The Aligarh Bulletin of Mathematics
Volume 44, Number 1 (2025), 57 - 65

ISSN: 0304-9787

Fuzzy relation and its max-product composition

Kshetrimayum Mangijaobi Devi

Department of Mathematics
Waikhom Mani Girls' College, Thoubal-795138
Manipur, India
Email: mangijaobiksh@gmail.com

(Received: March 13, 2025 Accepted: May 29, 2025)

Abstract

In this paper, we define the concepts of fuzzy sets, fuzzy relations and related definitions of max-product composition. We introduce associative properties in max-product composition of three binary fuzzy relations. Further, we present the reversal law of two binary fuzzy relations. An example is also provided to illustrate by using a 3×3 order of matrix.

1 Introduction

Fuzzy relations extend classical relations in mathematics, allowing for partial membership of elements, rather than a binary membership. In classical set theory, an element either belongs to a set or it does not (crisp relations), but fuzzy relations deal with situations where elements can have degrees of membership ranging between 0 and 1. This concept is crucial for modeling and reasoning in scenarios where boundaries are not clearly defined or where there is ambiguity, vagueness, or uncertainty.

Zadeh [13] first introduced the concept of fuzzy relations. It is not only a significant part of fuzzy set theory but also plays a vital role in image processing, control systems, artificial intelligence, data mining, machine learning and decision-making process [14]. Many researchers are stimulating these applications to set up diverse fuzzy relations and to fulfill various requirements in practice. Therefore,

Keywords and phrases: Fuzzy sets, fuzzy relation, max-product composition.
2020 AMS Subject Classification: 03E72, 03B52.

it is considered one of the most interesting topics for introducing the properties of fuzzy relations, both from a theoretical perspective and their applications.

In 2012, Dorugade et al. [2] presented a sample approach to predict life of a component indirectly from vague information investigating the compositional rule of inference of fuzzy relation and also a MATLAB program has been developed for computation of success degrees of outcome. Gowrishankar et al. [3] also introduced the properties of composition of fuzzy relations.

Also, the paper verifies the properties of composition of fuzzy relations using some numerical values of 2×2 order of matrix. Shakhatreh and Qawasmeh [11] presented several concepts and definitions related to the max-min composition of fuzzy relations. They also proved max-min composition of three fuzzy relations by using associativity. Kumar and Gangwal [4] examined the concept of fuzzy relations in the context of medical diagnosis and they introduced its application to such problems by extending the Sanchez's approach. Lee and Hur [6] introduced the concepts of a bipolar fuzzy reflexive, symmetric and transitive relation. Panda and Jena [8] discussed about some linguistic relation which across some definitions and established some operations, properties and composition of fuzzy relations. Latha Devi and Velammal [5] construct an appropriate mathematical model utilising Z-Fuzzy Subset (ZFS) which can be implemented as a computer aided medical diagnosis system to deal with application of Z-fuzzy relationship to medical diagnosis. Bezdek and Harris [1] explored some relationship among fuzzy partitions and similarity relations. Also finally they showed that every fuzzy c-partition of a finite data set produces a psuedo-metric of the type described. Zimmermann [15] described the basic mathematical framework of fuzzy set theory as well as the most important applications of this theory to other theories and techniques. Shakhatreh and Hayajneh [10] introduced a certain collection of fuzzy subsets of a nonempty set X and gives raise to a useful new fuzzy partition. This new fuzzy partition gives us a way of generating a fuzzy equivalence relation as in the ordinary case. Zadeh [12] defined the fuzzy sets. Further, proved a separation theorem for convex fuzzy sets without requiring that the fuzzy sets be disjoint. Li and Zhang [7] first characterized aggregation functions to ensure preserve properties of I-transitive and two aggregation functions of fuzzy preference relations. Focusing on T-S consistency, further, they investigate how aggregation functions maintain T-S consistent fuzzy preference relations. In this paper, we propose the max-product of three binary fuzzy relations by using the associative property. Further, we present the reversal law of two binary fuzzy relations. We also remind the previous definitions of fuzzy sets and fuzzy relations. In the last part of this paper, an example is also provided to illustrate by using a 3×3 order of matrix.

