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A common fixed point result of rational type in dislocated quasi metric spaces

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Abstract

This paper presents a proof of a common fixed point theorem for a pair of self-maps, achieved by eliminating the continuity requirement in dislocated quasi metric spaces. This work extends and generalizes established fixed point results in the literature. An example is provided to support the findings of this paper.

1 Introduction

In 1922, Banach [2] presented one of the most important fixed point results. There has been a number of generalizations of the Banach contraction principle in various directions. Dislocated metric space is one of the generalizations which is studied under the name of metric domains in the context of domain theory (see [1]). The study of dislocated topologies has beneficial applications in the pursuit of developing logic programming (see [8]).

In 2000, Hitzler and Seda [8] introduced the concept of dislocated metric space, wherein the self-distance of any point is not necessarily equal to zero. They also presented a variant of the well-known Banach contraction principle applicable in complete dislocated metric spaces. As a generalization of dislocated metric space, Zeyada et al. [14] introduced the concept of complete dislocated quasi-metric space

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in 2006, expanding upon the work of Hitzler et al. [8] in such spaces. In this new notion, the symmetric property is omitted. In 2010, Isufati [9] presented a fixed point result of Dass and Gupta [4] rational type contraction in complete dislocated quasi-metric spaces and generalized the result of Zeyada et al. [14]. In 2017, Bairagi et al. [3] proved a fixed point result of rational type contraction in complete dislocated quasi-metric space.

The objective of this paper is to prove a common fixed point theorem for a pair of self-maps by omitting continuity requirement in dislocated quasi metric spaces. It is to extend and generalize the result of Bairagi et al. [3] and then to unify some well-known fixed point results in the literature. The readers are encouraged to consult the recent literature for further insights into the advances in dislocation theory and its related domains, as detailed in [5–7, 10, 12, 13]. These works provide valuable perspectives on both foundational developments and emerging directions in the field.

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}^+$ the set of non-negative real numbers.

2 Preliminaries

We need to retrieve the following relevant definitions and results in the sequel.

Definition 2.1 ([14]). *Consider a non-empty set X and let $\rho : X \times X \longrightarrow \mathbb{R}^+$ be a function satisfying the following conditions:*

$$(i)\ \rho(u,v) = \rho(v,u) = 0 \Rightarrow u = v$$
$$(ii)\ \rho(u,v) \leq \rho(u,w) + \rho(w,v), \forall\ u,v,w \in X.$$

Then, ρ is called a dislocated quasi-metric (simply ρ_q-metric) on X , and the pair (X,ρ) is called a dislocated quasi-metric space (in short, ρ_q-metric space). In addition, if ρ satisfies $\rho(u,v) = \rho(v,u)$, for all $u,v, \in X$, then it is called a dislocated metric.

The dislocated metric and the dislocated quasi metric are both used to describe metrics on sets; however, a dislocated quasi metric does not always have to be a dislocated metric in order to be considered a metric (see [7]).

In the following, X represents the dislocated quasi-metric space (X,ρ).

Definition 2.2 ([14]). *A sequence $\{u_n\}$ dislocated quasi-converges (for short, ρ_q-*

converges) to u *if* $\lim_{n\to\infty} \rho(u_n, u) = \lim_{n\to\infty} \rho(u, u_n) = 0$ *. In this case,* u *is called a* ρ_q-*limit of* $\{u_n\}$ *.*

Lemma 2.1 (**[14]**)**.** *In a* ρ_q-*metric space,* ρ_q-*limits are unique.*

Definition 2.3 ([14])**.** *A sequence* $\{u_n\}$ *in* ρ_q-*metric space* X *is called Cauchy if for each* $\epsilon > 0$ *, there exists* $n_0 \in \mathbb{N}$ *such that* $\rho(u_m, v_n) < \epsilon$ *or* $\rho(u_n, v_m) < \epsilon$ *for all* $m, n \geq n_0$*.*

Definition 2.4 ([14])**.** *A* ρ_q-*metric space* X *is called complete if every Cauchy sequence in it is* ρ_q-*convergent.*

Lemma 2.2 ([14])**.** *Every subsequence of* ρ_q-*convergent sequence to a point* u_0 *is* ρ_q-*convergent to* u_0*.*

It is obvious that the converse of Lemma 2.2 may not be true.

In 2017, Bairagi et al. [3] proved the following theorem in the context of dislocated quasi metric space.

