$.

(b) $p(\hbar,\hbar)\leq p(\hbar,\aleph)\ for\ all\ \hbar,\aleph\in A$.

(c) $p(\hbar,\aleph)=p(\aleph,\hbar)\ for\ all\ \hbar,\aleph\in A$.

(d) $p(\hbar,\aleph)\leq p(\hbar,\varrho)+p(\varrho,\aleph)-p(\varrho,\varrho)\ for\ all\ \hbar,\aleph,\varrho\in A$.

Then, the map p *is called partial-metric on* A *and* (A,p) *is partial metric space.*

Definition 1.4. *[27] Considering* A *as a set having at least one element and* $p^b:A\times A\rightarrow[0,+\infty)$ *satisfying:*

(a) $p^b(\hbar,\aleph)=p^b(\hbar,\hbar)=p^b(\aleph,\aleph)\iff\hbar=\aleph\ for\ all\ \hbar,\aleph\in A$.

(b) $p^b(\hbar,\hbar)\leq p^b(\hbar,\aleph)\ for\ all\ \hbar,\aleph\in A$.

(c) $p^b(\hbar,\aleph)=p^b(\aleph,\hbar)\ for\ all\ \hbar,\aleph\in A$.

(d) $p^b(\hbar,\aleph)\leq b[p^b(\hbar,\varrho)+p^b(\varrho,\aleph)-p^b(\varrho,\varrho)]\ for\ all\ \hbar,\aleph,\varrho\in A$.

Then function p^b *is a partial b-metric on* A *and the space* (A,p^b) *is called a partial b-metric space* $(PbMS)$.

We know that if b = 1 then every partial metric space is always a partial b-metric space and also with zero self-distance and with the same coefficient, every b-metric space is always a partial b-metric space.

Example 1.1. *[27] Consider* $A=[0,+\infty),s\geq 1$, *be a set and* $p:A\times A\rightarrow[0,+\infty)$ *given by*

$$p^b(\hbar,\aleph)=\left(\max\{\hbar,\aleph\}\right)^s+\mod|\hbar-\aleph|^s\Big)\ \textit{for all}\ \hbar,\aleph\in A.$$

Then, (A,p^b) *is called partial metric space with co-efficient* $b=2^s$, *but it is neither a b- metric nor a partial metric spaces.*

Definition 1.5. *[13] Given a partial b-metric space (A, p^b). In A, let $\{\hbar_r\}$ be a sequence, and $\hbar \in A$. Then*

(a) *In (A, p^b), the sequence $\{\hbar_r\}$ is known to be convergent and converging to $\hbar$, if $p^b(\hbar_r, \hbar) < \epsilon$ for each $r > r_0$, and there exists $r_0 \in N$, for all $\epsilon > 0$ and the representation of this fact is given by $\lim_{r\to\infty} \hbar_r = \hbar$ or $\hbar_r \to \hbar$ as $r \to \infty$.*

(b) *In (A, p^b), the sequence $\{\hbar_r\}$ is known to be a Cauchy sequence if $p^b(\hbar_r, \hbar_{r+k}) < \epsilon$ for all $r > r_0, k > 0$ for every $\epsilon > 0$ and there exists $r_0 \in N$, or equivalently, if $\lim\limits_{r\to\infty} p^b(\hbar_r, \hbar_{r+k}) = 0$ for all $k > 0$.*

(c) *If every Cauchy sequence in A converges to some $\hbar \in A$, then (A, p^b) is a complete partial b-metric space.*

Proposition 1.1. *[24] In a partial b-metric space (A, p^b), let $\{\hbar_r\}$ be a Cauchy sequence with $\lim_{r\to\infty} p^b(\hbar_r, \hbar) = 0$, where $\hbar \in A$. Then, for all $z \in A$, $\lim_{r\to\infty} p^b(\hbar_r, z) = p^b(\hbar, z)$. That is, if $z \neq \hbar$, then z is not the point of convergence of $\{\hbar_r\}$.*

Definition 1.6. *[12] Given a nonempty set A and two mappings $F : A \to A$ and $\theta : A \times A \to [0, \infty)$, then the map F is a θ-admissible if $\theta(\hbar, \aleph) \geq 1 \implies \theta(F\hbar, F\aleph) \geq 1$ for every $\hbar, \aleph \in A$.*

Definition 1.7. *[12] Suppose that (A, p^b) is a partial b-metric space, and $\theta : A \times A \to [0, +\infty)$. If $\hbar_r \to \hbar$ for any sequence $\{\hbar_r\}$ in A and $\theta(\hbar_r, \hbar_{r+1}) \geq 1$, there exists a subsequence $\{\hbar_{r_k}\}$ of $\{\hbar_r\}$ satisfying $\theta(\hbar_{r_k}, \hbar) \geq 1$ for all $k \in \mathbb{N}$, then A is said to be $\theta-$ regular partial b-metric space.*

2 Main results

Using $(\theta - \phi)-$rational type contraction mapping, we show some special theorems having unique fixed point in the space (A, p^b) in our study. The auxiliary map ϕ, as defined by Hamed H Alsulami et al. [2], will be considered throughout this study. Assume that Φ is a family of mappings $\phi : [0, +\infty) \to [0, +\infty)$ that satisfy the properties given below:

(a) ϕ is strictly increasing and upper semi-continuous.