2 Preliminaries

First, we will discuss some definitions and preliminaries concepts related to fuzzy sets, fuzzy relations and max-product composition.

Definition 2.1. *[9] If U is a universe of discourse and u be any particular element of U, then a fuzzy set $\tilde{P}$ defined on U may be written as a collection of ordered pairs $\tilde{P} = \{(u, \mu_{\tilde{P}}(u)) : u \in U\}$ where, each pair $(u, \mu_{\tilde{P}}(u))$ is called a singleton.*

Definition 2.2. *[11] Let $\tilde{P}$ and $\tilde{Q}$ be two fuzzy sets in $X \neq \phi$ is said to be equality of two sets iff $\mu_{\tilde{P}}(x) = \mu_{\tilde{Q}}(x)$ for all $x \in X$.*

Definition 2.3. *[11] Let P and Q be nonempty sets. Then, $\tilde{R} = \{((p, q), \mu_{\tilde{R}}(p, q)) : (p, q) \in P \times Q\}$ is called a fuzzy relations in $P \times Q$.*

Definition 2.4. *[9] Let $\tilde{P}_1$ and $\tilde{P}_2$ be two fuzzy relations on (U, V) and (V, W), respectively. Then, the max-product composition is denoted as $\tilde{P}_1 \circ \tilde{P}_2$ is defined as*

$$\tilde{P}_1 \circ \tilde{P}_2(u, w) = \{[(u, w), max\{\mu_{\tilde{P}_1}(u, v) \cdot \mu_{\tilde{P}_2}(v, w)\}]\} \text{ for all } u \in U, v \in V, w \in W.$$

3 Main results

We now proceed to prove the following theorems.

Theorem 3.1. *Let $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$ be a fuzzy relation in $T \times U$, $U \times V$ and $V \times W$. Then, $\tilde{P}_1 \circ (\tilde{P}_2 \circ \tilde{P}_3) = (\tilde{P}_1 \circ \tilde{P}_2) \circ \tilde{P}_3$.*

Proof. Here we need to prove that

$$\mu_{\tilde{P}_1 \circ (\tilde{P}_2 \circ \tilde{P}_3)}(t, u, v, w) = \mu_{(\tilde{P}_1 \circ \tilde{P}_2) \circ \tilde{P}_3}(t, u, v, w) \text{ for all } t \in T, u \in U, v \in V, w \in W.$$

We have

$$\begin{aligned}
\mu_{\tilde{P}_1 \circ \tilde{P}_2}(t, v) &= \{[(t, v), max\{\mu_{\tilde{P}_1}(t, u) \cdot \mu_{\tilde{P}_2}(u, v)\}]\}, \text{ for all } t \in T, u \in U, v \in V \\
\mu_{\tilde{P}_2 \circ \tilde{P}_3}(u, w) &= \{[(u, w), max\{\mu_{\tilde{P}_2}(u, v) \cdot \mu_{\tilde{P}_3}(v, w)\}]\}, \text{ for all } u \in U, v \in V, w \in W \\
\mu_{\tilde{P}_1 \circ (\tilde{P}_2 \circ \tilde{P}_3)}(t, w) &= \{[(t, w), max\{\mu_{\tilde{P}_1}(t, u) \cdot \mu_{\tilde{P}_2 \circ \tilde{P}_3}(u, w)\}]\} \\
&= \{[(t, w), max\{\mu_{\tilde{P}_1}(t, u) \cdot \{[(u, w), max\{\mu_{\tilde{P}_2}(u, v) \cdot \mu_{\tilde{P}_3}(v, w)\}]\}\}]\} \\
&\quad \text{for all } t \in T, u \in U, v \in V, w \in W.
\end{aligned}$$

