The Aligarh Bulletin of Mathematics
Volume 43, Number 2 (2024), 67-90

ISSN: 0304-9787

A note on approximate controllability for second-order non-autonomous stochastic neutral integro-differential equations

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(Received: October 10, 2024 Accepted: January 1, 2025)

Abstract

This article focuses on investigating the existence of mild solutions and the approximate controllability of a second-order non-autonomous stochastic neutral integro-differential equation with infinite delay. Our findings are derived using the properties of the resolvent operator associated with the second-order abstract non-autonomous differential equation and fixed point theorems. An example is provided to demonstrate the obtained result.

1 Introduction

Second-order differential equations find extensive applications in mathematical and physical model problems, including the vibration of springs, electric circuits, population growth, and damping. Numerous authors have investigated the existence, controllability, and stability of second-order differential equations [5, 6, 23, 31]. The notion of controllability stands as a cornerstone in mathematical control

Keywords and phrases: Non-autonomous, stochastic system, controllability, neutral system, resolvent operator.

2020 AMS Subject Classification: 93B05, 34G20, 34K30, 34K40, 34K45.

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theory, serving pivotal roles in diverse control problems, such as time-optimal control problems [4], irreducibility of transition semigroups [7], and stabilization of unstable systems through feedback control [6]. Controllability generally refers to the ability to steer the dynamical control systems from initial to final states using admissible controls. Two fundamental theories of controllability emerge: approximate controllability and exact controllability. While most criteria in the literature are formulated for finite-dimensional systems, many unresolved issues persist in the case of infinite-dimensional systems concerning controllability. In the case of infinite-dimensional systems, controllability can be distinguished into approximate and exact controllability. Numerous authors have investigated the existence, controllability, and approximate controllability of stochastic differential equations in both first and second order [5, 7, 14, 20, 21, 23, 25–28, 30, 33].

In recent years, various authors [1–3, 8, 16, 17] have explored the existence of abstract second-order initial value problems for non-autonomous systems. Grimmer et al. [9, 10] have examined analytic resolvent operators for integral equations in Banach spaces, specifically focusing on resolvent operators for such equations. Henríquez et al. [13] studied the existence of solutions for a second-order abstract functional Cauchy problem with nonlocal conditions. Additionally, in [15], Henríquez investigated the existence of solutions for non-autonomous abstract Cauchy problems of second-order integro-differential equations using resolvent operators instead of cosine family operators. Henríquez also explored the existence of solutions for non-autonomous second-order functional differential equations with infinite delay by employing the Leray-Schauder alternative fixed point theorem in [16].

In nature, randomness inevitably disrupts everything. Stochastic differential equations have emerged as a more adept tool for describing real-world dynamics, leading to their rapid development. Therefore, it is crucial to account for stochastic factors. Scholars have become increasingly intrigued by the issue of approximate controllability in stochastic differential equations. The study of approximate controllability in second-order neutral stochastic nonautonomous integrodifferential equations employing resolvent operators has not been studied yet. This is the motivation of our work.

In this paper, we study approximate controllability of the following neutral integro-differential stochastic equation:

$$\begin{cases} \dfrac{d^2}{dt^2}\big[\vartheta(t)+\varphi(t,\vartheta_t)\big] = A(t)\big[\vartheta(t)+\varphi(t,\vartheta_t)\big] \\ +\displaystyle\int_0^t \zeta(t,s)\big[\vartheta(s)+\varphi(s,\vartheta_s)\big]ds + B\upsilon(t) \\ +\varrho\big(t,\vartheta(t)\big)+\tau\big(t,\vartheta(t)\big)\dfrac{dw(t)}{dt}, \quad t\in J=[0,\ell], \\ \vartheta(t)=\mu(t)\in L_2(\Omega,\wp), \quad t\in J_0=(-\infty,0], \\ \vartheta'(0)=\alpha^1\in H, \end{cases} \tag{1.1}$$

where the state function $\vartheta(\cdot)$ takes the values in a real separable Hilbert space $(H,\|\cdot\|)$. The operator $A(t):D(A(t))\subseteq H\to H$ is closed and its domain is dense in H. The operator $\zeta(t,s):D(\zeta)\subseteq H\to H$ are closed linear operators and its domain is independent of (t,s). The function $\vartheta_t:(-\infty,0]\to H$ defined by $\vartheta_t(s)=\vartheta(t+s),\ s\in(-\infty,0]$ belongs to the some abstract phase space $\wp$ described axiomatically. Also the linear operator $B:U\to H$ is bounded, where U is a real separable Hilbert space, the control $\upsilon(\cdot)\in L^2(J,U)$, a Hilbert space of admissible control functions. $w(\cdot)$ is a K-valued Wiener process with a covariance operator Q defined on a complete probability space $(\Omega,\Upsilon,\mathbb{P})$ with a normal filtration $\{\Upsilon_t\}_{t\geq 0}$ and K is another real separable Hilbert space. Moreover, $L(K,H)$ denotes the set of all bounded linear operators. The functions $\varphi:J\times\wp\to H$, $\varrho:J\times H\to H$, $\tau:J\times H\to L_Q(K,H)$ satisfy some conditions. μ and α^1 are H-valued random variables.