(b) For every $r>0$, $[\phi^r(r)]_{r\in\mathbb{N}}$ converging to 0 as $r\to+\infty$.

(c) For every $r>0$, $\phi(r)<r$.

Definition 2.8. *Consider (A,p^b) as $(PbMS)$ and a map $\theta: A\times A\to[0,+\infty)$. A map $F: A\to A$ is called a $(\theta-\phi)$ - rational contractive mapping of type-I if there exists a map $\phi\in\Phi$ such that for all $\hbar,\aleph\in A$, the statement given below are satisfied:*

$$\theta(\hbar,\aleph)p^b(F\hbar,F\aleph)\le\phi(M(\hbar,\aleph)),$$

where

$$\begin{aligned}M(\hbar,\aleph)=\max\Big\{&p^b(\hbar,\aleph),\frac{p^b(\hbar,\aleph)+p^b(\hbar,F\hbar)}{2}\\&\frac{p^b(\hbar,F\hbar)+p^b(\aleph,F\aleph)}{2},\frac{p^b(\hbar,F\hbar)p^b(\aleph,F\aleph)}{1+p^b(\hbar,\aleph)}\\&\frac{p^b(\hbar,F\hbar)p^b(\aleph,F\aleph)}{1+p^b(F\hbar,F\aleph)}\Big\}.\end{aligned}\tag{1}$$

Theorem 2.1. *Let (A,p^b) be a complete $(PbMS)$. Let $F: A\to A$ be a map and $\theta: A\times A\to[0,+\infty)$ be also a map. Considering the statement given below are satisfied:*

(a) F is a θ-admissible map.

(b) F is a $(\theta-\phi)$ - rational contractive mapping of type-I.

(c) There exists $\hbar_0\in A$ such that $\theta(\hbar_0,F\hbar_0)\ge 1$.

(d) Either A is θ-regular or F is continuous.

Then,it follows that F has a fixed point $\hbar^*\in A$, and $\{F^r\hbar_0\}$ converges to $\hbar^*$. Moreover, we have $\theta(\hbar,\aleph)\ge 1$, if for all $\hbar,\aleph\in F(A)$. Then $\hbar=\aleph$, i.e., it has a unique fixed point in A.

Proof. Let $\hbar_0\in A$ satisfies $\theta(\hbar_0,F\hbar_0)\ge 1$. We define a sequence $\{\hbar_r\}\in A$ as $\hbar_r=F^r\hbar_0=F\hbar_{r-1}$ for all $r\in\mathbb{N}$.

We know that $\hbar_{r_0}$ is a fixed point of F if $\hbar_{r_0}=\hbar_{r_0+1}$, for some $r_0\in\mathbb{N}$. Now we assume that $\hbar_r\ne\hbar_{r+1}\ \forall\ r\in\mathbb{N}$.

As F is θ-admissible

$$\theta(\hbar_0, F\hbar_0) = \theta(\hbar_0, \hbar_1) \geq 1,$$

implies

$$\theta(F\hbar_0, F\hbar_1) = \theta(\hbar_1, \hbar_2) \geq 1,$$

and thus,

$$\theta(F\hbar_1, F\hbar_2) = \theta(\hbar_2, \hbar_3) \geq 1,$$

$$\vdots$$

therefore

$$\theta(\hbar_r, \hbar_{r+1}) \geq 1,$$

for all $r \geq 0$. Taking $\hbar = \hbar_r, \aleph = \hbar_{r+1}$ on (1), we have

$$p^b(\hbar_{r+1}, \hbar_{r+2}) = p^b(F\hbar_r, F\hbar_{r+1}) \leq \theta(\hbar_r, \hbar_{r+1})p^b(F\hbar_r, F\hbar_{r+1}) \leq \phi(M(\hbar_r, \hbar_{r+1}),$$

where,

$$\begin{aligned} M(\hbar_r, \hbar_{r+1}) &= \max\Big\{p^b(\hbar_r, \hbar_{r+1}), \frac{p^b(\hbar_r, \hbar_{r+1}) + p^b(\hbar_r, F\hbar_r)}{2}, \\ &\quad \frac{p^b(\hbar_r, F\hbar_r) + p^b(\hbar_{r+1}, F\hbar_{r+1})}{2}, \\ &\quad \frac{p^b(\hbar_r, F\hbar_r)p^b(\hbar_{r+1}, F\hbar_{r+1})}{1 + p^b(\hbar_r, \hbar_{r+1})}, \frac{p^b(\hbar_r, F\hbar_r)p^b(\hbar_{r+1}, F\hbar_{r+1})}{1 + p^b(F\hbar_r, F\hbar_{r+1})}\Big\} \\ &= \max\Big\{p^b(\hbar_r, \hbar_{r+1}), \frac{p^b(\hbar_r, \hbar_{r+1}) + p^b(\hbar_r, \hbar_{r+1})}{2} \\ &\quad \frac{p^b(\hbar_r, \hbar_{r+1}) + p^b(\hbar_{r+1}, \hbar_{r+2})}{2}, \\ &\quad \frac{p^b(\hbar_r, \hbar_{r+1})p^b(\hbar_{r+1}, \hbar_{r+2})}{1 + p^b(\hbar_r, \hbar_{r+1})}, \frac{p^b(\hbar_r, \hbar_{r+1})p^b(\hbar_{r+1}, \hbar_{r+2})}{1 + p^b(\hbar_{r+1}, \hbar_{r+2})}\Big\} \\ &= \max\Big\{p^b(\hbar_r, \hbar_{r+1}), p^b(\hbar_{r+1}, \hbar_{r+2}), \\ &\quad \frac{p^b(\hbar_r, \hbar_{r+1}) + p^b(\hbar_{r+1}, \hbar_{r+2})}{2}\Big\}. \end{aligned} \tag{2}$$

