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A new characterization of groups $^2\mathrm{E}_6(\mathrm{q})$

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Abstract: In this paper, we prove that the simple groups $^2E_6(q)$, $q = 2^n$ and $\frac{q^6-q^3+1}{3}$ where is prime numbers can be uniquely determined by its order and the largest elements order.

1 Introduction

For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the largest element of the set $\pi_e(G)$ of element orders of G is denoted by $k(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, we assume that $2 \in \pi_1$.
If H be a finite group such that $|G| = |H|$ and $k(G) = k(H)$ implies that $H \cong G$, then we say G is recognizable by its order and the largest elements order. In the way, the authors try to characterize some finite simple groups by using less quantities and have successfully characterized simple $L_3(q)$ and $U_3(q)$, where q is some special power of prime, by using three numbers: the order of group, the largest and the second largest element orders, of which some results can be seen in [16]. Also, in [3], Chen and He proved the group $L_2(q)$ where $q = p^n < 125$ is recognizable by largest element order and group order, also in [4], Chen and He proved K_4 -groups of type $L_2(p)$ are reconizable only by using the order of a group and the

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largest element order, where p is a prime but not of the form 2^n-1. Moreover, the Authors in ([2], [6], [7], [8], [9], [10], [11], [12], [13], [20]) proved that groups such as, the simple sporadic groups, $PGL(2,q)$, $PSU_3(3^n)$, symplectics groups $PSP(8,q)$, simple groups $C_4(q)$, the simple groups ${}^2D_8(2^n)^2$, symplectic groups $PSP(4,2^n)$, simple groups ${}^2D_n(3)$, projective special linear groups $PSL(5,2)$ and $PSL(4,5)$ and suzuki group $Sz(q)$, where $q-1$ or $q \pm \sqrt{2q}+1$ is a prime number by largest element order proved.
In this paper, we prove that the groups ${}^2E_6(q)$, $q=2^n$, where $\frac{q^6-^3+1}{3}$ is a prime number is recognizable by the largest elements order and order of the group. In fact, we prove the following main theorem:

Main Theorem. Let G be a group such that $k(G) = k({}^2E_6(q))$ and $|G| = |{}^2E_6(q)|$, $q=2^n$, and where $\frac{q^6-q^3+1}{3}$ is a prime number. Then, $G \cong {}^2E_6(q)$.

2 Notations and preliminaries

Lemma 2.1. *[17] Let H be a finite soluble group all of whose elements are of a power prime order. Then, $|\pi(H)| \leq 2$.*

Lemma 2.2. *[15] Let G be a Frobenius group of even order with kernel K and complement H. Then,*

1. *$t(G)=2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K|-1$;*
3. *K is nilpotent;*
4. *Every subgroup of H of order p and q (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of H of odd order is cyclic and a Sylow 2-subgroup of H is either cyclic or a generalized quaternion group. If H is non-solvable then H has a subgroup of index at most 2 isomorphic to $SL_2(5) \times M$, where M has cyclic Sylow p-subgroups and order coprime to 2, 3 and 5.*

Definition 2.1. *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H, respectively.*

Lemma 2.3. *[3] Let G be a 2-Frobenius group of even order. Then,*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4. *[23] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements hold:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group;*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.5. *[24] Let q, k, l be natural numbers.Then*

1. *$(q^k - 1, q^l - 1) = q^{(k,\, l)} - 1$.*
2. $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,\, l)} + 1; & \textit{if both } \frac{k}{(k,\, l)} \textit{ and } \frac{l}{(k,\, l)} \textit{ are odd,} \\ (2, q+1); & \textit{otherwise.} \end{cases}$
3. $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,\, l)} + 1; & \textit{if } \frac{k}{(k,\, l)} \textit{ is even and } \frac{l}{(k,\, l)} \textit{ is odd,} \\ (2, q+1); & \textit{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3 Proof of the main theorem

In this section, we prove the main theorem. We denote groups $^2E_6(q)$, $q = 2^n$ and prime number $\frac{q^6 - q^3 + 1}{3}$ by E, p respectively. To prove the main theorem, we will prove several lemmas as follows. In the way, we note that $|E| =$

$\frac{q^{36}(q^{12}-1)(q^9+1)(q^8+1)(q^6-1)(q^5+1)(q^2-1)}{3}$ and also $k(E)=\frac{(q+1)(q^2+1)(q^3-1)}{3}$. Hence, we have the following theorem.

