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On some fixed point results in E-fuzzy cone metric spaces

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Abstract

The contraction requirement in the Banach contraction principle requires that a function be continuous. Numerous authors circumvent this obligation and attenuate the hypotheses using metric spaces equipped with a partial order. This work presents many tripled fixed point theorems for functions exhibiting mixed monotone features in cone metric spaces, which are broader than partially ordered metric spaces.

1 Introduction and preliminaries

One of the most significant and researched fixed point theorems in nonlinear analysis is the well-known Banach's contraction principle, which is a typical procedure. Readers may refer to related works on fuzzy concepts and applications

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in [7–20, 24–27]. It says that $\forall\ k \in S$ and S is a complete metric space, if $F : S \to S$ is a contraction on S, then $u \in S$ is a unique fixed point of F and $\lim_n F^n a = u \forall\ a \in S$. A fuzzy cone metric space (or $FCMS$) can be conceptualized in a variety of ways [22, 23, 28]. The majority of scholarly works in this domain duly acknowledge the foundational contribution of Oner et al. [21] to the formulation of fuzzy cone metric spaces, within which they successfully established a Hausdorff topology. Subsequently, it was demonstrated in [23] that the topology induced by a fuzzy cone metric space, as defined by Oner, is metrizable. Furthermore, in [21], the concept of Banach contraction was extended within the framework of complete fuzzy cone metric spaces through the application of shifting distance techniques. Studies involving algebraic systems solved via fixed point principles in C^*-algebra valued metric space [5] illustrate the broad applicability of contraction-type results, motivating further generalisation in fuzzy cone settings.

Definition 1.1. *[21] A sequence $\{a_n\}$ in a $FCMS$ $(S, \zeta, *)$ is Cauchy if for any $0 < \epsilon < 1$ and any $t > 0$ there exists a natural number n_0 such that $\zeta(a_n, a_m, t) > 1 - \epsilon$ for all $n, m > n_0$.*

In this study, we show that the Banach Contraction Principle and the Meir-Keeler Fixed Point Theorem can be applied in the context of $FCMS$, with a slight modification to the definition originally introduced by T. Oner et al [19].

Definition 1.2. *[25] A continuous t- norm is defined as a binary operation $* : [0, 1] \to [0, 1]$ if it satisfies the following conditions.*

1. *(T_1) $*$ is commutative and associative,*

2. *(T_2) $*$ is continuous,*

3. *(T_3) $p * 1 = 1$ for every $p \in [0, 1]$,*

4. *(T_4) $p * q \leq r * s$ when $p \leq\ r$ and $q \leq s$, with $p, q, r, s \in [0, 1]$.*

Definition 1.3. *[9] A fuzzy metric space is an ordered triple $(S, \zeta, *)$ where ζ is a fuzzy set on $S \times S \times (0, \infty) \to (0, 1]$, S is a non-empty set and $*$ is a continuous t-norm such that*

1. *(F1) $\zeta(p, q, t) > 0$,*

2. *(F2) $\zeta(p, q, t) = 1$ if and only if when p equals q,*

3. *(F3)* $\zeta(p, q, t) = T(q, p, t)$,

4. *(F4)* $\zeta(p, q, t) * \zeta(q, e, c) \leq \zeta(p, e, t + c)$,

5. *(F5)* $\zeta(p, q, .) : (0, +\infty) \to (0, 1]$ *is left continuous, for all* $p, q, e \in S$ *and* $t, c > 0$.