Since,
$$\frac{p^b(\hbar_r,\hbar_{r+1})p^b(\hbar_{r+1},\hbar_{r+2})}{1+p^b(\hbar_r,\hbar_{r+1})}\leq p^b(\hbar_{r+1},\hbar_{r+2}),$$
and
$$\frac{p^b(\hbar_r,\hbar_{r+1})p^b(\hbar_{r+1},\hbar_{r+2})}{1+p^b(\hbar_{r+1},\hbar_{r+2})}\leq p^b(\hbar_r,\hbar_{r+1}).$$
If for some r, we have

$$M(\hbar_r,\hbar_{r+1})=p^b(\hbar_{r+1},\hbar_{r+2}),$$

then

$$\begin{aligned} p^b(\hbar_{r+1},\hbar_{r+2}) &\leq \phi(M(\hbar_r,\hbar_{r+1})) \\ &= \phi(p^b(\hbar_{r+1},\hbar_{r+2})) \\ &< p^b(\hbar_{r+1},\hbar_{r+2})), \end{aligned} \tag{3}$$

this is not true.

If for some r, we have

$$\begin{aligned} M(\hbar_r,\hbar_{r+1}) &= \frac{p^b(\hbar_r,\hbar_{r+1})+p^b(\hbar_r,\hbar_{r+2})}{2} \\ &\leq \max\Big\{p^b(\hbar_r,\hbar_{r+1}),p^b(\hbar_{r+1},\hbar_{r+2})\Big\}. \end{aligned}$$

If
$$\max\{p^b(\hbar_r,\hbar_{r+1}),p^b(\hbar_{r+1},\hbar_{r+2})\}=p^b(\hbar_{r+1},\hbar_{r+2}).$$
Then,

$$\begin{aligned} M(\hbar_r,\hbar_{r+1}) &\leq p^b(\hbar_{r+1},\hbar_{r+2}) \\ &\leq \phi(M(\hbar_r,\hbar_{r+1})) \\ &\leq \phi(p^b(\hbar_{r+1},\hbar_{r+2})) \\ &\leq p^b(\hbar_{r+1},\hbar_{r+2}), \end{aligned}$$

this is not true. So, we get

$$M(\hbar_r,\hbar_{r+1})=p^b(\hbar_r,\hbar_{r+1})\ for\ all\ r\in\mathbb{N}.$$

$$\begin{aligned} p^b(\hbar_{r+1}, \hbar_{r+2}) &\le \phi(M(\hbar_r, \hbar_{r+1}) \\ &= \phi(p^b(\hbar_r, \hbar_{r+1}). \end{aligned}$$

By the property (c) of ϕ, one can deduce

$$p^b(\hbar_{r+1}, \hbar_{r+2}) < p^b(\hbar_r, \hbar_{r+1}) \; for \; every \; r \in \mathbb{N}, \tag{4}$$

from the above two inequalities, we deduce that

$$p^b(\hbar_{r+1}, \hbar_{r+2}) \le \phi^r(p^b(\hbar_0, \hbar_1)) \; for \; all \; r \in \mathbb{N},$$

by the property (b) of ϕ, one can take

$$\lim_{r\to\infty} p^b(\hbar_{r+1}, \hbar_{r+2}) = 0. \tag{5}$$

Now, we will show, $\hbar_r \neq \hbar_s$, for all $r \neq s$. Presume the opposite that $\hbar_r = \hbar_s$ for some $s, r \in \mathbb{N}$ with $r \neq s$.

Since $p^b(\hbar_k, \hbar_{k+1}) > 0$, for all $k \in \mathbb{N}$, we can presume that $s > r + 1$ without lost of generality.

Again substituting $\hbar = \hbar_r = \hbar_s$ and $\aleph = \hbar_{r+1} = \hbar_{s+1}$ in (1) which gives

$$\begin{aligned} p^b(\hbar_r, \hbar_{r+1}) &= p^b(\hbar_r, F\hbar_r) = p^b(\hbar_s, F\hbar_s) = p^b(F\hbar_{s-1}, F\hbar_s) \\ &\le \theta(\hbar_{s-1}, \hbar_s) p^b(F\hbar_{s-1}, F\hbar_s) \le \phi(M(\hbar_{s-1}, \hbar_s)), \end{aligned} \tag{6}$$