Now, suppose $\mu_{\tilde{P}_1}(t,u) = p$, $\mu_{\tilde{P}_2}(u,v) = q$, $\mu_{\tilde{P}_3}(v,w) = r$.

$$\begin{aligned} \mu_{\tilde{P}_1 \dot{o} (\tilde{P}_2 \dot{o} \tilde{P}_3)}(t,w) &= max\{p, max\{q,r\}\} \\ &= max\{p,q,r\}. \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} \mu_{(\tilde{P}_1 \dot{o} \tilde{P}_2) \dot{o} \tilde{P}_3}(t,w) &= \{[(t,w), max\{\mu_{\tilde{P}_1 \dot{o} \tilde{P}_2}(t,v) \cdot \mu_{\tilde{P}_3}(v,w)\}]\} \\ &= \{[(t,w), max\{[(t,v), max\{\mu_{\tilde{P}_1}(t,u) \cdot \mu_{\tilde{P}_2}(u,v)\}]\} \cdot \mu_{\tilde{P}_3}(v,w)]\} \\ &\quad \text{for all } t \in T, u \in U, v \in V, w \in W. \end{aligned}$$

Now, suppose $\mu_{\tilde{P}_1}(t,u) = p$, $\mu_{\tilde{P}_2}(u,v) = q$, $\mu_{\tilde{P}_3}(v,w) = r$.

$$\begin{aligned} \mu_{(\tilde{P}_1 \dot{o} \tilde{P}_2) \dot{o} \tilde{P}_3}(t,w) &= max\{max\{p,q\}, r\} \\ &= max\{p,q,r\}. \end{aligned} \tag{3.2}$$

From Equations (3.1) and (3.2), we have

$$\mu_{\tilde{P}_1 \dot{o} (\tilde{P}_2 \dot{o} \tilde{P}_3)}(t,w) = \mu_{(\tilde{P}_1 \dot{o} \tilde{P}_2) \dot{o} \tilde{P}_3}(t,w).$$

Hence,

$$\tilde{P}_1 \dot{o} (\tilde{P}_2 \dot{o} \tilde{P}_3) = (\tilde{P}_1 \dot{o} \tilde{P}_2) \dot{o} \tilde{P}_3$$

. □

Example 3.1. *Let* $\tilde{P}_1 = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{bmatrix} 1 & .7 & .3 \\ .4 & 1 & .8 \\ .7 & .5 & 1 \end{bmatrix} \end{array}$, $\tilde{P}_2 = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} .7 & 0 & 1 \\ .4 & .5 & 0 \\ .7 & .5 & 1 \end{bmatrix} \end{array}$,

and $\tilde{P}_3 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \begin{bmatrix} .7 & .5 & .3 \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix} \end{array}$. *Then, prove that* $\tilde{P}_1 \dot{o} (\tilde{P}_2 \dot{o} \tilde{P}_3) = (\tilde{P}_1 \dot{o} \tilde{P}_2) \dot{o} \tilde{P}_3$.

Proof. We have to prove that

$\mu_{\tilde{P}_1 \underset{\cdot}{o} (\tilde{P}_2 \underset{\cdot}{o} \tilde{P}_3)}(t,u,v,w) = \mu_{(\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2) \underset{\cdot}{o} \tilde{P}_3}(t,u,v,w)$ for all $t \in T, u \in U, v \in V, w \in W$.

Then by using max-product of fuzzy relation, we have,

$$\tilde{P}_2 \underset{\cdot}{o} \tilde{P}_3 = \begin{matrix} \\ u_1 \\ u_2 \\ u_3 \end{matrix} \begin{matrix} v_1 \quad v_2 \quad v_3 \\ \begin{bmatrix} .7 & 0 & 1 \\ .4 & .5 & 0 \\ .7 & .5 & 1 \end{bmatrix} \end{matrix} \underset{\cdot}{o} \begin{matrix} \\ v_1 \\ v_2 \\ v_3 \end{matrix} \begin{matrix} w_1 \quad w_2 \quad w_3 \\ \begin{bmatrix} .7 & .5 & .3 \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix} \end{matrix}.$$