Definition 1.4. *[20] Let* ζ *be a fuzzy set on* $S \times S \times (0, \infty]$, *let* $S \neq \phi$, *and let* $*$ *be a continuous* t*-norm, and* ζ *such that*

1. *(EF1)* $\zeta(p, q, t) > 0$,

2. *(EF2)* $\zeta(p, q, t) = 1$ *if and only if when* p *equals* q,

3. *(EF3)* $\zeta(p, q, t) = \zeta(q, p, t)$,

4. *(EF4)* $\zeta(p, q, t) * \zeta(q, z, s) \leq \zeta(p, z, t + s)$,

5. *(EF5)* $\zeta(p, q, .) : (0, +\infty) \to (0, 1]$ *is left continuous.*

6. *(EF6) For some* $r >$ *positive, the collection* $\{\zeta(p, q, .) : (0, r) \to (0, 1]; (p, q) \in S^2\}$ *is uniformly equicontinuous,*

for all $t, s > 0$ *and* $p, q, z \in S$. *Then, the ordered triple* $(S, \zeta, *)$ *is called a* E*-fuzzy metric space.*

Definition 1.5. *[21] If* P *is a cone of* E, S *is an arbitrary set,* $*$ *is a continuous* t*-norm and* ζ *is a fuzzy on* $X^2 \times int(P)$, *then a* 3*-tuple* $(S, \zeta, *)$ *is considered fuzzy cone metric space satisfying the following conditions, for all* $t, s \in int(P)$, $a, b, z \in S$ *and* $t > 0, s > 0$.

1. *(FCM1)* $\zeta(p, q, t) > 0$,

2. *(FCM2)* $\zeta(p, q, t) = 1$ *if and only if when* p *equals* q,

3. *(FCM3)* $\zeta(p, q, t) = \zeta(q, p, t)$,

4. *(FCM4)* $\zeta(p, q, t) * \zeta(q, z, s) \leq \zeta(p, z, t + s)$,

5. *(FCM5)* $\zeta(p, q, .) : int(P) \to [0, 1]$ *is continuous.*

The following class of fuzzy cone metric spaces will be examined in this research.

Definition 1.6. *Let ζ be a fuzzy set on $S \times S \times (0, \infty]$, let $S \neq \phi$, and let $*$ be a continuous t-norm, and ζ such that*

1. *(EFCM1) $\zeta(p, q, t) > 0$,*
2. *(EFCM2) $\zeta(p, q, t) = 1$ if and only if when p equals q,*
3. *(EFCM3) $\zeta(p, q, t) = \zeta(q, p, t)$,*
4. *(EFCM4) $\zeta(p, q, t) * \zeta(q, z, s) \leq \zeta(p, z, t + s)$,*
5. *(EFCM5) $\zeta(p, q, .) : int(P) \to (0, 1]$ is left continuous.*
6. *(EFCM6) The collection $\{\zeta(p, q, .) : (0, r) \to (0, 1]; (p, q) \in S^2\}$ is uniformly equicontinuous, for some r positive,*

for all $t, s > 0$ and $p, q, z \in S$. Then, the ordered triple $(S, \zeta, *)$ is called a E-fuzzy cone metric space (or $E - FCMS$).

Definition 1.7. *[21] In a fuzzy cone metric space $(S, \zeta, *)$. Then,*

1. *A sequence $\{a_n\}_n$ converges to $a \in S$ iff $\zeta(a_n, a, t) \to 1$ as $n \to +\infty$ for every $t > 0$;*
2. *$\{a_n\}_n \in S$ is a Cauchy Sequence if and only if for every ϵ belonging to $(0, 1)$ and t positive, there exists n_0 such that $\zeta(a_n,\ x_m,\ t) > 1 - t$, for all m, $n \geq 0$;*
3. *If each Cauchy sequence in a $FCMS$ converges to an element p in S, then $FCMS$ is said to be complete.*

Now we need to prove the following lemma:

Lemma 1.1. *Consider a sequence $\{a_p\}_p$ in S such that $\lim_{p\to\infty} \zeta(a_p, a_{p+1}, t) = 1$, $\forall\ t > 0$, let $(S, \zeta, *)$ represent a E-fuzzy cone metric space (or $E - FCMS$) and let $\bar{\zeta}$ be the continuous extension of ζ up to $[0, \infty)$. Then,*

$$\lim_{p\to\infty} \bar{\zeta}(a_p, a_{p+1}, 0) = 1. \tag{1.1}$$

Proof. Since the function $t \mapsto \zeta(a,b,t)$ is positive, non-decreasing and continuous on the interval $(0,+\infty)$, $\bar{\zeta}$ is well-defined for any $a, b \in S$. Suppose $\{t_p\}_p$ is a sequence of positive numbers that decreases monotonically and converges to 0. Similarly, let $\{a_p\}_p$ be a sequence in S such that $\lim_{p\to\infty} \zeta(a_p, a_{p+1}, t) = 1$ for all $t > 0$. This means that for every $t > 0$ and for every $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $\forall\ p \geq p_0$; $1 - \zeta(a_p, a_{p+1}, t) < \epsilon$. Consequently, for $t > 0$ and $\forall\ \epsilon > 0$, $\exists p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, and for all $k \in \mathbb{N}$,

$$\begin{aligned} 1-&\zeta(a_p, a_{p+1}, 0) + \zeta(a_p, a_{p+1}, 0) - \zeta(a_p, a_{p+1}, t_k) \\ &+ \zeta(a_p, a_{p+1}, t_k) - \zeta(a_p, a_{p+1}, t) < \frac{\epsilon}{2}. \end{aligned} \tag{1.2}$$

Thus, for every $t > 0$ and for every ϵ positive, $\exists$ a $p_0 \in \mathbb{N}$ such that $\forall\ k \in \mathbb{N}$ and for every $p \geq p_0$,

$$\begin{aligned} 1 - \bar{\zeta}(a_p, a_{p+1}, 0) <& \frac{\epsilon}{2} + \mid \bar{\zeta}(a_p, a_{p+1}, 0) - \zeta(a_p, a_{p+1}, t_k) \mid \\ &+ \mid \zeta(a_p, a_{p+1}, t_p) - \zeta(a_p, a_{p+1}, t) \mid . \end{aligned} \tag{1.3}$$

Alternatively, based on the fact that $\lim_k \zeta(a_p, a_{p+1}, t_k) = \bar{\zeta}(a_p, a_{p+1}, 0)$ and assumption $(EFCM6)$, we deduce that $\exists\ t_0$ positive: $\forall\ \epsilon$ positive, $\exists\ p_0 \in \mathbb{N}$, $\forall\ p \geq p_0$, $\exists\ k_0 \in \mathbb{N}$, such that

$$\mid \bar{\zeta}(a_p, a_{p+1}, 0) - \zeta(a_p, a_{p+1}, t_k) \mid \leq \frac{\epsilon}{4} \tag{1.4}$$

$$\mid \zeta(a_p, a_{p+1}, t_k) - \zeta(a_p, a_{p+1}, t) \mid \leq \frac{\epsilon}{4} \tag{1.5}$$

Such that for every k greater than k_0 and t less than t_0. Hence, by relation 1.3, 1.4, 1.5, it gives $\forall\ t > 0$, $\exists\ p_0 \in \mathbb{N}$ such that for every $p \geq p_0$, we have $1 - \bar{\zeta}(a_p, a_{p+1}, 0) < \epsilon$, and this means

$$\lim_p \bar{\zeta}(a_p, a_{p+1}, 0) = 1, \tag{1.6}$$

which completes the proof. □

2 Main results

Variants of generalised F-contractions such as $(\alpha, F)-$Geraghty contractions have been fruitfully examined in non-Archemedean fuzzy frameworks [4], further emphasizing the versatility of fuzzy metric extensions.

Now, we will present our main results.

Theorem 2.1. *Consider the E- fuzzy cone metric space* $(S, \zeta, *)$ *to be complete. Consider a contractive mappings* $\chi : S \to S$ *with a contractive constant k, which means there exists* $k \in]0,1[$ *such that*

$$\frac{1}{\zeta(\chi a, \chi b, t)} - 1 \leq k((\frac{1}{\zeta(a, b, t)} - 1), \tag{2.1}$$

$\forall\, a, b \in S$ *and* $\forall\, t > 0$. *Consequently, the sequence* $\{\chi^n a\}$ *converges to* a^* *and* χ *has a unique fixed point* a^* *for all* $a \in S$.