where,

$$\begin{aligned} M(\hbar_{s-1}, \hbar_s) &= \max\Big\{p^b(\hbar_{s-1}, \hbar_s), \frac{p^b(\hbar_{s-1}, \hbar_s) + p^b(\hbar_{s-1}, F\hbar_{s-1})}{2} \\ &\quad \frac{p^b(\hbar_{s-1}, F\hbar_{s-1}) + p^b(\hbar_s, F\hbar_s)}{2}, \frac{p^b(\hbar_{s-1}, \hbar_{s-1}) p^b(\hbar_s, \hbar_s)}{1 + p^b(\hbar_{s-1}, \hbar_s)}, \\ &\quad \frac{p^b(\hbar_{s-1}, F\hbar_{s-1}) p^b(\hbar_s, F\hbar_s)}{1 + p^b(F\hbar_{s-1}, F\hbar_s)}\Big\} \end{aligned}$$

$$= \max\Big\{p^b(\hbar_{s-1}, \hbar_s), p^b(\hbar_s, \hbar_{s+1}), \frac{p^b(\hbar_{s-1}, \hbar_s) + p^b(\hbar_s, \hbar_{s+1})}{2}\Big\}. \tag{7}$$

If

$$M(\hbar_{s-1}, \hbar_s) = p^b(\hbar_{s-1}, \hbar_s)$$

then, (6) implies

$$p^b(\hbar_r, \hbar_{r+1}) \leq \phi(p^b(\hbar_{s-1}, x_s)) \leq \phi^{s-r}(p^b(\hbar_r, \hbar_{r+1})). \tag{8}$$

If

$$M(\hbar_{s-1}, \hbar_s) = p^b(\hbar_s, \hbar_{s+1})$$

from (6), we get

$$p^b(\hbar_r, \hbar_{r+1}) \leq \phi(p^b(\hbar_s, \hbar_{s+1}) \leq \phi^{s-r+1}(p^b(\hbar_r, \hbar_{r+1})). \tag{9}$$

If

$$\begin{aligned} M(\hbar_{s-1}, \hbar_s) &= \frac{p^b(\hbar_{s-1}, \hbar_s) + p^b(\hbar_s, \hbar_{s+1})}{2} \\ &< \max\{p^b(\hbar_{s-1}, \hbar_s), p^b(\hbar_s, \hbar_{s+1)}\}. \end{aligned}$$

If

$$max\{p^b(\hbar_{s-1}, \hbar_s), p^b(\hbar_s, \hbar_{s+1})\} = p^b(\hbar_{s-1}, \hbar_s)$$

(6) implies

$$p^b(\hbar_r, \hbar_{r+1}) \leq \phi(p^b(\hbar_{s-1}, \hbar_s)) \leq \phi^{s-r}(p^b(\hbar_r, \hbar_{r+1})). \tag{10}$$

If

$$max\{p^b(\hbar_{s-1}, \hbar_s), p^b(\hbar_s, \hbar_{s+1})\} = p^b(\hbar_s, \hbar_{s+1})$$

(6) implies

$$p^b(\hbar_r, \hbar_{r+1}) \leq \phi(p^b(\hbar_s, \hbar_{s+1})) \leq \phi^{s-r+1}(p^b(\hbar_r, \hbar_{r+1})). \tag{11}$$

By using property (c) of ϕ, the inequalities (8), (9), (10), (11) imply

$$p^b(\hbar_r, \hbar_{r+1}) < p^b(\hbar_r, \hbar_{r+1}),$$

this is not true. We now show $\{\hbar_r\}$ is a cauchy sequence, i.e.,

$$\lim_{r\to\infty} p^b(\hbar_r, \hbar_{r+k}) = 0 \text{ for all } k \in \mathbb{N}.$$

We have already shown for $k = 1$ in (5).
Take arbitrary, $k \geq 2$.

Using the property (d) of (PbMS), we have

$$\begin{aligned}
p^b(\hbar_r, \hbar_{r+k}) &= p^b(\hbar_r, \hbar_{r+2s+1}) \\
&\leq b[p^b(\hbar_r, \hbar_{r+1}) + p^b(\hbar_{r+1}, \hbar_{r+2s+1}) - p^b(\hbar_{r+1}, \hbar_{r+1})] \\
&\leq b[p^b(\hbar_r, \hbar_{r+1}) + p^b(\hbar_{r+1}, \hbar_{r+2s+1})] \\
&\leq b[p^b(\hbar_r, \hbar_{r+1})] + b[p^b(\hbar_{r+1}, \hbar_{r+2s+1})] \\
&\leq b[p^b(\hbar_r, \hbar_{r+1})] + b^2[p^b(\hbar_{r+1}, \hbar_{r+2}) \\
&\quad + p^b(\hbar_{r+2}, \hbar_{r+2s+1}) - p^b(\hbar_{r+2}, \hbar_{r+2})] \\
&\leq b[p^b(\hbar_r, \hbar_{r+1})] + b^2[p^b(\hbar_{r+1}, \hbar_{r+2}) + p^b(\hbar_{r+2}, \hbar_{r+2s+1})] \\
&\leq b[p^b(\hbar_r, \hbar_{r+1})] + b^2[p^b(\hbar_{r+1}, \hbar_{r+2})] + b^2[p^b(\hbar_{r+2}, \hbar_{r+2s+1})] \\
&\quad \vdots \\
&\leq b[p^b(\hbar_r, \hbar_{r+1})] + b^2[p^b(\hbar_{r+1}, \hbar_{r+2})] + b^3[p^b(\hbar_{r+2}, \hbar_{r+3})] \\
&\quad + b^4[p^b(\hbar_{r+3}, \hbar_{r+4})] + \cdots + b^{2s+1}[p^b(\hbar_{r+2s}, \hbar_{r+2s+1})] \\
&\leq b\phi^r p^b(\hbar_o, \hbar_1) + b^2\phi^{r+1} p^b(\hbar_o, \hbar_1) + b^3\phi^{r+2} p^b(\hbar_o, \hbar_1) \\
&\quad + b^4\phi^{r+3} p^b(\hbar_o, \hbar_1) + \cdots + b^{2s+1}\phi^{r+2s} p^b(\hbar_o, \hbar_1) \\
&\leq b\phi^r[1 + b\phi + b^2\phi^2 + b^3\phi^3 + \cdots + b^{2s}\phi^{2s}] p^b(\hbar_o, \hbar_1) \\
&\leq b\phi^r \sum_{i=0}^{i=2s} b^i\phi^i p^b(\hbar_o, \hbar_1) \to 0 \ as \ r \to \infty.
\end{aligned}$$