$$Thus, \tilde{P}_2 \underset{\cdot}{o} \tilde{P}_3 = \begin{matrix} \\ u_1 \\ u_2 \\ u_3 \end{matrix} \begin{matrix} w_1 \quad w_2 \quad w_3 \\ \begin{bmatrix} .49 & .8 & .4 \\ .28 & .30 & .4 \\ .49 & .8 & .4 \end{bmatrix} \end{matrix},$$

$$\tilde{P}_1 \underset{\cdot}{o} (\tilde{P}_2 \underset{\cdot}{o} \tilde{P}_3) = \begin{matrix} \\ t_1 \\ t_2 \\ t_3 \end{matrix} \begin{matrix} u_1 \quad u_2 \quad u_3 \\ \begin{bmatrix} 1 & .7 & .3 \\ .4 & 1 & .8 \\ .7 & .5 & 1 \end{bmatrix} \end{matrix} \underset{\cdot}{o} \begin{matrix} \\ u_1 \\ u_2 \\ u_3 \end{matrix} \begin{matrix} w_1 \quad w_2 \quad w_3 \\ \begin{bmatrix} .49 & .8 & .4 \\ .28 & .3 & .4 \\ .49 & .8 & .4 \end{bmatrix} \end{matrix}.$$

$$Thus, \tilde{P}_1 \underset{\cdot}{o} (\tilde{P}_2 \underset{\cdot}{o} \tilde{P}_3) = \begin{matrix} \\ t_1 \\ t_2 \\ t_3 \end{matrix} \begin{matrix} w_1 \quad w_2 \quad w_3 \\ \begin{bmatrix} .49 & .8 & .4 \\ .392 & .64 & .4 \\ .49 & .8 & .4 \end{bmatrix} \end{matrix}. \tag{3.3}$$

And

$$\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2 = \begin{matrix} \\ t_1 \\ t_2 \\ t_3 \end{matrix} \begin{matrix} u_1 \quad u_2 \quad u_3 \\ \begin{bmatrix} 1 & .7 & .3 \\ .4 & 1 & .8 \\ .7 & .5 & 1 \end{bmatrix} \end{matrix} \underset{\cdot}{o} \begin{matrix} \\ u_1 \\ u_2 \\ u_3 \end{matrix} \begin{matrix} v_1 \quad v_2 \quad v_3 \\ \begin{bmatrix} .7 & 0 & 1 \\ .4 & .5 & 0 \\ .7 & .5 & 1 \end{bmatrix} \end{matrix}.$$

$$\text{Thus, } \tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2 = \begin{matrix} & v_1 & v_2 & v_3 \\ t_1 \\ t_2 \\ t_3 \end{matrix} \begin{bmatrix} .7 & .35 & 1 \\ .56 & .5 & .8 \\ .7 & .5 & 1 \end{bmatrix},$$

$$(\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2) \underset{\cdot}{o} \tilde{P}_3 = \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \overset{\begin{matrix} v_1 & v_2 & v_3 \end{matrix}}{\begin{bmatrix} .7 & .35 & 1 \\ .56 & .5 & .8 \\ .7 & .5 & 1 \end{bmatrix}} \underset{\cdot}{o} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \overset{\begin{matrix} w_1 & w_2 & w_3 \end{matrix}}{\begin{bmatrix} .7 & .5 & .3 \\ .5 & .6 & .8 \\ .3 & .8 & .4 \end{bmatrix}}.$$

$$Thus, (\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2) \underset{\cdot}{o} \tilde{P}_3 = \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \overset{\begin{matrix} w_1 & w_2 & w_3 \end{matrix}}{\begin{bmatrix} .49 & .8 & .4 \\ .392 & .64 & .4 \\ .49 & .8 & .4 \end{bmatrix}}. \tag{3.4}$$