Proof. Let $a \in S$ and $a_n = \chi^n a, (n \in \mathbb{N})$ suppose t be positive and n belonging to natural number, $\mathbb{N}$. By 2.1, we get

$$\frac{1}{\zeta(a_{n+1}, a_{n+2}, t)} - 1 \leq k(\frac{1}{\zeta(a_n, a_{n+1}, t)} - 1), \tag{2.2}$$

for all t greater than 0 and for every n belonging to $\mathbb{N}$ this indicates that

$$\lim_{n\to\infty} \zeta(a_n, a_{n+1}, t) = 1, \forall t > 0. \tag{2.3}$$

In order to show that $\{a_n\}_n$ is a Cauchy sequence, one can proceed by contradiction. Since $t \mapsto \zeta(a, b, t)$ is a function which non decreasing, $\exists\, \epsilon \in (0, 1)$ and $\exists\, \xi$ positive such that $\forall\, p$ belong to $\mathbb{N}$ and

$$\exists\, n_p(\geq p) < m_p \in \mathbb{N}$$

such that

$$\zeta(a_{m_p}, a_{n_P}, t) \leq 1 - \epsilon, \tag{2.4}$$

$\forall\, t < \xi$, Let $t_0 < \min\{\xi, r\}$. As a result of (2.3) and the previous inequality, it can be expressed that $\exists\, \epsilon \in (0,1)$ and $\forall\, p$ belong to $\mathbb{N}$, $\exists\ n_p(\geq p) < m_p \in \mathbb{N}$;

$$\begin{aligned} &\zeta(a_{m_p}, a_{n_p}, t_0) \leq 1 - \epsilon, \\ &\zeta(a_{m_p-1}, a_{n_p}, t_0) > 1 - \epsilon. \end{aligned} \tag{2.5}$$

Given the continuity of the function $t \mapsto \zeta(a, b, t)$ and the condition $\zeta(a_{m_p-}, a_{n_p}, t_0) > 1 - \epsilon$, we can select $q_0 \in \mathbb{N}$ such that

$$\zeta(a_{m_p-1}, a_{n_p}, t_0 - \frac{1}{q_0}) > 1 - \epsilon. \tag{2.6}$$

As a result of assumptions (T4) and (EFCM4) and relation 2.5 and 2.6, it follows that

$$\begin{aligned} 1 - \epsilon &\geq \zeta(a_{m_p}, a_{n_p}, t_0) \\ &\geq \zeta(a_{m_p}, a_{m-p-1}, \frac{1}{q_0}) * \zeta(a_{m_p-1}, a_{n_p}, t_0 - \frac{1}{q_0}) \\ &\geq \bar{\zeta}(a_{m_p}, a_{m_p-1}, 0) * (1 - \epsilon). \end{aligned} \tag{2.7}$$

Based on our hypotheses (T2)-(T3), limit (2.3), and Lemma 1, it follows

$$\lim_{p\to\infty} \zeta(a_{m_p}, a_{n_p}, t_0) = 1 - \epsilon. \tag{2.8}$$

Assuming for every p_1 positive, there exists $p \geq p_1$ so that

$$\zeta(a_{m_p+1}, a_{n_p+1}, t_0) \leq 1 - \epsilon.$$

This means that, considering (2.1) and (2.8), that $\{a_n\}_n$ has two subsequences $\{a_{n_p}\}_p$ and$\{a_{m_p}\}_p$ which satisfy

$$\lim_{p\to\infty} \zeta(a_{m_p}, a_{n_p}, t_0) = \lim_{p\to\infty} \zeta(a_{m_p+1}, a_{n_p+1}, t_0) = 1 - \epsilon. \tag{2.9}$$