Theorem 2.1. *Let* (X, ρ) *be a complete* ρ_q-*metric space and let* $P, Q : X \to X$ *be a pair of continuous self-maps satisfying the following condition:*

$$\begin{aligned} \rho(Pu, Qv) \quad &\leq \quad a_1 \frac{\rho(v, Qv)\rho(u, Pu)}{\{1 + \rho(u, Pu)\}\{1 + \rho(v, Qv)\}} + a_2 \frac{\rho(u, v)\rho(u, Pu)}{1 + \rho(u, Pu)} \\ &\quad + a_3 \frac{\rho(u, v)\rho(v, Qv)}{1 + \rho(u, v)}, \end{aligned}$$

for all $u, v \in X$*, where* $a_i \geq 0$ *and* $a_1 + a_2 + a_3 < 1$*.*

Then, there exists a unique common fixed point in X for P and Q.

3 Main results

Now, we prove the following main result:

Theorem 3.1. *Let* (X, ρ) *be a complete* ρ_q*-metric space, and let* $P, Q : X \to X$ *be a pair of self-maps satisfying the following condition:*

$$\rho(Pu, Qv) \quad \leq \quad a_1 \frac{\rho(v, Qv)\rho(u, Pu)}{\{1 + \rho(u, Pu)\}\{1 + \rho(v, Qv)\}} + a_2 \frac{\rho(u, v)\rho(u, Pu)}{1 + \rho(u, Pu)}$$

$$+a_3 \frac{\rho(u,v)\rho(v,Qv)}{1+\rho(u,v)} + a_4\rho(u,v), \tag{3.1}$$

$\forall x, y \in X$, where $a_i \geq 0$ and $a_1 + a_2 + a_3 + a_4 < 1$.

Then, there exists a unique common fixed point in X for P and Q.

Proof. Let us choose $u_0 \in X$ arbitrary. Define a sequence $\{u_n\}$ in X such that $u_{2n+1} = Pu_{2n}$ and $u_{2n+2} = Qu_{2n+1}$ for all $n \in \mathbb{N}_0$.

We consider $\rho(u_{2n+1}, u_{2n+2}) = \rho(Pu_{2n}, Qu_{2n+1})$.

Using (3.1), we have

$$\begin{aligned}
\rho(u_{2n+1}, u_{2n+2}) \quad \leq \quad & a_1 \frac{\rho(u_{2n+1}, Qu_{2n+1})\rho(u_{2n}, Pu_{2n})}{\{1+\rho(u_{2n}, Pu_{2n})\}\{1+\rho(u_{2n+1}, Qu_{2n+1})\}} \\
& + a_2 \frac{\rho(u_{2n}, u_{2n+1})\rho(u_{2n}, Pu_{2n})}{1+\rho(u_{2n}, Pu_{2n})} \\
& + a_3 \frac{\rho(u_{2n}, u_{2n+1})\rho(u_{2n+1}, Qu_{2n+1})}{1+\rho(u_{2n}, u_{2n+1})} + a_4\rho(u_{2n}, u_{2n+1}), \\
& \leq a_1 \frac{\rho(u_{2n}, u_{2n+1})}{1+\rho(u_{2n}, u_{2n+1})} + a_2\, \rho(u_{2n}, u_{2n+1}) \\
& + a_3\, \rho(u_{2n+1}, u_{2n+2}) + a_4\, \rho(u_{2n}, u_{2n+1}) \\
& \leq a_1\, \rho(u_{2n}, u_{2n+1}) + a_2\, \rho(u_{2n}, u_{2n+1}) + a_3\, \rho(u_{2n+1}, u_{2n+2}) \\
& + a_4\, \rho(u_{2n}, u_{2n+1})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (1-a_3)\, \rho(u_{2n+1}, u_{2n+2}) \quad & \leq \quad (a_1+a_2+a_4)\, \rho(u_{2n}, u_{2n+1}) \\
\Rightarrow \rho(u_{2n+1}, u_{2n+2}) \quad & \leq \quad \left(\frac{a_1+a_2+a_4}{1-a_3}\right) \rho(u_{2n}, u_{2n+1}) \\
\Rightarrow \rho(u_{2n+1}, u_{2n+2}) \quad & \leq \quad \lambda\, \rho(u_{2n}, u_{2n+1}), \text{ where } \lambda = \frac{a_1+a_2+a_4}{1-a_3} < 1.
\end{aligned}$$

Similarly, we have $\rho(u_{2n}, u_{2n+1}) \leq \lambda\, \rho(u_{2n-1}, u_{2n})$. So, we obtain $\rho(u_{2n+1}, u_{2n+2}) \leq \lambda^2\, \rho(u_{2n-1}, u_{2n})$. Proceeding in this way, we have $\rho(u_{2n+1}, u_{2n+2}) \leq \lambda^{2n+1}\rho(u_0, u_1)$. We claim that $\{u_n\}$ is a Cauchy sequence in X. Now, for $n,\ k \in \mathbb{N}$, we see that

$$\begin{aligned}
\rho(u_n, u_{n+k}) \quad \leq \quad & \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \rho(u_{n+2}, u_{n+3}) + \ldots + \rho(u_{n+k-1}, u_{n+k}) \\
& \leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + \ldots + \lambda^{n+k-1})\, \rho(u_0, u_1) \\
& \leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + \ldots)\, \rho(u_0, u_1) \\
& = (\frac{\lambda^n}{1-\lambda})\, \rho(u_0, u_1).
\end{aligned}$$