Hence, from Equations (3.3) and (3.4), we get

$$\tilde{P}_1 \underset{\cdot}{o} (\tilde{P}_2 \underset{\cdot}{o} \tilde{P}_3) = (\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2) \underset{\cdot}{o} \tilde{P}_3 \qquad \square$$

Theorem 3.2. *Let $\tilde{P}_1$ and $\tilde{P}_2$ be two fuzzy relation in $T \times U$, $U \times V$ respectively. Then, $[\tilde{P}_1(T,U) \underset{\cdot}{o} \tilde{P}_2(U,V)]^{-1} = \tilde{P}_2^{-1}(V,U) \underset{\cdot}{o} \tilde{P}_1^{-1}(U,T)$.*

Proof. We need to prove that

$$\mu_{[\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2]^{-1}}(v,t) = \mu_{[\tilde{P}_2]^{-1} \underset{\cdot}{o} [\tilde{P}_1]^{-1}}(v,t) \text{ for all } t \in T, u \in U, v \in V.$$

We have

$$\mu_{\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2}(t,v) = \{[(t,v), max\{\mu_{\tilde{P}_1}(t,u) \cdot \mu_{\tilde{P}_2}(u,v)\}]\} \text{ for all } t \in T, u \in U, v \in V$$

$$\begin{aligned} \mu_{[\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2]^{-1}}(v,t) &= \mu_{\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2}(t,v) \\ &= \{[(t,v), max\{\mu_{\tilde{P}_1}(t,u) \cdot \mu_{\tilde{P}_2}(u,v)\}]\} \text{ for all } t \in T, u \in U, v \in V \end{aligned} \tag{3.5}$$

$$\begin{aligned}\mu_{\tilde{P}_2^{-1}}(v,u) \underset{\cdot}{o} \mu_{\tilde{P}_1^{-1}}(u,t) &= max\{\mu_{\tilde{P}_2^{-1}}(v,u).\mu_{\tilde{P}_1^{-1}}(u,t)\} \\ &= max\{\mu_{\tilde{P}_2}(u,v).\mu_{\tilde{P}_1}(t,u)\} \quad [\because \mu_{\tilde{P}_1^{-1}}(u,t) = \mu_{\tilde{P}_1}(t,u)].\end{aligned} \tag{3.6}$$

From Equations (3.5) and (3.6), we get

Hence, $[\tilde{P}_1(T,U) \underset{\cdot}{o} \tilde{P}_2(U,V)]^{-1} = \tilde{P}_2^{-1}(V,U) \underset{\cdot}{o} \tilde{P}_1^{-1}(U,T)$. □

Example 3.2. *Let* $\tilde{P}_1 = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & .7 & .3 \\ .4 & 1 & .8 \\ .7 & .5 & 1 \end{bmatrix} \end{array}$ *and* $\tilde{P}_2 = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} .7 & 0 & 1 \\ .4 & .5 & 0 \\ .7 & .5 & 1 \end{bmatrix} \end{array}$.

Then, prove that $[\tilde{P}_1(T,U) \underset{\cdot}{o} \tilde{P}_2(U,V)]^{-1} = \tilde{P}_2^{-1}(V,U) \underset{\cdot}{o} \tilde{P}_1^{-1}(U,T)$.

Proof. We have to prove that

$$\mu_{[\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2]^{-1}}(v,t) = \mu_{[\tilde{P}_2]^{-1} \underset{\cdot}{o} [\tilde{P}_1]^{-1}}(v,t) \text{ for all } t \in T, u \in U, v \in V.$$