Now, we assume that $\exists\, p_1$ greater than or equal to 0 so that

$$\zeta(a_{m_p+1}, a_{n_p+1}, t_0) > 1 - \epsilon$$

for every $p_1 \geq p$. One can assert that

$$\lim_p \zeta(a_{m_p+1}, a_{n_p+1}, t_0) = 1 - \epsilon.$$

Let us assume this is not the case, i.e., $\exists\ \gamma$ positive and two subsequences $\{a_{n_p}\}_p$ and $\{a_{m_p}\}_p$ verifying

$$\zeta(a_{m_p+1}, a_{n_p+1}, t_0) > \gamma + (1 - \epsilon), \tag{2.10}$$

$\forall\ p$ belonging to $\mathbb{N}$. Having q belonging to $\mathbb{N}$ satisfying

$$\zeta(a_{m_p+1}, a_{n_p+1}, t_0 - \frac{1}{q}) > \gamma + (1 - \epsilon),$$

we obtain

$$\begin{aligned}
1 - \epsilon &\geq \zeta(a_{m_p}, a_{n_p}, t_0) \\
&\geq \zeta(a_{m-p}, a_{m_p+1}, \frac{1}{2q}) * \zeta(a_{m-p}, a_{n_p+1}, t_0 - \frac{1}{q}) \\
&* \zeta(a - n - p + 1, a_{n_p}, \frac{1}{2q}) \\
&\geq \bar{\zeta}(a_{m_p}, a_{m_p+1}, 0) * [\gamma + (1 - \epsilon)] * \bar{\zeta}(a_{n_p+1}, a_{n_p}, 0) \\
&\longrightarrow \gamma + (1 - \epsilon), \ \ as\ p \longrightarrow \infty.
\end{aligned} \tag{2.11}$$

This leads to a contradiction. Therefore,

$$\lim_p \zeta(a_{m_p+1}, a_{n_p+1}, t_0) = 1 - \epsilon. \tag{2.12}$$

Relation (2.8), (2.9) and (2.12) lead to contradiction with (2.1). So, $\{a_n\}_n$ is a Cauchy sequence in the complete $FCMS\ S$ and one can conclude that $\exists\ a^* \in S$ such that

$$\lim_n \zeta(a_n, a^*, t) = 1, \tag{2.13}$$

$\forall\ t > 0$, and by 2.1, we get

$$\frac{1}{\zeta(\chi a_n, \chi a^*, t)} - 1 \leq k(\frac{1}{\zeta(a_n, a^*, t)} - 1), \tag{2.14}$$

$\forall\, n \in \mathbb{N}$ and $\forall\, t > 0$. Taking the limit, considering the limit in (2.13), it becomes that $\zeta(a^*, \chi a^*, t) = 1$, which, with hypothesis (EFCM2) and (2.1), gives that a^* is the unique fixed point of mapping χ. Hence, we proof the theorem. □

Applications of fixed point theorems in solving nonlinear equations in mechanics, such as beam theory, have been rigorously addressed in modified metric space [2],which parallels the methodology used herein.

Theorem 2.2. *Consider the E- fuzzy cone metric space $(S, \zeta, *)$ to be complete. Let the mapping $\chi : S \to S$ be a fuzzy cone Meir-Keeler type, that is, there exists δ positive, for every ϵ belonging to $(0, 1)$ so that*

$$\epsilon - \delta < \zeta(a, b, t) \leq \epsilon \Rightarrow \zeta(\chi a, \chi b, t) > \epsilon, \quad \forall\, a, b \in S\, \forall\, t > 0. \tag{2.15}$$

Then, $\{\chi^n a\}$ converges to a^* and χ has a unique fixed point a^*, $\forall\, a \in S$.