This paper is divided into four sections. The first two sections contain the introduction, notations, some required definitions, assumptions, and lemmas. The third section is concerned with the existence of a mild solution and the system's controllability. In the end, an example is provided as an application.

2 Preliminaries and assumptions

Let $(\Omega,\Upsilon,\mathbb{P})$ be a complete probability space. $\Upsilon_t,\ t\in J$ is a normal filtration and Υ_0 contains all $\mathbb{P}$-null sets. w denotes Q-Wiener process on $(\Omega,\Upsilon,\mathbb{P})$ with a bounded nuclear covariance operator Q, where $Qe_n=\lambda_n e_n,\ n=1,2,\dots,$ and $Tr(Q)=\sum_{n=1}^{\infty}\lambda_n$, where λ_n denotes the bounded sequence of non-negative real numbers and $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in K. We define $w(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}w_n(t)e_n$, where $w_n(t),\ n=1,2,\dots$ are mutually independent one-dimensional standard Brownian motions over probability space $(\Omega,\Upsilon,\mathbb{P})$. For each

$\psi \in L(K, H)$, we define

$$\|\psi\|_Q^2 = Tr(\psi Q \psi^*) = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \psi e_n \right\|^2.$$

If the value of $\|\psi\|^2$ is finite, then ψ is said to be a Q-Hilbert Schmidt operator. Let the space of all Q-Hilbert-Schmidt operators $\psi : K \to H$ be denoted as $L_Q(K, H)$. The completion $L_Q(K, H)$ of $L(K, H)$ equipped with the norm $\|\cdot\|_Q$, where $\|\psi\|_Q^2 = <\psi, \psi>$ forms a Hilbert space.

The space $D(A)$, equipped with the graph norm induced by the operator $A(t)$ is a Banach space. We will assume that all of these norms are equivalent. A straightforward condition for ensuring this property is the existence of a $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, such that $(\lambda I - A(t))^{-1} : H \to D(A)$ is a bounded linear operator. Hereafter, we denote by $[D(A)]$, the vector space $D(A)$ equipped with any of these equivalent norms. We define the norm as follows:

$$\|\vartheta\|_{[D(A)]} = \|\vartheta\| + \|A(t)\vartheta\|, \quad \vartheta \in D(A).$$

Recently, there has been growing interest in examining the abstract second-order nonautonomous initial value problem:

$$\begin{aligned} &\vartheta''(t) = A(t)\vartheta(t) + f(t), \quad 0 \le s,\ t \le \ell, \\ &\vartheta(s) = \vartheta^0, \quad \vartheta'(s) = \vartheta^1, \end{aligned} \tag{2.1}$$

where $A(t) : D(A) \subseteq H \to H,\ t \in [0, \ell]$, is a densely defined closed linear operator and $f : [0, \ell] \to H$ is an appropriate function. For a detailed discussion on this type of problem, we refer readers to [3, 12, 18, 19, 29, 30].

In most of the works, the authors discussed that the existence of solutions to system (2.1) is related to the existence of an evolution operator $S(t, s)$ for the homogeneous system:

$$\vartheta''(t) = A(t)\vartheta(t), \quad 0 \le t \le \ell.$$

Let us assume that $t \to A(t)\vartheta$ is continuous for every $\vartheta \in D(A)$. We also assume that $A(\cdot)$ generates the family $\{S(t, s)\}_{0 \le s \le t \le \ell}$ as discussed by Kozak [19, Definition 2.1], (see also to Henriquez [16, Definition 1.1]). Here, we only consider that $S(\cdot, \cdot)\vartheta$ is continuously differentiable for every $\vartheta \in H$ with its derivative uniformly bounded on bounded intervals. This particularly implies that there exists a constant $M_1 > 0$ such that

$$\|S(t+h,s) - S(t,s)\| \leq M_1|h|, \quad \text{for all} \quad s,t,t+h \in [0,\ell].$$

We introduce another operator $C(t,s) = -\frac{\partial S(t,s)}{\partial s}$. Let $f : [0,\ell] \to H$ be an integrable function.

For each fixed $0 \leq s \leq \ell$, we define the mild solution ϑ of system (2.1) by

$$\vartheta(t) = C(t,s)\vartheta^0 + S(t,s)\vartheta^1 + \int_s^t S(t,\tau)f(\tau)d\tau, \quad s \leq t \leq \ell.$$

Next, we consider the second order integro-differential equation:

$$\begin{aligned} &\vartheta''(t) = A(t)\vartheta(t) + \int_s^t \zeta(t,\tau)\vartheta(\tau)d\tau, \quad s \leq t \leq \ell, \\ &\vartheta(s) = 0, \quad \vartheta'(s) = x \in H, \quad 0 \leq s \leq \ell. \end{aligned} \tag{2.2}$$

This particular problem was addressed in [18]. We denote $\Delta = \{(t,s) : 0 \leq s \leq t \leq \ell\}$. Below, we outline certain conditions that the operator $\zeta(\cdot)$ satisfies, as presented in [18].

