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Decay of solutions for a higher-order parabolic type Kirchhoff equation

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Abstract: In this work, we consider the higher-order parabolic type Kirchhoff equation in a bounded domain. Firstly, we establish the global existence of solutions. Finally, we prove the decay of solutions using the Komornik's lemma.

1 Introduction

1.1 Setting of the problem

In this work, we consider the following initial-boundary value problem for a class of higher-order parabolic-type Kirchhoff equation

$$\left(1+|z|^{p-2}\right) z_t + M\left(\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2\right) \mathcal{A} z = |z|^{q-2} z,\ (x,t) \in \Omega \times (0,T), \quad (1.1)$$

with the initial-boundary conditions

$$z(x,0) = z_0(x),\ x \in \Omega,$$

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and

$$\frac{\partial^i z(x,t)}{\partial \nu^i} = 0,\ i = 0,1,2,...,m-1,\ (x,t) \in \partial\Omega \times (0,T),$$

where $\mathcal{A} = (-\Delta)^m$, $m \geq 1$ is a natural number, Ω is a bounded domain in R^n ($n \geq 1$) with smooth boundary $\partial\Omega$, ν is unit outward normal vector on $\partial\Omega$; $p, q > 2$ and

$$M(s) = 1 + s^\gamma,\ \gamma \geq 1.$$

The problem (1.1) is a generalization of a model introduced by Kirchhoff [7]. More precisely, Kirchhoff proposed a model given by the equation for $f = g = 0$,

$$\rho h z_{tt} + \delta z_t + g(z_t) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L (z_x)^2\, dx\right\} z_{xx} + f(z), \tag{1.2}$$

for $0 < x < L$, $t \geq 0$, where $z(x,t)$ is the lateral displacement, E the Young modulus, ρ the mass density, h the cross-section area, L the length, ρ_0 the initial axial tension, δ the resistance modulus, and f and g the external forces. Moreover, (1.2) is called a degenerate equation when $\rho_0 = 0$ and nondegenerate one when $\rho_0 > 0$.

1.2 Literature overview

When $m = 1$, (1.1) becomes the following Kirchhoff-type parabolic equation

$$\left(1 + |z|^{p-2}\right) z_t - M\left(\|\nabla z\|^2\right) \Delta z = |z|^{q-2} z. \tag{1.3}$$

Ouaoua et al. [9] studied the global existence and decay of solutions of the problem (1.3). Later, Khaldi et al. [6] studied by taking variables instead of constants p and q in (1.2).

Pişkin and Ekinci [14] studied the blow up and growth for the following equation with initial-boundary conditions

$$z_t - \Delta z_t - M(\|\nabla z\|^2)\Delta z + |z|^{q-2} z_t = |z|^{p-2} z.$$

Pişkin and Cömert [12] studied the following nonlinear Kirchhoff-type parabolic equation with logarithmic source term

$$z_t - M(\|\nabla z\|^2)\Delta z - \Delta z_t = |z|^{q-2} z \ln |z|.$$

They established the existence and decay solutions.

Ishige et al. [4] considered the Cauchy problem for nonlinear higher-order heat equation as follows

$$z_t + \mathcal{A}z = |z|^q .$$

They proved the existence of solutions for Cauchy problem.

Xiao and Li [18] considered initial boundary value problem for nonlinear higher-order heat equations of

$$z_t + \mathcal{A}z_t + \mathcal{A}z = f(z).$$

They established the existence of solutions.

Moreover, numerous researchers have studied the existence, decay and blow up of solutions for the higher-order partial differential equations [2, 3, 5, 10, 11, 13, 15, 17, 19, 20]. Motivated by the above studies, in this work, we investigate the global existence and decay solutions for the eq. (1.1). This paper is organized as follows: In Section 2, we introduce some lemmas which will be needed later. In Section 3, we prove the global existence of solutions. In Section 4, we prove the decay of solutions using the Komornik's lemma.

2 Preliminaries

In this part, we shall give some lemmas, which will be used throughout this work. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Lemma 2.1. *(Sobolev-Poincaré inequality) [1, 16] Let p be a number with*

$$\begin{cases} 2 \le p < +\infty, & \text{if} \;\; n \le 2m, \\ 2 \le p \le \frac{2n}{n-2m}, & \text{if} \;\; n > 2m. \end{cases}$$

Then, there is constant C depending on Ω and p such that

$$\|u\|_p \le C \left\| \mathcal{A}^{\frac{1}{2}} u \right\|, \; \forall u \in H_0^m(\Omega).$$

Lemma 2.2. *[8]. Let $F : R^+ \longrightarrow R^+$ be a non-increasing function and assume that there are two constants $\alpha > 0$ and $C > 0$ such that*

$$\int_t^{\infty} F^{\alpha+1}(s)\, ds \le CF^{\alpha}(0)\, F(s)\, , \; \forall t \in R^+.$$

Then,

$$F(t) \leq \eta (C + \alpha t)^{\frac{-1}{\alpha}}, \ \forall t \geq C,$$

where η is a positive constant.