Theorem 3.1. *Let G be a group and $E := {}^2E_6(q)$, $q=2^n$ and $p=\frac{q^6-q^3+1}{3}$ is prime number. Then, $k(G)=k(E)$ and $|G|=|E|$ if and only if $G\cong E$.*

Lemma 3.1. *p is an isolated vertex in $\Gamma(G)$.*

Proof. We prove is an isolated vertex of $\Gamma(G)$. On opposite, there is prime number t in $\pi(G)$ such that $t\neq p$ and $tp\in\pi_e(G)$. Thus, we deduce that $tp\geq 2p\geq 2(\frac{q^6-q^3+1}{3})>\frac{(q+1)(q^2+1)(q^3-1)}{3}$. Hence, $k(G)>\frac{(q+1)(q^2+1)(q^3-1)}{3}$, which is impossible. So, we conclude that p is an isolated vertex of $\Gamma(G)$ and $t(G)\geq 2$. Now, Lemma 2.4 implies that G satisfies one of the following cases. □

Lemma 3.2. *G is nonsoluble.*

Proof. Let r be a prime divisor of $\frac{q^6-q^3+1}{3}$ and also $r\neq 3$, $r\neq p$. If G were soluble. Then, there would exist a $\{p,r,s\}$-Hall subgroup H of G. Since, F does not contain any element of orders pr, ps, rs. Thus all of elements of $\{p,r,3\}$-Hall subgroup H of G. Since, E does not contain any elements of orders pr, $3p$, $3r$. Thus, all of elements of H would be of prime power order. But this contradicts by Lemma 2.1. So, G is nonsoluble. □

Lemma 3.3. *The group G is neither a Frobenius group nor 2-Frobenius group.*

Proof. By Lemma 3.2, G is nonsoluble. Now, we prove that G is not a Frobenius group. On the contrary, we assume G be a Frobenius group with kernel K and complement H. Then, by Lemma 2.2, $t(G)=2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K|-1$. Since, H be a nonsoluble Frobenius complement, by Lemma2.2, H has a normal subgroup H_0 of index ≤ 2 such that $H_0\cong SL(2,5)\times Z$ where every sylow subgroup of Z cyclic and $\pi(Z)\cap\{2,3,5\}=\emptyset$. But $5\in\pi_e(G)$, which is a contradiction. Hence, G is not a Frobenius group. The other case is impossible as G is not 2-Frobenius group. Similarily. □

Lemma 3.4. *G is isomorphic to E.*

Proof. By the third case of Lemma 2.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, and also K/H is a non-abelian simple group. In the other hand, every odd order component of G is the odd order component of K/H. Since, $p \mid K/H$ so $t(k/H) \geq 2$. So according to the classification of the finite simple groups, we know that the possibilites for K/H are alternating group A_m, $m \geq 5$, one of the 26 sporadic groups, simple groups of Lie type. First, we assume that $G \cong E$. Then, we can easily prove that $k(G) = k(E)$ and $|G| = |E|$. Now, we need prove sufficient condition, that is, if $k(G) = k(E)$ and $|G| = |E|$, then $G \cong E$. For this purpose, we know by [18], $k(E) = \frac{(q+1)(q^2+1)(q^3-1)}{3}$, where this is an odd component of E and also $|E| = \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{3}$. Now, since K/H is a non-abelian simple group. So, K/H is isomorphic one of the following groups. □

Step 1. Let $K/H \cong A_m$, where $m \geq 5$ and $m = r, r+1, r+2$. Then, by [23] $\pi(A_m) \subseteq r, r-2$ and $|A_m| \mid |G|$, we consider $m \geq \frac{(q+1)(q^2+1)(q^3-1)}{3}$. In the way, since $|A_m| \nmid |G|$, so we have a contradiction.
Step 2. If K/H is isomorphic to sporadic groups, then by [18], $k(S) = \{5, 7, 11, 17, 19, 23, 31, 37, 59\}$. Now, we consider $\frac{(q+1)(q^2+1)(q^3-1)}{3} = 5, 7, 11, 17, 19, 23, 31, 37, 59$. In the way, for example, if $\frac{(q+1)(q^2+1)(q^3-1)}{3} = 7$, then we can easily, see these equation is impossible. If $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1 = 7$, then we have a contradiction. Similarily, for other groups, we have a contradiction.
Step 3. In this case, we consider K/H is isomorphic to a the group of Lie-type.
Case 3.1. $K/H \ncong B_n(q')$, where $n > 2$ and $C_n(q')$ with $n > 3$, and also q' is a prime power. For this purpose, we consider $K/H \cong B_n(q')$. Now, by [18], $k(B_n(q')) = q'^n + q'$ and $|B_n(q')| = \frac{1}{(2,q'-1)} q'^{n^2} \prod_{i=1}^{n}(q'^{2i}-1)$. Since, $|B_n(q')| \mid |G|$. So, $\frac{1}{(2,q'-1)} q'^{n^2} \prod_{i=1}^{n}(q'^{2i}-1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{3}$. Now, we consider $q'^n + q' = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. So, $3q'(q'^{n-1}+1) = (q-1)(q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1)$, which is impossible. For, $K/H \ncong C_n(q')$, similarily, we have a contradiction.
Case 3.2. If $K/H \cong {}^3D_4(q')$, then by [18], $k({}^3D_4(q')) = (q'^3-1)(q'+1)$. Also, we have $|{}^3D_4(q')| = q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1)$. Since $|{}^3D_4(q')| \mid |G|$. So, $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{3}$. Now, we consider $(q'^3 - 1)(q' + 1) = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. So, $3(q'^4 + q'^3 - q' - 1) = (q-1)(q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1)$. Thus, $q - 1 = 3$ and $q'^4 + q'^3 - q' - 1 = q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1$. As a result, $q'(q'^3 + q'^2 - 1) =$