(B1) For every $0 \leq s \leq t \leq \ell$ the operator $\zeta(t,s) : [D(A)] \to H$ is a bounded linear operator. Furthermore, for each $\vartheta \in D(A)$, the map $\zeta(\cdot,\cdot)\vartheta$ is continuous, and it satisfies the inequality

$$\|\zeta(t,s)\vartheta\| \leq b\,\|\vartheta\|_{[D(A)]}\,,$$

where $b > 0$ is a constant independent of s and t in Δ.

(B2) There exists a positive constant $L\zeta$ such that for every $\vartheta \in D(A)$ and $0 \leq s \leq t_1 \leq t_2 \leq \ell$, the following inequality holds:

$$\left\|\zeta(t_2,s)\vartheta - \zeta(t_1,s)\vartheta\right\| \leq L_\zeta|t_2 - t_1|\left\|\vartheta\right\|_{[D(A)]}.$$

(B3) There exists a positive constant b_1 such that for each $\vartheta \in D(A)$ and $0 \leq t_1 \leq t \leq \ell$, the inequality

$$\left\|\int_{t_1}^t S(t,s)\zeta(s,t_1)\vartheta ds\right\| \leq b_1\|\vartheta\|,$$

holds.

Under these conditions, it has been demonstrated that there exists a family $(\Re(t,s))_{t\geq s}$ associated with the problem described by equation (2.2).

Definition 2.1. *A two-parameter family $\Re(t,s)t \geq s$ on H is defined as a resolvent operator for the system (2.2) if it satisfies the following conditions:*

(i) The mapping $\Re : \Delta \to L(H)$ is strongly continuous, $\Re(t,\cdot)\vartheta$ is continuously differentiable for all $\vartheta \in H$, and it meets the conditions:

$$\Re(s,s) = 0, \quad \frac{\partial}{\partial t}\Re(t,s)\Big|_{t=s} = I, \quad \textit{and} \quad \frac{\partial}{\partial s}\Re(t,s)\Big|_{s=t} = -I.$$

(ii) Given $\vartheta \in D(A)$, the mapping $\Re(\cdot,s)\vartheta$ is a solution for the system (2.2), expressed as:

$$\frac{\partial^2}{\partial t^2}\Re(t,s)\vartheta = A(t)\Re(t,s)\vartheta + \int_s^t \zeta(t,\tau)\Re(\tau,s)\vartheta d\tau,$$

for all $0 \leq s \leq t \leq \ell$.

From condition (i), it follows that there exist positive constants M_1 and M_2, such that

$$\|\Re(t,s)\| \leq M_1, \quad \left\|\frac{\partial}{\partial s}\Re(t,s)\right\| \leq M_2, \quad (t,s) \in \Delta.$$

Furthermore, the linear operator

$$\mathcal{F}(t,\sigma)\vartheta = \int_\sigma^t \zeta(t,s)\Re(s,\sigma)\vartheta ds, \quad \vartheta \in D(A),\ 0 \leq \sigma \leq t \leq \ell,$$

can be extended to H. The expression $\mathcal{F} : \Delta \to L(H)$ is strongly continuous and satisfies

$$\Re(t,\sigma)\vartheta = S(t,\sigma) + \int_\sigma^t S(t,s)\mathcal{F}(s,\sigma)\vartheta ds, \quad \text{for all} \quad \vartheta \in H. \tag{2.3}$$

As a result of this property, $\Re(\cdot)$ is uniformly Lipschitz continuous, meaning there exists a constant $L_\Re > 0$ such that

$$\|\Re(t+h,s) - \Re(t,s)\| \leq L_\Re|h|, \quad \text{for all} \quad t, t+h, s \in [0,\ell]. \tag{2.4}$$

Consider $g : [0, \ell] \to H$ as an integrable function. The non-homogeneous problem

$$\begin{aligned} &\vartheta''(t) = A(t)\vartheta(t) + \int_0^t \zeta(t,\tau)\vartheta(\tau)d\tau + g(t), \quad 0 \le t \le \ell, \\ &\vartheta(0) = \vartheta^0, \quad \vartheta'(0) = \vartheta^1, \end{aligned} \tag{2.5}$$

was discussed in [18]. Now, we introduce the mild solution for the system (2.5).

Definition 2.2. *Suppose $\vartheta^0, \vartheta^1 \in H$. Let $\vartheta : [0, \ell] \to H$ be defined by*

$$\vartheta(t) = -\frac{\partial \Re(t,s)\vartheta^0}{\partial s}\Big|_{s=0} + \Re(t,0)\vartheta^1 + \int_0^t \Re(t,s)g(s)ds.$$

This function $\vartheta(t)$ is termed as the mild solution for the system (2.5).

It's evident that the function $\vartheta(\cdot)$ in Definition 2.2 is continuous.