Then, by using max-product of fuzzy relation, we have

$$\tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2 = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & .7 & .3 \\ .4 & 1 & .8 \\ .7 & .5 & 1 \end{bmatrix} \end{array} \underset{\cdot}{o} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} .7 & 0 & 1 \\ .4 & .5 & 0 \\ .7 & .5 & 1 \end{bmatrix} \end{array}.$$

$$\text{Thus, } \tilde{P}_1 \underset{\cdot}{o} \tilde{P}_2 = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} .7 & .35 & 1 \\ .56 & .5 & .8 \\ .7 & .5 & 1 \end{bmatrix} \end{array}.$$

$$\text{Therefore, } [\tilde{P}_1 \underset{\cdot}{\circ} \tilde{P}_2]^{-1} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} .7 & .56 & .7 \\ .35 & .5 & .5 \\ 1 & .8 & 1 \end{bmatrix} \end{array}. \tag{3.7}$$

$$[\tilde{P}_2]^{-1} \underset{\cdot}{\circ} [\tilde{P}_1]^{-1} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & 3 \end{array} \\ \begin{bmatrix} .7 & .4 & .7 \\ 0 & .5 & .5 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \underset{\cdot}{\circ} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 1 & .4 & .7 \\ .7 & 1 & .5 \\ .3 & .8 & 1 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} .7 & .56 & .7 \\ .35 & .5 & .5 \\ 1 & .8 & 1 \end{bmatrix} \end{array}. \tag{3.8}$$

Hence, from Equations (3.7) and (3.8), we get

$$[\tilde{P}_1 \underset{\cdot}{\circ} \tilde{P}_2]^{-1} = [\tilde{P}_2]^{-1} \underset{\cdot}{\circ} [\tilde{P}_1]^{-1}. \qquad \square$$

4 Conclusion

We proposed the max-product of three binary fuzzy relations by using the associative property and proved Theorem 3.1. Also, we present the reversal law of two binary fuzzy relations and proved Theorem 3.2. Examples are also provided to illustrate our results.

Acknowledgment
The author would like to thank the referee for his valuable suggestions, which greatly improved this paper.

References

[1] J. C. Bezdek, J. D. Harris, *Fuzzy partitions and relations; an axiomatic basis for clustering*, Fuzzy Sets Systems, **2** (1978), 111-127.

[2] N. H. Dorugade, G. S. Nhivekar, R. R. Mudholkar, *Application of fuzzy composition for electronics component*, IJAERS, **1**(2) (2012), 138-141.

[3] C. Gowrishankar, R. Dharshinee, K. Geetha, *Properties of composition of fuzzy relations and its verifications*, Int. J. Manage. Humanit., **4**(5)(2020).

[4] S. Kumar, C. Gangwal, *A study of fuzzy relation of its application in medical diagnosis*, Asian Res. J. Math., **17**(4)(2021), 6-11.

[5] J. A. L. Devi, G. Velammal, *Application of Z-fuzzy relationship to medical diagnosis*, Afr. J. Bial. Sci, **6**(9)(2024).

[6] J. G. Lee, K. Hur, *Bipolar fuzzy relations*, Mathematics, **7**(2019),1044.

[7] D. Li, Y. Zhang, *Aggregation of I-transitive fuzzy relations and fuzzy preference relations*, Comput. Appl. Math., **43**(208)(2024).

[8] G. K. Panda, S. P. Jena, *On the theory of fuzzy relations*, Int. J. Processes Eng. Human., **2**(4)(2016).

[9] S. K. Pundir, R. Pundir, *Fuzzy sets and their applications*, Pragati Prakashan tenth edition, (2022).

[10] M. A. Shakhatreh, M. A. Hayajneh, *SH fuzzy partition and fuzzy equivalence relation*, Int. J. Contemp. Math. Sciences, **5**(2010), 381-392.

[11] M. A. Shakhatreh, T. A. Qawasmeh, *Associativity of max-min composition of three fuzzy relations*, Ital. J. Pure Appl. Math., **44**(2020), 224-228.

[12] L. A. Zadeh, *Information and control*, Fuzzy Sets, **8**(1965), 338-353.

[13] L. A. Zadeh, *Similarity relations and fuzzy orderings,* Inform. Sci., **3**(1971), 177-200.

[14] L. A. Zadeh, *Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic*, Fuzzy Sets Systems, **90**(1997), 111-127.

[15] H. J. Zimmermann, *Fuzzy set theory*, Wiley Interdisip. Rev. Comput. Stat., **2** (2010), 317-332.