$2(2^{5n-1}+2^{4n}+3(2^{3n-1})+3(2^{2n-1})+2^n+1$ which is impssible.

Case 3.3. $K/H \cong E_6(q'), E_7(q'), E_8(q'), F_4(q')$. For example, if $K/H \cong F_4(q')$, then by [18], $k(F_4(q')) = (q'^3-1)(q'+1)$. Also$|F_4(q')| = q'^{24}(q'^2-1)(q'^6-1)(q'^8-1)(q'^{12}-1)$. Since, $|F_4(q')| \mid |G|$. So, $q'^{24}(q'^2-1)(q'^6-1)(q'^8-1)(q'^{12}-1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{3}$. For this purpose, we consider $(q'^3-1)(q'+1) = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. Then, similar to the proof case 3.2, we have a contradiction. Similarly, for $K/H \ncong E_6(q')$,$E_7(q')$, $E_8(q')$, we have a contradiction.

Case 3.4. If $K/H \cong^2 E_6(q')$, then by [18]. $k(^2E_6(q') = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. Also, we have $|^2E_6(q')| = \frac{q'^{36}(q'^{12}-1)(q'^9+1)(q'^8-1)(q'^6-1)(q'^5+1)(q'^2-1)}{(3,q'+1)}$. Now, we consider $\frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)} = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. First, if $(3, q'-1) = 1$, then $3(q'^6+q'^5-q'-1) = (q-1)(q^5+2q^4+3q^3+3q^2+2q+1)$, which is a contradiction.

Case 3.5. If $K/H \cong^2 G_2(3^{2m+1})$, where $m \geq 1$, then by [18], $k(^2G_2(3^{2m+1}))) = 3^{2m+1}+3^{m+1}+1$. Also, we know that $|^2G_2(3^{2m+1}| = q'^3(q'^3+1)(q'-1)$. Since, $|^2G_2(3^{2m+1}| \mid |G|$. So, $q'^3(q'^3+1)(q'-1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1))}{3}$. For this purpose, we consider $3^{2m+1}+3^{m+1}+1 = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. So, $3^{m+1}(3^m+1) = (q-1)(q^5+2q^4+3q^3+3q^2+2q+1)$. Hence, $3(3^{2m+1}+2^{m+1}+1) = (q-1)(q^5+2q^4+3q^3+3q^2+2q+1)$. Now, since $(3, 3^{2m+1}+2^{m+1}+1) = 1$, so we deduce $q-1 = 3$, also $3^{2m+1}+2^{m+1}+1 = q^5+2q^4+3q^3+3q^2+2q+1$, which is a contradiction.

Case 3.6. If $K/H \cong^2 B_2(q')$, where $q' = 2^{2m+1}$, $m \geq 1$, then by [18], $k(^2B_2(q') = q'+\sqrt{2q'}+1$, also $|^2B_2(q')| = q'^2(q'^2+1)(q'-1)$. Since $|^2B_2(q')| \mid |G|$. So, $q'^2(q'^2+1)(q'-1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1))}{3}$. Now, we consider $q'+\sqrt{2q'}+1 == \frac{(q+1)(q^2+1)(q^3-1)}{3}$. So, $3(2^{2m+1}+2^{m+1}+1) = (q-1)(q^5+2q^4+3q^3+3q^2+2q+1)$, hence $3 = q-1$ and $2^{2m+1}+2^{m+1}+1) = q^5+2q^4+3q^3+3q^2+2q+1$, which is a contradiction.