To investigate a retarded functional differential equation with infinite delay, it's essential to define the system with states within a suitably defined phase space $\wp$. Here, we adopt an axiomatic definition of the phase space as introduced by Hale and Kato [11]. The phase space $\wp$ is a vector space of functions from $(-\infty, 0]$ into H, equipped with the seminorm $\|\cdot\|_\wp$, which satisfies the following conditions:

(A1) If $\vartheta : (-\infty, \ell) \to H$ for $\ell > 0$ is continuous on $[0, \ell)$ and $\vartheta_0 \in \wp$, then for every $t \in [0, \ell)$, the subsequent conditions are satisfied:
(i) ϑ_t belongs to $\wp$.
(ii) $\|\vartheta(t)\|^2 \le K_1 \|\vartheta_t\|_\wp^2$.
(iii) $\|\vartheta_t\|_\wp^2 \le K_2(t) \sup\left\{\|\vartheta(s)\|^2 : 0 \le s \le t\right\} + K_3(t)\|\vartheta_0\|_\wp^2$,
where $K_1 > 0$ is a constant; K_2 and K_3 are functions defined on $[0, \infty)$, with $K_2(\cdot)$ being continuous and $K_3(\cdot)$ being locally bounded.

(A2) For $\vartheta(\cdot)$ as described in (A1), ϑ_t represents $\wp$-valued functions in $[0, \ell)$.

(A3) The space $\wp$ is complete.

The stochastic process $\vartheta_t : \Omega \to \wp$ for $t \ge 0$, is defined as $\vartheta_t = \{\vartheta(t+s)(w) : s \in (-\infty, 0]\}$. The collection of all H- valued, square integrable, random variables, denoted by $L_2(\Omega, H)$ forms a Banach space equipped with the norm

$$\|\vartheta(\cdot)\|_{L_2} = \left(\mathbb{E}\|(\cdot, w)\|_H^2\right)^{\frac{1}{2}},$$

where $\mathbb{E}$ is the expectation defined as

$$\mathbb{E}(h) = \int_{\Omega} h(w)d\mathbb{P}.$$

The set of all continuous maps from J_0 into $L_2(\Omega, H)$ i.e. $C(J_0, L_2(\Omega, H))$ is Banach space with

$$\sup_{t\in J_0} \mathbb{E}\big\|\vartheta(t)\big\|^2 < \infty.$$

Consider Z as the set all continuous processes ϑ that belongs to the space $C(J_0, L_2$ $(\Omega, \wp))$ with $\phi \in \wp$. The space Z is closed and form a Banach space with the norm defined as

$$\|\vartheta\|_Z = \left(\sup_{t\in J} \mathbb{E}\big\|\vartheta_t\big\|_{\wp}^2\right)^{\frac{1}{2}},$$

where $\|\vartheta_t\|_{\wp}^2 \leq \tilde{N}\mathbb{E}\|\phi\|_{\wp}^2 + \tilde{K}\sup\big\{\mathbb{E}\|\vartheta(s)\|^2 : 0 \leq s \leq \ell\big\}$, $\tilde{N} = \sup_{t\in J} N(t),\ \tilde{K} = \sup_{t\in J} K(t)$.

Lemma 2.1. *[22] For any $\vartheta_\ell \in L^2(\Omega, H)$, there exists a function $\varpi \in L^2_\Upsilon(J, L_2^0)$ such that*

$$\vartheta_\ell = \mathbb{E}\vartheta_\ell + \int_0^\ell \varpi(s)dw(s).$$

To define the mild solution of the neutral system (1.1), we take $\alpha^2 \in H$ such that $\frac{d}{dt}\varphi(t, \vartheta_t)\big|_{t=0} = \alpha^2$. Consequently, we define the mild solution of (1.1) as follows:

Definition 2.3. *[24] A stochastic process ϑ is said to be a mild solution of (1.1) if it satisfies the following conditions:*

(i) $\vartheta(t)$ is measurable and Υ_t-adapted.

(ii) ϑ is continuous on $[0, \ell]$, with $\vartheta_0 = \mu \in L_2(\Omega, \wp)$, and $\frac{d}{dt}[\vartheta(t)+\varphi(t, \vartheta_t)]\big|_{t=0} = \alpha^1 + \alpha^2$.

(iii) ϑ *satisfies the equation*

$$\begin{aligned}\vartheta(t) &= -\frac{\partial\Re(t,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0} + \Re(t,0)[\alpha^1+\alpha^2] - \varphi(t,\vartheta_t) \\ &\quad + \int_0^t \Re(t,s)\big[\varrho\big(s,\vartheta(s)\big) + B\upsilon(s)\big]ds + \int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)dw(s)\end{aligned}$$

for $t \in J$.

Let $\vartheta(\ell;\vartheta_0,\upsilon)$ be the state value of (1.1) at the terminal time $t=\ell$, corresponding to the control υ, with $\vartheta(t)=\mu(t)$ for $t\in\wp$. We define

$$\mathcal{R}(\ell,\vartheta_0) = \left\{\vartheta(\ell;\vartheta_0,\upsilon) : \upsilon \in L^2(I,U)\right\},$$

which is referred to as the reachable set of (1.1) at $t=\ell$. The closure of this set is denoted by $\overline{\mathcal{R}(\ell,\vartheta_0)}$.

Definition 2.4. *System (1.1) is considered to be approximately controllable on* J *if the reachable set* $\mathcal{R}(\ell,\vartheta_0)$ *is dense in* H*, that is* $\overline{\mathcal{R}(\ell,\vartheta_0)} = H$.