We have $\lambda < 1$ and $\lambda^n \to 0$ as $n \to \infty$ and so, $\rho(u_n, u_{n+k}) \to 0$. Similarly, we can show that $\rho(u_{n+k}, u_n) \to 0$. Thus $\{u_n\}$ is a Cauchy sequence in X. It follows that completeness of X implies existence of $x \in X$ such that $\lim_{n\to\infty} \rho(u_n, x) = \lim_{n\to\infty} \rho(x, u_n) = 0$. Also the subsequences $\{u_{2n+1}\}$ and $\{u_{2n+2}\}$ of the sequence $\{u_n\}$ converge to x.

Now we show that $Px = Qx = x$.
We have

$$\begin{aligned}\rho(x, Px) \;&\le\; \rho(x, u_{2n}) + \rho(u_{2n}, Px)\\ &= \rho(x, u_{2n}) + \rho(Qu_{2n-1}, Px).\end{aligned}$$

With (3.1), we get

$$\begin{aligned}\rho(x, Px) \;&\le\; \rho(x, u_{2n}) + a_1 \frac{\rho(x, Px)\rho(u_{2n-1}, Qu_{2n-1})}{\{1 + \rho(u_{2n-1}, Qu_{2n-1})\}\{1 + \rho(x, Px)\}}\\ &+ a_2 \frac{\rho(u_{2n-1}, x)\rho(u_{2n-1}, Qu_{2n-1})}{1 + \rho(u_{2n-1}, Qu_{2n-1})}\\ &+ a_3 \frac{\rho(u_{2n-1}, x)\rho(x, Px)}{1 + \rho(u_{2n-1}, x)} + a_4\rho(u_{2n-1}, x).\\ &\le \rho(x, u_{2n}) + a_1\, \rho(x, Px) + a_2\, \rho(u_{2n-1}, x)\\ &+ a_3\, \rho(x, Px) + a_4\, \rho(u_{2n-1}, x).\end{aligned}$$

Taking limit $n \to \infty$, we get

$$\begin{aligned}\rho(x, Px) \;&\le\; a_1\, \rho(x, Px) + a_3\, \rho(x, Px)\\ &\Rightarrow \rho(x, Px) = 0, \text{ since } a_1 + a_3 < 1.\end{aligned}$$

Also we have

$$\begin{aligned}\rho(Px, x) \;&\le\; \rho(Px, u_{2n}) + \rho(u_{2n}, x)\\ &= \rho(Px, Qu_{2n-1}) + \rho(u_{2n}, x).\end{aligned}$$

In view of (3.1), we have

$$\begin{aligned}\rho(x, Px) \;&\le\; a_1 \frac{\rho(u_{2n-1}, Qu_{2n-1})\rho(x, Px)}{\{1 + \rho(u_{2n-1}, Qu_{2n-1})\}\{1 + \rho(x, Px)\}} + a_2 \frac{\rho(x, u_{2n-1})\rho(x, Px)}{1 + \rho(x, Px)}\\ &+ a_3 \frac{\rho(x, u_{2n-1})\rho(u_{2n-1}, Qu_{2n-1})}{1 + \rho(x, u_{2n-1})} + a_4\, \rho(x, u_{2n-1}) + \rho(u_{2n}, x).\end{aligned}$$

$$\leq a_1\,\rho(x,Px) + a_2\,\rho(x,u_{2n-1}) + a_3\,\rho(u_{2n-1},u_{2n}) + a_4\,\rho(x,u_{2n-1}) + \rho(u_{2n},x).$$

Taking limit $n \to \infty$, we get

$$\begin{aligned}\rho(Px,x) &\leq a_1\,\rho(x,Px)\\ &\Rightarrow \rho(Px,x)=0, \text{ since } \rho(x,Px)=0.\end{aligned}$$

Therefore, $\rho(x,Px) = \rho(Px,x) = 0$ which implies $Px = x$. In same vein, it is evident that $Qx = x$. It so follows that $Px = Qx = x$, therefore, a common fixed point of P and Q is x.