Case 3.7. If $K/H \cong G_2(q')$, then by [18], $k(G_2(q') = q'^2+q'+1$, also $|G_2(q')| = q'^6(q'^6-1)(q'^2-1)$. Since $|G_2(q')| \mid |G|$. So, $q'^6(q'^6-1)(q'^2-1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1))}{3}$. For this purpose, we consider $q'^2+q'+1 = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. As a result, $3(q'^2+q'+1) = (q-1)(q^5+2q^4+3q^3+3q^2+2q+1)$, so $q-1 = 3$ and $q'(q'+1) = 2(2^{5n-1}+2^{4n-1}+3(2^{3n-1})+3(2^{2n-1})+2^{n+1}$, which is a contradiction. Since, $(q', q'+1) = 1$, so $q' = 2$ and $q'+1 = 2^{5n-1}+2^{4n-1}+3(2^{3n-1})+3(2^{2n-1})+2^{n+1}$, which is impossible.

Case 3.8. If $K/H \cong^2 A_n(q')$, where $n > 2$, then by [18], $k(^2A_n(q') = \frac{q'^{n+1}-1}{(3,\ q'+1)}$.

Also, we know that $|^2A_n(q')| = \frac{1}{(n+1,\, q'+1)} q'^{n(n+1)/2} \prod_{i=1}^{n}(q'^{i+1} - (1^{i+1})$. Since, $|^2A_n(q')| \mid |G|$. So, $\frac{1}{(n+1,\, q'+1)} q'^{n(n+1)/2} \prod_{i=1}^{n}(q'^{i+1} - (1^{i+1}) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1))}{3}$. For this purpose, we consider $\frac{q'^{n+1}-1}{(3,\, q+1)} = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. Now, if $(3, q'+1) = 1$, then $3(q'^{n+1} - 1) = (q-1)(q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1)$, so $3 = q - 1$ and $q'^{n+1} - 1 = q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1$, which is a contradiction. Similarly for $(3, q' + 1) = 3$, we have a contradiction.

Case 3.9. If $K/H \cong D_n(q')$, where $n \geq 4$, then by [18], $k(D_n(q') = \frac{(q'^{n-1}+1)(q'+1}{(4,q'-1)}$. Also, we know that $|D_n(q')| = \frac{1}{(4,\, q'^n-1)} q'^{n(n-1)}(q'^n - 1) \prod_{i=1}^{n-1}(q'^{2i} - 1)$. Since, $|D_n(q')| \mid |G|$. So, $\frac{1}{(4,\, q'^n-1)} q'^{n(n-1)}(q'^n - 1) \prod_{i=1}^{n-1}(q'^{2i} - 1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1))}{3}$. For this purpose, we consider $\frac{(q'^{n-1}+1)(q'+1}{(4,\, q'-1)} = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. Now, if $(4, q' - 1) = 1$, then $3(q'^n + q'^{n-2} + q' + 1) = 1 = (q-1)(q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1)$, which is a contradiction.

Case 3.10. If $K/H \cong L_{n+1}(q')$, where $n \geq 1$, then by [18] $k(L_{n+1}(q')) = \frac{q'^{n+1}-1}{q'-1(q'-1,n+1)}$. Also, we know that $|L_{n+1}(q')| = \frac{1}{(n+1,\, q'-1)} q'^{n(n+1)/2}(q'^n - 1) \prod_{i=1}^{n}(q'^{i+1} - 1)$. Since $|L_{n+1}(q')| \mid |G|$. So $\frac{1}{(n+1,\, q'-1)} q'^{n(n+1)/2}(q'^n - 1) \prod_{i=1}^{n}(q'^{i+1} - 1) \mid \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1))}{3}$. For this purpose, we consider $\frac{q'^{n+1}-1}{q'-1(q'-1,\, n+1)} = \frac{(q+1)(q^2+1)(q^3-1)}{3}$. Now, if $(q' - 1, n + 1) = 1$ then $3(q' - 1)(q'^n + q'^{n-1} + \cdots) = (q - 1)(q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1)$, which is impossible.

Case 3.11. So, $K/H \cong^2 E_6(q)$, as a result $|K/H| = |E|$. On the other hand, we know that $H \trianglelefteq K \trianglelefteq G$. Also, $k(K/H) \mid k(E)$, hence $\frac{(q'+1)(q'^2+1)(q'^3-1)}{3}$. Hence, $n = n'$. Now, since $|K/H| = |E|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong E$.

References

[1] G.Y. Chen, *On the structure of Frobenius groups and 2-Frobenius groups*, J. Southwest China Norm. Univ., **20**(5)(1995), 485-487.

[2] G. Y. Chen, L. G. He and J. H. Xu, *A new characterization of sporadic simple groups*, Ital. J. Pure Appl. Math., **30**(2013), 373-392.