Consider the linear system described by:

$$\begin{aligned}&\frac{d^2}{dt^2}\big[\vartheta(t)+\varphi(t,\vartheta_t)\big] = A(t)\vartheta(t) + \int_0^t \zeta(t,s)\big[\vartheta(s) \\ &+\varphi(s,\vartheta_s)\big]ds + B\upsilon(t), \quad t\in[0,\ell], \\ &\vartheta(0)=\mu(0), \quad \vartheta'(0)=\alpha^1,\end{aligned} \tag{2.6}$$

It is convenient at this point to introduce the following controllability and resolvent operators associated with the system (2.6):

$$\begin{aligned}\Gamma_0^\ell &= \int_0^\ell \Re(\ell,s)BB^*\Re^*(\ell,s)ds : H \to H, \\ \mathcal{V}(\epsilon,\Gamma_0^\ell) &= (\epsilon I + \Gamma_0^\ell)^{-1} : H \to H,\end{aligned}$$

where B^* and $\Re^*$ denote the adjoints of B and $\Re$, respectively. It is straightforward that the operator Γ_0^ℓ is a linear bounded operator.

We also impose the following resolvent condition:

(H0) $\varepsilon\mathcal{V}(\varepsilon,\Gamma_0^\ell) = \varepsilon(\varepsilon I+\Gamma_0^\ell)^{-1} \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology.

According to [32], Hypothesis (H0) is satisfied if and only if the linear system (2.6) is approximately controllable on $[0,\ell]$.

3 Controllability result

To formulate the controllability problem in a way that makes the fixed point theorem easily applicable, we assume that the corresponding linear system (2.6) is approximately controllable. We will demonstrate that system (1.1) is approximately controllable, provided we can show that for all $\epsilon > 0$, there exists a continuous function $\vartheta \in C([0,\ell], H)$ such that

$$v^{\epsilon}(t,\vartheta) \quad = \quad B^{*}\Re^{*}(\ell,t)\mathcal{V}(\epsilon,\Gamma_0^{\ell})p(\vartheta), \tag{3.1}$$

where

$$\begin{aligned} p(\vartheta) \quad &= \mathbb{E}\vartheta_{\ell} + \textstyle\int_0^{\ell} \varpi(s)dw(s) + \frac{\partial \Re(\ell,s)\left[\mu(0)+\varphi(0,\mu)\right]}{\partial s}\Big|_{s=0} - \Re(\ell,0)\left[\alpha^1+\alpha^2\right] \\ &+\varphi(\ell,\vartheta_\ell) - \textstyle\int_0^{\ell}\Re(\ell,s)\varrho\big(s,\vartheta(s)\big)ds - \int_0^{\ell}\Re(\ell,s)\tau\big(s,\vartheta(s)\big)dw(s), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \vartheta(t) \quad &= \quad -\frac{\partial \Re(t,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0} + \Re(t,0)[\alpha^1+\alpha^2] - \varphi(t,\vartheta_t) \\ &+ \int_0^t \Re(t,s)\varrho\big(s,\vartheta(s)\big), ds + \int_0^t \Re(t,s)Bv^{\epsilon}(s,\vartheta), ds \\ &+ \int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)\, dw(s). \end{aligned} \tag{3.3}$$

To establish the result, we require the following hypotheses:

(H1) The resolvent operator $\Re(t,s)$, for $(t,s) \in \Delta$, is compact.

(H2) The operator B is bounded, i.e., $\|B\| \leq M_B$, where M_B is a positive constant.

(H3) The function $\varphi : [0,\ell] \times \wp \to H$ satisfies the following conditions:

- (i) For each $\vartheta \in \wp$, the function $\varphi(\cdot,\vartheta) : [0,\ell] \to H$ is strongly measurable.
- (ii) For every $t \in [0,\ell]$, $\varphi(t,\cdot) : \wp \to H$ is continuous.
- (iii) The function $\varphi : [0,\ell] \times \wp \to H$ satisfies the Lipschitz conditions, i.e., there exists a constant $0 < L_{\varphi} < 1$ such that

$$\|\varphi(t_1,\vartheta_1) - \varphi(t_2,\vartheta_2)\|^2 \leq L_{\varphi}\left[|t_1-t_2| + \|\vartheta_1-\vartheta_2\|_{\wp}^2\right],$$

for any $0 \leq t_1, t_2 \leq \ell$ and $\vartheta_1, \vartheta_2 \in \wp$. Moreover, there exists a function $f(\cdot) \in L^2(\mathbb{R})$ such that the inequality

$$\sup_{\|\vartheta\|^2 \leq r} \|\varphi(t, \vartheta)\| \leq f(r)$$

holds for every $\vartheta \in \wp$.

(H4) The function $\varrho : [0, \ell] \times H \to H$ satisfies the following conditions:

(i) For each $\vartheta \in H$, the function $\varrho(\cdot, \vartheta) : [0, \ell] \to H$ is strongly measurable.

(ii) For every $t \in [0, \ell]$, $\varrho(t, \cdot) : H \to H$ is continuous.

(iii) There exists a function $g(\cdot) \in L^2(\mathbb{R})$ such that the inequality

$$\sup_{\|\vartheta\|^2 \leq r} \|\varrho(t, \vartheta)\| \leq g(r)$$

holds for every $\vartheta \in H$.