Proof. Suppose $a \in S$ and $a_n = \chi^n a (n \in \mathbb{N})$ and $t > 0$. Obviously,

$$\zeta(a, \chi a, t) - \delta < \zeta(a, \chi a, t) \leq \zeta(a, \chi a, t), \tag{2.16}$$

$\forall\, \delta > 0$, and due to relation (2.15), we get $\zeta(\chi^2 a, , \chi a, t) > \zeta(a, \chi a, t)$. Recursively, we get a sequence

$$\{\zeta(a_n, a_{n+1}, t)\}_n \quad \in [0, 1]$$

verifying

$$\zeta(a_n, a_{n+1}, t) < \zeta(a_{n+1}, a_{n+2}, t), \forall n \in \mathbb{N}. \tag{2.17}$$

It is a sequence that is both bounded and non-decreasing. Then, $\exists\, v : (0, \infty) \to [0, 1]$ so that

$$\lim \zeta(a_n, a_{n+1}, t) = \sup_{n \in \mathbb{N}} \zeta(a_n, a_{n+1}, t) = v(t), \text{ for every } t \text{ greater than } 0. \tag{2.18}$$

For every t positive, we assert that $v(t) = 1$. Let this is not the case, meaning there exist t_0 positive such that $v(t_0)$ belonging to $(0, 1)$.Using limit from (2.18), for every $\delta \in (0, v(t_0)), \exists\, n_0 \in \mathbb{N}$ so that

$$v(t_0) - \delta < \zeta(a_n, a_{n+1}, t_0) \leq v(t_0), \tag{2.19}$$

$\forall\, n \geq n_0$, which, combined with condition (2.15), leads to the conclusion that

$$\zeta(a_{n+1}, a_{n+2}, t_0) > v(t_0).$$

This clearly contradicts condition (2.18). Hence,

$$\lim_{n\to\infty} \zeta(a_n, a_{n+1}, t) = 1, \forall\, t > 0. \tag{2.20}$$

Next, one can prove in similar approach as in the Theorem 2.1 to demonstrate that $\{a_n\}_n$ is a Cauchy sequence in the complete $FCMS$. This leads to the conclusion that there exists an element $a^* \in S$ such that

$$\lim_{n\to\infty} \zeta(a^*, a_n, t) = 1, \tag{2.21}$$

Conversely, for every n belonging to $\mathbb{N}$ and all δ belonging to $(0, \zeta(a^*, a_n, t))$, we have

$$\zeta(a^*, a_n, t) - \delta < \zeta(a^*, a_n, t) \leq \zeta(a^*, a_n, t), \tag{2.22}$$

condition (2.15) assures that

$$1 \geq \zeta(\chi a^*, \chi a_n, t) > \zeta(a^*, a_n, t), \tag{2.23}$$

which, with the limit in (2.21), gives $\lim \zeta(\chi a^*, \chi a_n, t) = 1$, and finally

$$a^* = \chi a^* \tag{2.24}$$

To prove uniqueness, assuming $\exists\, b^*(\neq a^*) \in S$ so that $b^* = \chi b^*$. It is evident

$$\forall \delta \in (0, \zeta(a^*, b^*, t)), \zeta(a^*, b^*, t) - \delta < \zeta(a^*, b^*, t) \leq \zeta(a^*, b^*, t).$$

Here, by (2.15),

$$\zeta(\chi a^*, \chi b^*, t) > \zeta(a^*, b^*, t)$$

or

$$\zeta(a^*, b^*, t) > \zeta(a^*, b^*, t),$$

a contradiction, and this achieves the proof. □

Corollary 2.1. *Let us consider a complete metric space (S, d), Let χ be a Meir-Keeler mapping on S, that is, for each ε positive, $\exists\ \beta$ positive such that for every $a, b \in S$,*

$$\varepsilon \leq d(a,b) < \varepsilon + \beta \Longrightarrow d(\chi(a), \chi(b)) < \varepsilon. \tag{2.25}$$

Let ζ be a function on $S \times S \times (0, +\infty)$ defined by

$$\zeta(a,b,t) = \frac{t+1}{t+1+d(a,b)}. \tag{2.26}$$

Then,

(i) $(S, \zeta, \cdot)$ is an $E - FCMS$, where $\cdot$ represents the product t-norm.