3 Global existence

In this part, we prove the global existence of solutions.

The energy function $E(t)$ and the Nehari's functional $I(t)$ can be defined respectively by

$$E(t) = E(z) = \frac{1}{2} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \frac{1}{2(\gamma+1)} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} - \frac{1}{q} \|z\|_q^q \tag{3.1}$$

and

$$I(t) = I(z) = \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} - \|z\|_q^q. \tag{3.2}$$

Lemma 3.1. *Suppose that z be a solution of (1.1). Then, the energy function $E(t)$ is a nonincreasing function for $t \in [0, T]$ and*

$$E'(t) = -\|z_t\|^2 - \int_\Omega |z|^{p-2} |z_t|^2 \, dx \leq 0, \tag{3.3}$$

and

$$E(t) \leq E(0).$$

Proof. Multiplying the equation in (1.1) by z_t and integrating over the domain Ω, to obtain

$$\frac{d}{dt}\left[\frac{1}{2} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \frac{1}{2(\gamma+1)} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} - \frac{1}{q} \|z\|_q^q\right] = -\|z_t\|^2 - \int_\Omega |z|^{p-2} |z_t|^2 \, dx.$$

Then,

$$E'(t) = -\|z_t\|^2 - \int_\Omega |z|^{p-2} |z_t|^2 \, dx \leq 0.$$

Integrating (3.3) over $(0, t)$, we get

$$E(t) \leq E(0).$$

□

Lemma 3.2. *Let z be a solution of (1.1) and $q > 2(\gamma+1)$,*

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1, \tag{3.4}$$

here

$$\beta_1 = \alpha c_*^q \left(\frac{2q}{q-2} E(0)\right)^{\frac{q-2}{q}} \quad \textit{and} \quad \beta_2 = (1-\alpha) c_*^q \left(\frac{2(\gamma+1)q}{q-2(\gamma+1)} E(0)\right)^{\frac{q-2(\gamma+1)}{2(\gamma+1)}},$$

with $0 < \alpha < 1$, and c_* is the best embedding constant of $H_0^m(\Omega) \hookrightarrow L^q(\Omega)$, then

$$I(t) > 0,$$

for all $t \in [0, T]$.

Proof. By continuity, there exist T_* such that

$$I(t) \geq 0, \text{ for all } t \in [0, T_*]. \tag{3.5}$$

By direct comptutation, we obtain for all $t \in [0, T]$,

$$\begin{aligned}
E(t) &= E(z) = \frac{1}{2}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + \frac{1}{2(\gamma+1)}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2(\gamma+1)} - \frac{1}{q}\|z\|_q^q \\
&= \frac{1}{2}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + \frac{1}{2(\gamma+1)}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2(\gamma+1)} \\
&\quad -\frac{1}{q}\left(\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2(\gamma+1)} - I(t)\right) \\
&\geq \frac{q-2}{2q}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + \frac{q-2(\gamma+1)}{2(\gamma+1)q}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2(\gamma+1)} + \frac{1}{q} I(t).
\end{aligned}$$

By using (3.5), we have

$$\frac{q-2}{2q}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + \frac{q-2(\gamma+1)}{2(\gamma+1)q}\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2(\gamma+1)} \leq E(t), \ \forall t \in [0, T_*]. \tag{3.6}$$

By the definition of $E(t)$, we obtain

$$\begin{aligned}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2} &\leq \frac{2q}{q-2}E(t) \\ &\leq \frac{2q}{q-2}E(0) \qquad (3.7)\end{aligned}$$

and

$$\begin{aligned}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2(\gamma+1)} &\leq \frac{2q(\gamma+1)}{q-2(\gamma+1)}E(t) \\ &\leq \frac{2q(\gamma+1)}{q-2(\gamma+1)}E(0). \qquad (3.8)\end{aligned}$$