(H5) The function $\tau : [0, \ell] \times H \to L_2^0$ satisfy the following conditions:

(i) For each $\vartheta \in H$, the function $\tau(\cdot, \vartheta)$ is measurable.

(ii) For every $t \in [0, \ell]$, $\tau(t, \cdot) : H \to L_2^0$ is continuous.

(iii) There exists function $h(\cdot) \in L^2(\mathbb{R})$ such that the inequality

$$\sup_{\|\vartheta\|^2 \leq r} \|\tau(t, \vartheta)\| \leq h(r)$$

holds for every $\vartheta \in H$.

For $\epsilon > 0$, we consider the operator $\Phi^\epsilon : Z \to Z$ defined as follows:

$$(\Phi^\epsilon \vartheta)(t) = \begin{cases} -\dfrac{\partial \Re(t,s)[\mu(0) + \varphi(0, \mu)]}{\partial s}\Big|_{s=0} + \Re(t, 0)[\alpha^1 + \alpha^2] - \varphi(t, \vartheta_t) \\ + \displaystyle\int_0^t \Re(t, s)\varrho\big(s, \vartheta(s)\big)ds + \int_0^t \Re(t, s)Bv^\epsilon(s, \vartheta)ds \\ + \displaystyle\int_0^t \Re(t, s)\tau\big(s, \vartheta(s)\big)dw(s), \quad t \in [0, \ell], \\ \mu(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.4}$$

We will demonstrate that for all $\epsilon > 0$, the operator Φ^ϵ possesses a fixed point.

Theorem 3.1. *If the assumption (H0)- (H5) hold, then for each $0 < \epsilon \leq 1$, the operator Φ^ϵ has a fixed point the following condition is satisfied:*

$$\lim_{r\to\infty}(r - K_1 f(r) - K_2 g(r) - K_3 h(r)) \to \infty, \tag{3.5}$$

with

$$K_1 = 1 + M_B^4 M_1^4 \frac{\ell}{\epsilon}, \quad K_2 = M_1^2 \ell K_1, \quad K_3 = M_1^2 Tr(Q)\ell K_1.$$

Proof. We define the set $D(r) = \{\vartheta \in Z : \mathbb{E}\|\vartheta(t)\|^2 \leq r,\ t \in [0,\ell]\}$. It is obvious that $D(r)$ is a bounded, closed and convex set in Z.

We first show that for any $0 < \epsilon \leq 1$, there exists a positive constant $r_0 = r_0(\epsilon)$ such that $\Phi^\epsilon(D(r_0)) \subseteq D(r_0)$. In fact, assume that for any $r > 0$, there exists a $\vartheta^r \in D(r)$, denoted simply by ϑ, and a $t_r \in (0,\ell]$ such that $\mathbb{E}\|(\Phi^\epsilon\vartheta)(t_r)\|^2 > r$. Then, using the conditions (3.1) and (3.2), we obtain

$$\begin{aligned}
&\mathbb{E}\|\upsilon^\epsilon(t,\vartheta)\|^2 \\
= \;& \mathbb{E}\|B^*\Re^*(\ell,t)(\epsilon I + \Gamma_0^\ell)^{-1}p(\vartheta)\|^2 \\
\leq \;& M_B^2 M_1^2 \frac{1}{\epsilon}\mathbb{E}\|p(\vartheta)\|^2 \\
\leq \;& M_B^2 M_1^2 \frac{1}{\epsilon}\Big[\mathbb{E}\|\vartheta_\ell\|^2 + \mathbb{E}\left\|\int_0^\ell \varpi(s)dw(s)\right\|^2 + M_2^2\Big[\mathbb{E}\|\mu(0)\|^2 + \mathbb{E}\|\varphi(0,\mu)\|^2\Big] \\
& + M_1^2\Big[\mathbb{E}\|\alpha^1\|^2 + \mathbb{E}\|\alpha^2\|^2\Big] + f(r) + M_1^2\ell g(r) + M_1^2 Tr(Q)\ell h(r)\Big].
\end{aligned}$$

Here, without loss of generality, we assume by condition (H0) that

$$\|\mathcal{V}(\epsilon,\Gamma_0^\ell)\| \leq \frac{1}{\epsilon}, \quad \text{for all} \quad \epsilon \in (0,1].$$