Thus one can say the sequence $\{\hbar_r\}$is a Cauchy in (A, p^b). As, (A, p^b) is complete, there exists $\hbar^* \in A$ having

$$\lim_{r\to\infty} p^b(\hbar_r, \hbar^*) = 0. \tag{12}$$

Next, we shall demonstrate that the limit point $\hbar^*$ of the cauchy sequence $\{\hbar_r\}$ is the fixed point of F. At first, let's assume F is continuous. Now by (12),

$$\lim_{r\to\infty} p^b(F\hbar_r, F\hbar^*) = \lim_{r\to\infty} p^b(\hbar_{r+1}, F\hbar^*) = 0.$$

We deduce from Proposition 1.1 that

$$\hbar^* = F\hbar^*.$$

This means that $\hbar^*$ is the fixed point of F.
We now consider A is $\theta-$regular. Then there exists a subsequence $\{\hbar_{r_k}\}$ of $\{\hbar_r\}$ satisfying

$$\theta(\hbar_{r_k-1}, \hbar^*) \geq 1 \text{ for all } k \in \mathbb{N}.$$

Then, by (1) with $\hbar = \hbar_{r_k}$ and $\aleph = \hbar^*$, we can say

$$\begin{aligned} p^b(\hbar_{r_k+1}, F\hbar^*) &= p^b(F\hbar_{r_k}, F\hbar^*) \\ &\leq \theta(\hbar_{r_k}, \hbar^*) p^b(F\hbar_{r_k}, F\hbar^*) \\ &\leq \phi(M(\hbar_{r_k}, \hbar^*)), \end{aligned} \tag{13}$$

where,

$$\begin{aligned} M(\hbar_{r_k}, \hbar^*) = \max\Big\{& p^b(\hbar_{r_k}, \hbar^*), \frac{p^b(\hbar_{r_k}, \hbar^*) + p^b(\hbar_{r_k}, F\hbar_{r_k})}{2}, \\ & \frac{p^b(\hbar_{r_k}, F\hbar_{r_k}), p^b(\hbar^*, F\hbar^*)}{2}, \frac{p^b(\hbar_{r_k}, F\hbar_{r_k}) p^b(\hbar^*, F\hbar^*)}{1 + p^b(\hbar_{r_k}, \hbar^*)}, \\ & \frac{p^b(\hbar_{r_k}, F\hbar_{r_k}) p^b(\hbar^*, F\hbar^*)}{1 + p^b(F\hbar_{r_k}, F\hbar^*)}\Big\} \\ = \max\Big\{& p^b(\hbar_{r_k}, \hbar^*), \frac{p^b(\hbar_{r_k}, \hbar^*), + p^b(\hbar_{r_k}, \hbar_{r_k+1})}{2}, \\ & \frac{p^b(\hbar_{r_k}, \hbar_{r_k+1}), p^b(\hbar^*, F\hbar^*)}{2}, \frac{p^b(\hbar_{r_k}, \hbar_{r_k+1}) p^b(\hbar^*, F\hbar^*)}{1 + p^b(\hbar_{r_k}, \hbar^*)}, \\ & \frac{p^b(\hbar_{r_k}, \hbar_{r_k+1}) p^b(\hbar^*, F\hbar^*)}{1 + p^b(\hbar_{r_k+1}, F\hbar^*)}\Big\}. \end{aligned} \tag{14}$$