(ii) For every $\varepsilon > 0$, $\exists\ \beta > 0$. so

$$\varepsilon - \beta < \zeta(a,b,t) \leq \varepsilon \Longrightarrow \zeta(\chi a, \chi b, t) > \varepsilon, \tag{2.27}$$

for every $a, b \in S$ and for every $t > 0$.

Proof. Consider$(S, \zeta, \cdot)$ as a $FCMS$ (as described in [9] and [21]). The set of functions $\{t \mapsto \zeta(a,b,t); a, b \in S\}$ has a common Lipschitz constant of 1, making it uniformly equicontinuous.It implies, $(S, \zeta, \cdot)$ is an $E - FCMS$.
For the next assumption, it is enough to verify that for every both ε and t positive, $\beta \in (0, \epsilon)$, and for every $a, b \in S$, the following holds

$$\begin{aligned}\varepsilon - \beta < \zeta(a,b,t) \leq \varepsilon &\Longleftrightarrow \varepsilon - \beta < \frac{t+1}{t+1+d(a,b)} \leq \varepsilon \\ &\Longleftrightarrow \varepsilon - \beta < \frac{1}{1 + (\frac{1}{t+1})d(a,b)} \leq \varepsilon \\ &\Longleftrightarrow (t+1)(\frac{1}{\varepsilon} - 1) \leq d(a,b) < (t+1)(\frac{1}{1-\beta} - 1)\end{aligned} \tag{2.28}$$

Consider $\varepsilon_0 = (t+1)((\frac{1}{\varepsilon}) - 1)$ and $\beta_{\varepsilon_0} > 0$ so that

$$\varepsilon_0 \leq d(a,b) < \varepsilon_0 + \beta_{\varepsilon_0} \implies d(\chi a, \chi b) < \varepsilon_0. \tag{2.29}$$

Now, we select β in 2.28 so, $(t+1)((\frac{1}{(\varepsilon-\beta)})-1) < (t+1)((\frac{1}{\varepsilon})-1)+\beta_{\varepsilon_0}$. Hence, applying 2.28 and 2.29, becomes

$$\begin{aligned} \varepsilon - \beta < \zeta(a,b,t) \leq \varepsilon &\implies ((t+1)(\frac{1}{\varepsilon}-1) < d(a,b) \leq (t+1)(\frac{1}{\varepsilon}) + \beta_{\varepsilon_0} \\ &\implies d(\chi a, \chi b) < (t+1)(\frac{1}{\varepsilon}-1) \\ &\implies \frac{t+1}{t+1+d(\chi a, \chi b)} > \varepsilon \\ &\implies \zeta(\chi a, \chi b, t) > \varepsilon, \end{aligned} \tag{2.30}$$

which completes the result. □

3 Application

The following portion aims to provide an illustration showing that an integral equation has a solution, which can be obtained by using Theorem 2.1. Readers may refer to [6] for a common solution to a system of two integral equations including such integral equations. Real-world dynamical systems, such as the ascending motion of rockets, have also been effectively modeled in extended metric spaces [1], underscoring the practical utility of such abstract mathematical frameworks. Examine the integral equation,

$$a(c) = f(c) + \int_0^c Q(c,s,a(s))ds, \ \forall\, c \in [0,I], I > 0, \tag{3.1}$$

$S = C([0,I],\mathbb{R})$ which is Banach space consists of all continuous functions on the interval $[0,1]$, and is equipped with the supremum norm

$$\|a\| = \sup_{c\in[0,1]} \mid a(c) \mid, \ a \in C([0,I],\mathbb{R}) = S, \tag{3.2}$$

with induced metric

$$d(a,b) = \sup_{c\in[0,1]} \mid a(c) - b(c) \mid . \tag{3.3}$$

Now, let the $FCMS$ $(S, \zeta, *)$ equipped with the product t-norm, and assume it is complete

$$\zeta(a,b,t) = \frac{t}{t + d(a,b)},\ for\ all\ a, b \in C([0,I],\mathbb{R}) = S \text{ and } t > 0. \tag{3.4}$$