On the other hand, we get

$$\|z\|_q^q = \alpha\|z\|_q^q + (1-\alpha)\|z\|_q^q.$$

By the embedding of $H_0^m(\Omega) \hookrightarrow L^q(\Omega)$, we have

$$\begin{aligned}\|z\|_q^q &\leq \alpha c_*^q \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{q} + (1-\alpha)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{q} \\ &= \alpha c_*^q \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{q-2}.\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2} \\ &\quad + (1-\alpha)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{q-2(\gamma+1)}.\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2(\gamma+1)}.\end{aligned}$$

From (3.7) and (3.8), we have

$$\begin{aligned}\int_\Omega |z|^q\,dx &\leq \alpha c_*^q\left(\frac{2q}{q-2}E(0)\right)^{\frac{q-2}{2}}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2} \\ &\quad + (1-\alpha)c_*^q\left(\frac{2q(\gamma+1)}{q-2(\gamma+1)}E(0)\right)^{\frac{q-2(\gamma+1)}{2(\gamma+1)}}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2(\gamma+1)}, \forall t\in[0,T_*],\end{aligned}$$

which implies that

$$\int_\Omega |z|^q\,dx \leq \beta_1\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2} + \beta_2\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2(\gamma+1)}, \forall t\in[0,T_*]. \qquad (3.9)$$

Since $\beta_1 + \beta_2 < 1$, so

$$\int_\Omega |z|^q \, dx \leq \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)}, \ \forall t \in [0, T_*] .$$

This implies that

$$I(t) > 0, \ \forall t \in [0, T_*] .$$

By repeating the above procedure, we can extend T_* to T. □

Theorem 3.1. *Under the assumptions of above lemma, the local solution of (1.1) is global.*

Proof. It is sufficient to show that $\left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)}$ is bounded independently of t. By using (3.1) and (3.2), we have

$$\begin{aligned} E(t) &= \frac{1}{2} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \frac{1}{2(\gamma+1)} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} - \frac{1}{q} \|z\|_q^q \\ &= \frac{q-2}{2q} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \frac{q-2(\gamma+1)}{2(\gamma+1)q} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} + \frac{1}{q} I(t) . \end{aligned}$$

Since $I(t) \geq 0$, which implies that

$$\left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} \leq C E(t) ,$$

so

$$\left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2(\gamma+1)} \leq C E(0) ,$$

where $C = \max \left\{ \frac{2q}{q-2}, \ \frac{2(\gamma+1)q}{q-2(\gamma+1)} \right\}$. □

4 Decay of solutions

In this part, we prove the decay of solutions by using the Komornik's lemma.

Lemma 4.1. *Assume that the assumptions of Lemma 3.1 and $p > 2$, hold. Then, there exist a positive constant c such that*

$$\|z\|_p^p \leq c E(t) . \tag{4.1}$$

Proof. By direct computation, we get

$$\begin{aligned} \|z\|_p^p &\leq c_*^p \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^p \\ &\leq c_*^p \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{p-2} . \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 \\ &\leq c_*^p \left(\frac{2q}{q-2} E(0) \right)^{\frac{p-2}{2}} . \frac{2q}{q-2} E(t) \\ &\leq cE(t). \end{aligned}$$

□

Theorem 4.1. *Suppose that the assumptions of Lemma 3.1. Then, there exists constants $C > 0$, such that*

$$E(t) \leq E(0) \left(\frac{C + qt}{C + qC} \right)^{\frac{-1}{q}}, \text{ for all } t \geq C.$$

Proof. Multiplying the equation of (1.1) by $z(t) E^q(t), (q > 0)$ and integrating over $\Omega \times (S, T)$, we get

$$\begin{aligned} &\int_S^T \int_\Omega E^q(t) \left[z z_t - z \left(M \left(\int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx \right) \mathcal{A} z + |z_t|^{p-2} z \right) \right] dx dt \\ = &\int_S^T E^q(t) \int_\Omega |z|^q \, dx dt. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_S^T \int_\Omega E^q(t) \left[z z_t + \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 + \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{2\gamma} \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 + z |z|^{p-2} z_t \right] dx dt \\ = &\int_S^T E^q(t) \int_\Omega |z|^q \, dx dt. \end{aligned}$$