Let $k \to \infty$ in (14), we obtain

$$M(\hbar_{r_k}, \hbar^*) = p^b(\hbar^*, F\hbar^*).$$

Thus, on taking limit as $k \to \infty$, in inequality (13), we get

$$p^b(\hbar^*, F\hbar^*) \leq \phi(p^b(\hbar^*, F\hbar^*)) < p^b(\hbar^*, F\hbar^*),$$

which implies that

$$\hbar^* = F\hbar^*.$$

This means that the fixed point of F is $\hbar^*$.
Lastly, let's consider $\hbar^*$, $\aleph^*$ are the distinct fixed points of F i.e. $\hbar^* \neq \aleph^*$. Then by the given condition, $\theta(\hbar^*, \aleph^*) \geq 1$. Thus, from (1) with $\hbar = \hbar^*$ and $\aleph = \aleph^*$, we obtain

$$\begin{aligned} p^b(\hbar^*, \aleph^*) = p^b(F\hbar^*, F\aleph^*) &< \theta(\hbar^*, \aleph^*)p^b(F\hbar^*, F\aleph^*) \\ &< \phi(M(\hbar^*, \aleph^*)). \end{aligned}$$

where

$$\begin{aligned} M(\hbar^*, \aleph^*) = \max \Big\{ & p^b(\hbar^*, \aleph^*), \frac{p^b(\hbar^*, \aleph^*) + p^b(\hbar^*, F\hbar^*)}{2}, \\ & \frac{p^b(\hbar^*, F\hbar^*) + p^b(\aleph^*, F\aleph^*)}{2}, \frac{p^b(\hbar^*, F\hbar^*)p^b(\aleph^*, F\aleph^*)}{1 + p^b(\hbar^*, \aleph^*)}, \\ & \frac{p^b(\hbar^*, F\hbar^*)p^b(\aleph^*, F\aleph^*)}{1 + p^b(F\hbar^*, F\aleph^*)} \Big\} \\ = {} & p^b(\hbar^*, \aleph^*). \end{aligned}$$

Hence, we get

$$p^b(\hbar^*, \aleph^*) \leq \phi(p^b(\hbar^*, \aleph^*)) < p^b(\hbar^*, \aleph^*),$$

it is only possible if $p^b(\hbar^*, \aleph^*) = 0$, that is, $\hbar^* = \aleph^*$. Therefore, F has a unique fixed point. □

Definition 2.9. *Consider* (A, p^b) *be a partial b-metric space (PbMS) and* $\theta : A \times A \to [0, +\infty)$. *The mapping* $F : A \to A$ *is known to be* $(\theta - \phi)$ *- rational Contractive mapping of type-II if there exists a map* $\phi \in \Phi$ *such that for all* $\hbar, \aleph \in$

A, the conditions given below holds:

$$\theta(\hbar,\aleph)p^b(F\hbar,F\aleph)\leq\phi(M(\hbar,\aleph)),$$

where

$$M(\hbar,\aleph)=\max\Big\{p^b(\hbar,\aleph),p^b(\hbar,F\hbar),p^b(\aleph,F\aleph),\frac{p^b(\hbar,F\hbar)p^b(\aleph,F\aleph)}{1+p^b(\hbar,\aleph)+p^b(\hbar,F\aleph)+p^b(\aleph,F\hbar)},\frac{p^b(\hbar,F\hbar)p^b(\aleph,F\aleph)}{1+p^b(\hbar,F\hbar)+p^b(\aleph,F\hbar)+p^b(\aleph,F\aleph)}\Big\}.$$

Theorem 2.2. *Let (A,p^b) be a complete partial b-metric space, $F:A\to A$ be a map, and $\theta:A\times A\to[0,+\infty)$. Presume that the statement given below are hold:*

(a) The map F is $\theta-$ admissible self map.

(b) F is a $(\theta-\phi)$ - rational contractive mapping of type-II.

(c) For all $\hbar_0\in A$, $\theta(\hbar_0,F\hbar_0)\geq 1$.

(d) Either A is θ-regular or F is continuous.

Consequently, F has a fixed point. $(F^r\hbar_0)$ converges to $\hbar^*$ and $\hbar^*\in A$. Furthermore, if we have $\theta(\hbar,\aleph)\geq 1$ for all $\hbar,\aleph\in F(A)$, then $\hbar=\aleph$, i.e., it has unique fixed point in A.

Proof. The lines of the proof of Theorem 2.1 can be followed to complete the proof. □

Example 2.2. *Let $A=[0,+\infty)$ be endowed with the partial b-metric space with $b=2$.*

$$P_b(\hbar,\aleph)=\begin{cases}(\hbar+\aleph)^2, & if\ \hbar\neq\aleph\\ 0, & if\ \hbar=\aleph.\end{cases}$$

$$F(\hbar)=\begin{cases}\frac{1}{4}\hbar^2, & if\ \hbar\in[0,1)\\ \frac{1}{4}\hbar, & if\ \hbar\in[1,2)\\ \frac{1}{16}, & if\ \hbar\in[2,\infty).\end{cases}$$

Also define

$$\phi(t) = \frac{t}{2},$$

and

$$\theta(\hbar, \aleph) = 1 \textit{ for all } \hbar, \aleph \in A.$$

Clearly (A, p^b) *is a partial b-metric space and it is complete also.*

Now, Considering the following cases:

Case 1: Let $\hbar, \aleph \in [0, 1)$. *Therefore,*

$$\theta(\hbar, \aleph)p^b(F\hbar, F\aleph) = ((\frac{1}{4}\hbar^2 + \frac{1}{4}\aleph^2)^2 = \frac{1}{16}(\hbar^2 + \aleph^2)^2$$

$$\leq \frac{1}{2}(\hbar + \aleph)^2 = \frac{1}{2}p^b(\hbar, \aleph) = \phi p^b(\hbar, \aleph) \leq \phi(M(\hbar, \aleph)).$$