Then, we have

$$
\begin{aligned}
r &< \mathbb{E}\Big\|(\Phi^{\epsilon}\vartheta)(t_r)\Big\|^2 \\
&= \mathbb{E}\Big\| -\frac{\partial\Re(t,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0} + \Re(t,0)[\alpha^1+\alpha^2] - \varphi(t,\vartheta_t) \\
&\quad + \int_0^t \Re(t,s)\varrho\big(s,\vartheta(s)\big)ds + \int_0^t \Re(t,s)B\upsilon^{\epsilon}(s,\vartheta)ds \\
&\quad + \int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)dw(s)\Big\|^2 \\
&\leq \mathbb{E}\left\|\frac{\partial\Re(t,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0}\right\|^2 + \mathbb{E}\left\|\Re(t,0)[\alpha^1+\alpha^2]\right\|^2 + \mathbb{E}\left\|\varphi(t,\vartheta_t)\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^t \Re(t,s)\varrho\big(s,\vartheta(s)\big)ds\right\|^2 + \mathbb{E}\left\|\int_0^t \Re(t,s)B\upsilon^{\epsilon}(s,\vartheta)ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)dw(s)\right\|^2 \\
&\leq M_2^2\Big[\mathbb{E}\|\mu(0)\|^2 + \mathbb{E}\|\varphi(0,\mu)\|^2\Big] + M_1^2\Big[\mathbb{E}\|\alpha^1\|^2 + \mathbb{E}\|\alpha^2\|^2\Big] + f(r) \\
&\quad + M_1^2\ell g(r) + M_B^2M_1^2\ell \times M_B^2M_1^2\frac{1}{\epsilon}\bigg\{\mathbb{E}\|\vartheta_\ell\|^2 + \mathbb{E}\left\|\int_0^\ell \varpi(s)dw(s)\right\|^2 \\
&\quad + M_2^2\Big[\mathbb{E}\|\mu(0)\|^2 + \mathbb{E}\|\varphi(0,\mu)\|^2\Big] + M_1^2\Big[\mathbb{E}\|\alpha^1\|^2 + \mathbb{E}\|\alpha^2\|^2\Big] + f(r) \\
&\quad + M_1^2\ell g(r) + M_1^2Tr(Q)\ell h(r)\bigg\} + M_1^2Tr(Q)\ell h(r) \\
&= \left(1 + M_B^4M_1^4\frac{\ell}{\epsilon}\right)M_2^2\Big[\mathbb{E}\|\mu(0)\|^2 + \mathbb{E}\|\varphi(0,\mu)\|^2\Big] \\
&\quad + \left(1 + M_B^4M_1^4\frac{\ell}{\epsilon}\right)M_1^2\Big[\mathbb{E}\|\alpha^1\|^2 + \mathbb{E}\|\alpha^2\|^2\Big] \\
&\quad + M_B^4M_1^4\frac{\ell}{\epsilon}\left[\mathbb{E}\|\vartheta_\ell\|^2 + \mathbb{E}\left\|\int_0^\ell \varpi(s)dw(s)\right\|^2\right] \\
&\quad + \left(1 + M_B^4M_1^4\frac{\ell}{\epsilon}\right)f(r) + M_1^2\ell\left(1 + M_B^4M_1^4\frac{\ell}{\epsilon}\right)g(r) \\
&\quad + M_1^2Tr(Q)\ell\left(1 + M_B^4M_1^4\frac{\ell}{\epsilon}\right)h(r),
\end{aligned} \tag{3.6}
$$

or

$$\begin{aligned} r - K_1 f(r) - K_2 g(r) - K_3 h(r) &\leq \left(1 + M_B^4 M_1^4 \frac{\ell}{\epsilon}\right) M_2^2 \Big[\mathbb{E}\|\mu(0)\|^2 + \mathbb{E}\|\varphi(0,\mu)\|^2\Big] \\ &\quad + \left(1 + M_B^4 M_1^4 \frac{\ell}{\epsilon}\right) M_1^2 \Big[\mathbb{E}\|\alpha^1\|^2 + \mathbb{E}\|\alpha^2\|^2\Big] \\ &\quad + M_B^4 M_1^4 \frac{\ell}{\epsilon}\left[\mathbb{E}\|\vartheta_\ell\|^2 + \mathbb{E}\left\|\int_0^\ell \varpi(s) dw(s)\right\|^2\right] \\ &< \infty, \end{aligned}$$

which contradicts (3.5). Therefore, there exists a positive integer r_0 such that $\Phi^\epsilon(D(r_0)) \subseteq D(r_0)$.

Next, we will demonstrate that Φ^ϵ has a fixed point on $D(r_0)$. To achieve this, we decompose Φ^ϵ to Φ_1^ϵ and Φ_2^ϵ (i.e. $\Phi^\epsilon = \Phi_1^\epsilon + \Phi_2^\epsilon$), where the operators Φ_1^ϵ and Φ_2^ϵ are defined on $D(r_0)$ as follows:

$$(\Phi_1^\epsilon \vartheta)(t) = \begin{cases} -\dfrac{\partial \Re(t,s)}{\partial s}\varphi(0,\mu)\Big|_{s=0} + \Re(t,0)\alpha^2 - \varphi(t,\vartheta_t), & t \in [0,\ell], \\ 0, & t \in (-\infty, 0], \end{cases} \tag{3.7}$$

and

$$(\Phi_2^\epsilon \vartheta)(t) = \begin{cases} -\dfrac{\partial \Re(t,s)}{\partial s}\mu(0)\Big|_{s=0} + \Re(t,0)\alpha^2 + \displaystyle\int_0^t \Re(t,s)\varrho\big(s,\vartheta(s)\big)ds \\ + \displaystyle\int_0^t \Re(t,s) B v^\epsilon(s,\vartheta) ds + \int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)dw(s), \quad t \in [0,\ell], \\ \mu(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.8}$$

Next, we will demonstrate that Φ_1^ϵ is a contraction operator, while Φ_2^ϵ is completely continuous.