We add and substract the term

$$\int_S^T E^q(t) \int_\Omega \left[\beta_1 \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 + \beta_2 \left\| \mathcal{A}^{\frac{1}{2}} z \right\|_2^{2\gamma} \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 \right] dx dt$$

and use (3.9) to obtain

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega \left[\left|\mathcal{A}^{\frac{1}{2}}z\right|^2 dxdt\right] \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left[\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2\gamma}\left|\mathcal{A}^{\frac{1}{2}}z\right|^2\right] dxdt \\
&+\int_S^T E^q(t)\int_\Omega [zz_t]\, dxdt \\
&+\int_S^T E^q(t)\int_\Omega zz_t\left|z^{p-2}\right| dxdt \\
&= -\int_S^T E^q(t)\int_\Omega \left[\beta_1\left|\mathcal{A}^{\frac{1}{2}}z\right|^2+\beta_2\left\|\mathcal{A}^{\frac{1}{2}}z\right\|_2^{2\gamma}\left|\mathcal{A}^{\frac{1}{2}}z\right|^2-|z|\right] dxdt \le 0. \quad (4.2)
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\xi\int_S^T E^q(t)\int_\Omega \left[\frac{1}{2}\left|\mathcal{A}^{\frac{1}{2}}z\right|^2+\frac{1}{2(\gamma+1)}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2\gamma}\left|\mathcal{A}^{\frac{1}{2}}z\right|^2-\frac{|z|^q}{q}\right] dxdt \\
&\le (1-\beta_1)\int_S^T E^q(t)\int_\Omega \left[\frac{1}{2}\left|\mathcal{A}^{\frac{1}{2}}z\right|^2\right] dxdt \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left[\frac{1}{2(\gamma+1)}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2\gamma}\left|\mathcal{A}^{\frac{1}{2}}z\right|^2\right] dxdt \quad (4.3)
\end{aligned}$$

where $\xi=\min\{1-\beta_1,\ 1-\beta_2\}$. By use of (4.2), (4.3) and definition of $E(t)$, we have

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt \le &-\int_S^T E^q(t)\int_\Omega zz_t dxdt \\
&-\int_S^T E^q(t)\int_\Omega zz_t|z|^{p-2}\,dxdt. \quad (4.4)
\end{aligned}$$

We estimate the terms in the right-hand side of (4.4), for the first term, we use the following Young inequality, we have

$$XY \le \frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ X,Y\ge 0,\ \varepsilon>0$$

and $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$,

$$\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega zz_t dxdt \\
&\le \int_S^T E^q(t)\int_\Omega \left(\varepsilon c\,|z|^2 + c_\varepsilon\,|z_t|^2\right) dxdt. \qquad (4.5)
\end{aligned}$$

We use again the above Young inequality, we get

$$\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega zz_t\,|z|^{p-2}\,dxdt \\
&= -\int_S^T E^q(t)\int_\Omega |z|^{\frac{p-2}{2}}\,z_t\,|z|^{\frac{p-2}{2}}\,zdxdt \\
&\le \int_S^T E^q(t)\int_\Omega \left(\varepsilon c\,|z|^p + c_\varepsilon\,|z|^{p-2}\,z_t^2\right) dxdt. \qquad (4.6)
\end{aligned}$$

By (4.5) and (4.6), inequality (4.4), becomes

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt \le{}& \int_S^T E^q(t)\int_\Omega \left(\varepsilon c\,|z|^2 + c_\varepsilon\,|z|^2\,dxdt\right) \\
&+\int_S^T E^q(t)\int_\Omega \left(\varepsilon c\,|z|^p + c_\varepsilon\,|z|^{p-2}\,z_t^2 dxdt\right) \\
\le{}& \varepsilon c\int_S^T E^q(t)\int_\Omega \left[|z|^2+|z|^p\right] dxdt \\
&+c_\varepsilon\int_S^T E^q(t)\int_\Omega \left(|z_t|^2+|z|^{p-2}\,z_t^2\right) dxdt. \qquad (4.7)
\end{aligned}$$

By (3.10) and definition of $E'(t)$, we have

$$\xi\int_S^T E^{q+1}(t)\,dt \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon\int_S^T E^q(t)\int_\Omega \left(-E'(t)\right) dxdt. \qquad (4.8)$$

This implies

$$\xi \int_S^T E^{q+1}(t)\,dt \leq \ \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon \left[E^{q+1}(S) - E^{q+1}(T)\right]$$
$$\leq \ \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E^q(0)\,E(S)\,. \tag{4.9}$$

Choosing ε small enough such that, we arrive at

$$\int_S^T E^{q+1}(t)\,dt \leq cE^q(0)\,E(S)\,.$$

By taking T goes to ∞, we have

$$\int_S^\infty E^{q+1}(t)\,dt \leq cE^q(0)\,E(S)\,.$$

Komornik's integral inequality yields the desired result. □

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