Case 2: Let $\hbar, \aleph \in [1, 2)$ *with* $\hbar \leq \aleph$. *Then,*

$$\theta(\hbar, \aleph)p^b(F\hbar, F\aleph) = ((\frac{1}{4}\hbar + \frac{1}{4}\aleph)^2 = \frac{1}{16}(\hbar + \aleph)^2$$

$$\leq \frac{1}{2}(\hbar + \aleph)^2 = \frac{1}{2}p^b(\hbar, \aleph) = \phi p^b(\hbar, \aleph) \leq \phi(M(\hbar, \aleph)).$$

Case 3: Let $\hbar, \aleph \in [2, \infty)$ *with* $\hbar \leq \aleph$. *Then,*

$$\theta(\hbar, \aleph)p^b(F\hbar, F\aleph) = ((\frac{1}{16} + \frac{1}{16})^2 = \frac{1}{64}$$

$$\leq \frac{1}{2}(\frac{1}{4}(1 + 1)^2) = \frac{1}{2}p^b(\hbar, \aleph) = \phi p^b(\hbar, \aleph) \leq \phi(M(\hbar, \aleph)).$$

Case 4: Let $\hbar \in [0, 1)$ *and* $\aleph \in [1, 2)$. *Clearly* $\hbar \leq \aleph$. *Then,*

$$\theta(\hbar, \aleph)p^b(F\hbar, F\aleph) = (\frac{1}{4}\hbar^2 + \frac{1}{4}\aleph)^2 = \frac{1}{16}(\hbar^2 + \aleph)^2$$

$$\leq \frac{1}{2}(\hbar + \aleph)^2 = \frac{1}{2}p^b(\hbar, \aleph) = \phi p^b(\hbar, \aleph) \leq \phi(M(\hbar, \aleph)).$$

Case 5: Let $\hbar \in [0, 1)$ *and* $\aleph \in [2, \infty)$. *Clearly* $\hbar \leq \aleph$. *Then,*

$$\theta(\hbar,\aleph)p^b(F\hbar,F\aleph)=(\frac{1}{4}\hbar^2+\frac{1}{16})^2\leq(\frac{1}{4}\hbar+\frac{1}{16})^2\leq\frac{1}{4}\hbar+\frac{1}{4}\aleph)^2=\frac{1}{16}(\hbar+\aleph)^2$$

$$\leq\frac{1}{2}(\hbar+\aleph)^2=\frac{1}{2}p^b(\hbar,\aleph)=\phi p^b(\hbar,\aleph)\leq\phi(M(\hbar,\aleph)).$$

Case 6: Let $\hbar\in[1,2)$ *and* $\aleph\in[2,\infty)$. *Clearly* $\hbar\leq\aleph$. *Then,*

$$\theta(\hbar,\aleph)p^b(F\hbar,F\aleph)=(\frac{1}{4}\hbar+\frac{1}{16})^2\leq(\frac{1}{4}\hbar+\frac{1}{4})^2\leq\frac{1}{4}\hbar+\frac{1}{4}\aleph)^2=\frac{1}{16}(\hbar+\aleph)^2$$

$$\leq\frac{1}{2}(\hbar+\aleph)^2=\frac{1}{2}p^b(\hbar,\aleph)=\phi p^b(\hbar,\aleph)\leq\phi(M(\hbar,\aleph))$$

Therefore,

$$\theta(\hbar,\aleph)p^b(F\hbar,F\aleph)\leq\phi(M(\hbar,\aleph))\ \ \forall\hbar,\aleph\in A\ with\ \hbar\leq\aleph.$$

Therefore, all the criteria mention in Theorem 2.1 are satisfied and hence there unique fixed point of F.

3 Conclusion

To sum up, on utilizing $(\theta-\phi)-$ rational type contractive mappings, we bridged the gap between traditional partial b-metric and metric spaces, allowing for a broader class of spaces where fixed point theorems hold. These results have significant implications in mathematical analysis and its applications, providing a foundation for further studies in spaces that exhibit both partial metrics and more generalized contraction conditions. The extended framework ensures that fixed point results can be applied to a wider range of problems, such as in optimization and computational mathematics, where the traditional assumptions of a complete metric space may not hold. Overall, these advancements provide valuable insights and tools for addressing complex mathematical models.

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References

[1] H. Aydi, A. Felhi and S. Sahim, *Common fixed points in rectangular b-metric space using (E. A) property*, J. Adv. Math. Stud., **8**(2)(2015), 159-169.

[2] H. H. Alsulami, S. Chandok, M.-A. Taoudi, I. M. Erhan, *Some fixed point theorem for (α, ψ)-rational type contractive mappings*, Fixed Point Theory Appl., **2015**, DOI 10.1186/s13663-015-0332-3.

[3] I. A. Bakhtin, *The contraction principle in Quasi-metric spaces*, Funct. Anal., **30**(1989), 26-37.

[4] A. Branciari (Macerata), *A fixed point theorrem of Banach-Cassioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57**(2000), 31-37.