We argue that the common unique fixed point of P and Q is x. Since x is a common fixed point of P and Q, we have

$$\begin{aligned}\rho(x,x) &= \rho(Px,Qx)\\ &\leq a_1\,\frac{\rho(x,Qx)\rho(x,Px)}{\{1+\rho(x,Px)\}\{1+\rho(x,Qx)\}} + a_2\,\frac{\rho(x,x)\rho(x,Px)}{1+\rho(x,Px)}\\ &\quad + a_3\,\frac{\rho(x,x)\rho(x,Qx)}{1+\rho(x,x)} + a_4\rho(x,x)\\ &\leq a_1\,\rho(x,x) + a_2\,\rho(x,x) + a_3\rho(x,x) + a_4\,\rho(x,x).\end{aligned}$$

Therefore, $\rho(x,x) = 0$, since $a_1 + a_2 + a_3 + a_4 < 1$.

If possible, let there exists another common fixed point y of P and Q. Subsequently,

$$\begin{aligned}\rho(x,y) &= \rho(Px,Qy)\\ &\leq a_1\,\frac{\rho(y,Qy)\rho(x,Px)}{\{1+\rho(x,Px)\}\{1+\rho(y,Qy)\}} + a_2\,\frac{\rho(x,y)\rho(x,Px)}{1+\rho(x,Px)}\\ &\quad + a_3\,\frac{\rho(x,y)\rho(y,Qy)}{1+\rho(x,y)} + a_4\,\rho(x,y)\\ &= a_1\,\frac{\rho(y,y)\rho(x,x)}{\{1+\rho(x,x)\}\{1+\rho(y,y)\}} + a_2\,\frac{\rho(x,y)\rho(x,x)}{1+\rho(x,x)}\\ &\quad + a_3\,\frac{\rho(x,y)\rho(y,y)}{1+\rho(x,y)} + a_4\,\rho(x,y)\\ &= a_4\,\rho(x,y).\end{aligned}$$

Therefore, $\rho(x,y) = 0$, since $a_4 < 1$. Similarly, we have $\rho(y,x) = 0$. So, we obtain $\rho(x,y) = \rho(y,x) = 0$, which gives $x = y$. Hence, x is a unique common fixed point of P and Q. This completes the proof.

Now, we furnish the following example in support of Theorem 3.1.

Example 3.1. *Let $X = [0,1]$ endowed with the ρ_q- metric defined by $\rho(u,v) =| u |$ for all $u, v \in X$, be complete. Define maps $P, Q : X \to X$ by*

$$Pu = \begin{cases} \frac{u}{2}, & 0 \leq u < 1 \\ 0, & u = 1 \end{cases}$$

and

$$Qu = \begin{cases} u, & 0 \leq u < 1 \\ \frac{3}{4}, & u = 1 \end{cases}$$

for all $u \in X$.

In particular, we take $a_1 = \frac{1}{4}, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}$ and $a_4 = \frac{1}{6}$, so that $a_1 + a_2 + a_3 + a_4 = \frac{57}{60}$ which is < 1.

Four cases arise:

Case I: When $0 \leq u < 1$ and $0 \leq v < 1$, we have $\frac{1}{2} \mid u \mid \leq \frac{1}{4} + \frac{1}{3} \mid u \mid + \frac{1}{5} \mid v \mid + \frac{1}{6} \mid u \mid$

Case II: When $0 \leq u < 1$ and $v = 1$, we have $\frac{1}{2} \mid u \mid \leq \frac{1}{4} + \frac{1}{3} \mid u \mid + \frac{1}{5} + \frac{1}{6} \mid u \mid$

Case III: When $u = 1$ and $0 \leq v < 1$, we have $0 \leq \frac{1}{8} + \frac{1}{6} + \frac{1}{10} \mid v \mid + \frac{1}{6}$

Case IV: When $u = 1$ and $v = 1$, we have $0 \leq \frac{1}{16} + \frac{1}{6} + \frac{1}{10} + \frac{1}{6}$

From the above cases, we observe that $P(0) = 0 = Q(0)$.

All the conditions of Theorem 3.1 are satisfied and so, $u = 0$ is the unique common fixed point of P and Q in X.

Taking into account that P and Q are continuous in the Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let (X, ρ) be a complete ρ_q-metric space, and let $P, Q : X \to X$ be a pair of continuous self-maps satisfying the following condition:*

$$\begin{aligned} \rho(Pu, Qv) \quad \leq \quad & a_1 \frac{\rho(v, Qv)\rho(u, Pu)}{\{1 + \rho(u, Pu)\}\{1 + \rho(v, Qv)\}} + a_2 \frac{\rho(u, v)\rho(u, Pu)}{1 + \rho(u, Pu)} \\ & + a_3 \frac{\rho(u, v)\rho(v, Qv)}{1 + \rho(u, v)} + a_4 \, \rho(u, v) \end{aligned}$$

for all $u, v \in X$*, where* $a_i \geq 0$ *and* $a_1 + a_2 + a_3 + a_4 < 1$.

Then P *and* Q *have a unique common fixed point in* X.

Remark 3.1. *Corollary* 3.1 *reduces to Theorem* 2.1 *by setting* $a_4 = 0$.

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