Theorem 3.1. *Let χ be an integral operator on $C([0,I],\mathbb{R})$ be*

$$\chi a(c) = f(r) + \int_0^c Q(c,s,a(s))ds. \tag{3.5}$$

Assume there exists a function $g : [0,I]\times[0,I] \to [0,\infty)$ such that $g \in L^1([0,I],\mathbb{R})$ and let Q fulfill the following relations:

$$\mid Q(s,c,a(c)) - Q(s,c,b(c)) \mid \leq g(c,s) \mid a(s) - b(s) \mid, \tag{3.6}$$

for all $a, b \in C([0,I],\mathbb{R})$ and $\forall\ c, s \in [0,I]$ where

$$\sup_{c\in[0,I]} \int_0^c g(c,s)ds \leq k < 1. \tag{3.7}$$

Then, the integral equation equation 3.1 has a unique solution.

Proof. Let $a, b \in C([0,I],\mathbb{R})$ and consider

$$\begin{aligned} \mid \chi a(c) - \chi b(c) \mid &\leq \int_0^c \mid Q(c,s,a(s)) - Q(c,s,b(s)) \mid d(s) \\ &\leq \int_0^c g(c,s) \mid a(s) - b(s) \mid d(s) \\ &\leq d(a,b) \int_0^c g(c,s)d(s) \\ &\leq kd(a,b). \end{aligned} \tag{3.8}$$

So,

$$d(\chi a, \chi b) \leq kd(a, b). \tag{3.9}$$

Using 3.4, we can write

$$\begin{aligned}
\frac{1}{\zeta(a,b,t)} - 1 &= \frac{d(a,b)}{t} \\
\frac{1}{\zeta(\chi a,\chi b,t)} - 1 &= \frac{t + d(\chi a,\chi b) - t}{t} \\
&= \frac{d(\chi a,\chi b)}{t} \\
&\leq k\frac{d(a,b)}{t} \\
&\leq k(\frac{1}{T(a,b,t)} - 1)
\end{aligned} \tag{3.10}$$

since all the conditions of Theorem 2.1 are satisfied, hence (3.1) has a unique solution. □

4 Conclusion

The study of fixed point results in E-fuzzy cone metric spaces is an important area in mathematical analysis, particularly due to its applications in optimization, decision theory, and computational mathematics. Classical fixed point theorems can be applied to E-fuzzy cone metric spaces. This gives us a strong way to solve problems in areas that are not as well organized as regular metric or normed spaces. These spaces incorporate the concepts of fuzziness and cone structures, making them highly adaptable for modeling uncertainty and imprecision. The key findings show that we can establish fixed point theorems in E-fuzzy cone metric spaces under suitable contractive conditions. These results extend and generalize classical fixed point theorems like the Banach contraction principle to more abstract settings. Moreover, the role of the cone's properties, such as normality and solidness, is crucial in ensuring the existence and uniqueness of fixed points. Recent applications of fixed points theory to machine learning and neural network using $\phi-$interpolative contractions have been discussed in [3], indicating the growing relevance of these mathematical tools in AI research. Moreover, the exploration of E-fuzzy cone metric spaces enriches the field by offering new theoretical insights and expanding its applicability. Future research may focus on developing more generalized conditions and exploring applications in diverse areas such as machine learning, network theory, and engineering systems that involve uncertainty and fuzzy information.

Conflict of Interest:
There are no disclosed conflicts of interest for the authors. The study was conducted independently without any external funding, and the authors did not have any involvement from commercial entities that could have influenced the study design, data collection, analysis, interpretation, or publication.

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