[5] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta. Math. Inform. Univ. Ostrav., **1**(1)(1993), 5-11.

[6] S. Czerwik, *Nonlinear set valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ. Modena, **46**(2)(1998), 263-276.

[7] S. Chandok, D. Kumar, *Some common Fixed point results for rational type contraction mappings in complex valued metric spaces*, Journal of Operators, **2013**.

[8] D. Dhamodharan, Y. Roben, A. H. Ansari, *Fixed point theorems of C-class functions in S_b metric space*, Res. Fixed Point Theory Appl., **2018**.

[9] D. Dolicanin–Dekic, *On some ciric type results in partial b-metric spaces*, Filomat, **31**(11)(2017), 3473-3481.

[10] H. -S. Ding, V. Ozturk, S. Radenovic, *On some new fixed point results in b-rectangular metric spaces*, Arab J. Math Sci., **8**(2015), 378-386.

[11] H. -S. Ding, M. Imdad, S. Radenovic, J. Vujakovic, *On some fixed point results in b-metric, rectangular and b-rectangular metric spaces*, Arab J. Math. Sci., **22**(2016), 151-164.

[12] L. P. Devi, L. S. Singh, *Fixed Point theorem of* (α, ψ)*- Rational type Contractive mappings in Rectangular b-metric space*, Electron. J. Math. Anal. Appl., **9**(1)(2021), 226-238.

[13] L. P. Devi, L. S. Singh, *Fixed point theorems for* (α, ψ)*-Rational type contractions in generalised parametric* b*-metric space*, J. Adv, Math. Stud., **14**(2)(2021), 169-179.

[14] N. V. Dung, V. T. L. Hang, *Remarks on partial b-metric spaces and fixed-point theorems*, Mat. Vesnik, **69**(4)(2017), 231-240.

[15] R. George, S. Radenovic, K. P. Reshma, S. Shukla, *Rectangular b-metric splaces and contraction principles*, J. Nonlinear Sci. Appl., **8**(2015), 1005-1013.

[16] M. M. Hussein, M. G. Bin-Saad, A. A. Al-Sayad, *Fixed point Results of Rational type - contraction mappings in b-metric spaces with an application*, Earthline J. Math. Sci., **12**(2)(2023), 141-164.

[17] B. Khomdram, Y. Rohen, T. C. Singh, *Couple fixed point theorems in* G_b *-metric spaces satisfying some rational contractive conditions*, Springer Plus, **5**(2016), 1261.

[18] B. Khomdram, Y. Rohen, *Quadruple common fixed point theorems in* G_b*-metric spaces*, Int. J. Pure Appl. Math., **109**(2)(2016), 279-293.

[19] B. Khomdram, Y. Rohen, *Some common Couple Fixed Point Theorems in* S_b*-metric Spaces*, Fasc. Math., **60**(1)(2018), 79-92.

[20] E. Karapınar, A. Fulga, R. P. Agarwal, *A survey: F-contractions with related fixed poiny results*, J. Fixed Point Theory Appl., **2020**.

[21] M. S. Khan, N. Priyobarta, Y. Rohen, *Fixed points of generalised rational* (α^*, ψ)*-Geraghty contraction for multivalued mappings*, J. Adv. Math, Stud., **12**(2)(2019), 156-169.

[22] X. Liua, S. Changb, Y. Xiaoa, L. Zhaoa, *Some fixed point theorems concerning* (ψ, ϕ)*-type contraction in complete metric spaces*, J. Nonlinear Sci. Appl., **9**(2016), 4127-4136.

[23] S. G. Mathews, *Partial Metric Topology*, Ann. N. Y. Acad. Sci., **728**(1994), 183-197.

[24] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Some common fixed point results in ordered partial b-metric spaces*, J. Inequal. Appl., **2013**,

[25] N. Mlaiki, Y. Rohen, *Some couple fixed point theorems in partially ordered A_b metric space*, J. Nonlinear Sci. Appl., **10**(2017), 1731-1743.

[26] M. Nazam, M. Arshad, M. Abbas, *Existence of common fixed points of improved F- contractions on partial metric spaces*, Appl. Gen. Topol., **18**(2)(2017), 277-287.

[27] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math., **11**(2)(2014), 703-711.

[28] N. Priyobarta, Y. Rohen, N. Mlaiki, *Complex valued S_b Metric Spaces*, J. Math. Anal., **8**(3)(2017), 13-24.

[29] T. C. Singh, Y. R. Singh, K. A. Singh, *Fixed point theorems of weakly compatible mappings in b_2 - metric space satisfying (ϕ, ψ) - contractive conditions*, J. Math. Comput. Sci., **10**(4)(2020), 1150-1166.

[30] D. Singh, V. Chauhan, I. Altun, *Common fixed point of a power graphic (F; psi)-contraction pair on partial b-metric spaces with application*, Nonlinear Analysis: Model. Control, **22**(5)(2017), 662-678.

[31] L. Shanjit, Y. Rohen, *Best Proximity Point theorems in b-metric space satisfying rational contraction*, J. Nonlinear Sci. Appl., **2019**(2), doi:10.5899/2019/jnaa-00408.