[3] G. Y. Chen and L. G. He, *A new characterization of $L_2(q)$ where $q = p^n <$* 125 Ital. J. Pure Appl. math., **38**(2011), 125-134.

[4] G. Y. Chen and L. G. He, *A new characterization o simple K_4 -group with type $L_2(p)$* Adv. Math. (China) 2012, doi: 10.11845/sxjz.165b.

[5] H. Deng and W. J. Shi, *The characterization of Ree Groups $^2F_4(q)$ by their element orders,* J. Algebra, **217**(1999), 180-187.

[6] B. Ebrahimzadeh., A. Iranmanesh., A. Tehranian and H. Parvizi Mosaed, *A characterization of the suzuki groups by order and the largest elements order,* J. Sci. Islam. Repub. Iran, **27**(4)(2016), 353-355.

[7] B. Ebrahimzadeh, R. Mohammadyari, *A new characterization of projective special unitary groups $PSU_3(3^n)$,* Discuss¿ Math. Gen. Algebra Appl., **39** (2019), 35-41.

[8] B. Ebrahimzadeh, M. Y. Sadeghi, A. Iranmanesh, A. Tehranian, *A new characterization of symplectics groups $PSP(8, q)$,* An. Ştiint. Univ. Al. I. Cuza laşi. Mat. (N.S.), **66**, (1)(2020), 93-99.

[9] B. Ebrahimzadeh, R. Mohammadyari, M. Y. Sadeghi, *A new characterization of simple groups $C_4(q)$ by its order and the largest order of elements,* Acta. Comment. Univ. Tartu. Math., **23**(2)(2019), 283-290.

[10] B. Ebrahimzadeh, *Recognition of the simple groups $^2D_8(2^n)^2)$ by its order and the largest order of elements,* An. Univ. Vest Timiş. Ser. Mat.-Inform., **2**(2019), 1-8.

[11] B. Ebrahimzadeh, A. R. Khalili Asboei, *A characterization of symplectic groups related to Fermat primes,* Comment. Math. Univ. Carolin., **62**, (1)(2021), 33-40.

[12] B. Ebrahimzadeh, *A new characterization of simple groups $^2D_n(3)$,* Trans. Issue Math. Azerbaijan Natl. Acad. Sci., **41**(4)(2021), 57-62.

[13] B. Ebrahimzadeh, B. Azizi, *A characterization of projective special linear groups $PSL(5, 2)$ and $PSL(4, 5)$*, An. Ştiint. Univ. Al. I. Cuza laşi. Mat. (N.S.), **68**(1)(2022), 133-140.

[14] B. Ebrahimzadeh, On the Suzuki Groups, *Asian Journal of Pure and Applied Mathematics*, **3**, no.1, (2021), 67-71. *A characterization of the suzuki groups by order and the largest elements order*, J. Sci., Islam. Rep. Iran, **27**(4)(2016), 353-355.

[15] D. Gorenstein, *Finite groups*, Harper and Row, New York, (1980).

[16] L. G. He, G. Y. Chen, *A new characterization of* $L_3(q)$ $(q \leq 8)$ *and* $U_3(q)$ $(q \leq 11)$, J. Southwest Univ. (Nat. Sci. Ed.), **27**(33)(2011), 81-87.

[17] G. Higman, *Finite groups in which every element has prime power order*, J. London. Math. Soc., **32**(1957), 335-342.

[18] W. M. Kantor and A. Seress, *Large element orders and the characteristic of Lie-type simple groups*, J. Algebra., **322**(3)(2009), 802-832.

[19] A. S. Kondrat'ev, *Prime graph components of finite simple groups*, Math. Sbornik, **67**(1)(1990), 235-247.

[20] J. Li, W. J. Shi and D. Yu, *A characterization of some* $PGL(2, q)$ *by maximum element orders*, Bull. Korean Math. Soc., **322**(2009), 802-832.

[21] W. J. Shi, *Pure quantitative characterization of each finite simples groups*, J. PROG. NAT. Sci., **4**(3)(1994), 316-32.

[22] A. V. Vasilev, M. A. Grechkoseerva and V. D. Mazurrov, *Characterization of finite simple groups bye sepecrum and order*, *J. Algebra Logic*, **48**(6)(2009), 385-409.

[23] J. S. Williams, *Prime graph components of finite groups*, J. Algebra, **69**(2)(1981), 487-513.

[24] A. V. Zavarnitsine, *Recognition of the simple groups* $L_3(q)$ *by element orders*, J. Group Theory, **7**(1)(2004), 81-97.