To prove that Φ_1^ϵ is a contraction, let $\vartheta_1, \vartheta_2 \in D(r_0)$. Then, for each $t \in [0,\ell]$, we have

$$\mathbb{E}\|(\Phi_1^\epsilon \vartheta_1)(t) - (\Phi_1^\epsilon \vartheta_2)(t)\|^2 \leq \mathbb{E}\|\varphi(t,\vartheta_{1,t}) - \varphi(t,\vartheta_{2,t})\|^2 \leq L_2 \mathbb{E}\|\vartheta_{1,t} - \vartheta_{2,t}\|^2.$$

Hence, Φ_1^ϵ is a contraction operator on $D(r_0)$.

Next, we prove that Φ_2^ϵ is completely continuous in several steps.

To establish the complete continuity of Φ_2^ϵ, we will prove in several steps.

Step 1: We first prove that Φ_2^ϵ is continuous on $D(r_0)$.

Let $\{\vartheta^m\} \subseteq D(r_0)$ with $\vartheta^m \to \vartheta$ as $m \to \infty$ in Z for some $\vartheta \in D(r_0)$, i.e.,

$$\mathbb{E}\|\vartheta^m - \vartheta\|^2 \leq \sup_{t\in(-\infty,\ell]} \mathbb{E}\|\vartheta^m(t) - \vartheta(t)\|^2 \to 0, \quad m \to \infty.$$

Then, for all $s \in [0, \ell]$,

$$\begin{aligned}
\mathbb{E}\|\vartheta_s^m - \vartheta_s\|^2 &\leq \sup_{\nu\in(-\infty,0]} \mathbb{E}\|\vartheta^m(s+\nu) - \vartheta(s+\nu)\|^2 \\
&= \sup_{\nu\in(-\infty,0]} \mathbb{E}\|\vartheta^m(t) - \vartheta(t)\|^2 \\
&\leq \mathbb{E}\|\vartheta^m - \vartheta\|^2 \to 0.
\end{aligned}$$

By the definition of Φ_2^ϵ, we have

$$\begin{aligned}
\mathbb{E}\|(\Phi_2^\epsilon\vartheta^m)(t) - (\Phi_2^\epsilon\vartheta)(t)\|^2 &\leq \mathbb{E}\left\|\int_0^t \Re(t,s)\Big[\varrho\big(s,\vartheta^m(s)\big) - \varrho\big(s,\vartheta(s)\big)\Big]ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^t \Re(t,s)B\Big[\upsilon^\epsilon(s,\vartheta^m) - \upsilon^\epsilon(s,\vartheta)\Big]ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^t \Re(t,s)\Big[\tau\big(s,\vartheta^m(s)\big) - \tau\big(s,\vartheta(s)\big)\Big]dw(s)\right\|^2.
\end{aligned}$$

Therefore, by applying the Lebesgue dominated convergence theorem, we have

$$\mathbb{E}\|(\Phi_2^\epsilon\vartheta^m)(t) - (\Phi_2^\epsilon\vartheta)(t)\|^2 \to 0, \quad \text{as} \quad m \to \infty,$$

implying that Φ_2^ϵ is continuous.

Step 2: Clearly, $\Phi_2^\epsilon(D(r_0)) = \{\Phi_2^\epsilon\vartheta : \vartheta \in D(r_0)\}$ is bounded in Z.

Step 3: To prove that the family of functions $\{(\Phi_2^\epsilon\vartheta)(\cdot) : \vartheta \in D(r_0)\} \subseteq Z$ is equicontinuous on the interval $(-\infty, \ell]$, it suffices to demonstrate this on $(0, \ell]$. Let $\vartheta \in D(r_0), t_1, t_2 \in (0, \ell]$ and $\lambda > 0$ be small enough such that $0 < \lambda < t_1 <$

$t_2 < \ell$. Then

$$
\begin{aligned}
& \mathbb{E}\|(\Phi_2^\epsilon\vartheta)(t_2) - (\Phi_2^\epsilon\vartheta)(t_1)\|^2 \\
\leq\ & \mathbb{E}\left\| - \left(\frac{\partial \Re(t_2,s)}{\partial s} - \frac{\partial \Re(t_1,s)}{\partial s} \right) \mu(0)\Big|_{s=0} \right\|^2 \\
& + \mathbb{E}\left\| \left[\Re(t_2,0) - \Re(t_1,0) \right] \alpha^2 \right\|^2 \\
& + \mathbb{E}\left\| \int_0^{t_2} \Re(t_2,s)\varrho\big(s,\vartheta(s)\big)ds - \int_0^{t_1} \Re(t_1,s)\varrho\big(s,\vartheta(s)\big)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_0^{t_2} \Re(t_2,s)B\upsilon^\epsilon(s,\vartheta)ds - \int_0^{t_1} \Re(t_1,s)B\upsilon^\epsilon(s,\vartheta)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_0^{t_2} \Re(t_2,s)\tau\big(s,\vartheta(s)\big)dw(s) - \int_0^{t_1} \Re(t_1,s)\tau\big(s,\vartheta(s)\big)dw(s) \right\|^2 \\
\leq\ & \mathbb{E}\left\| - \left(\frac{\partial \Re(t_2,s)}{\partial s} - \frac{\partial \Re(t_1,s)}{\partial s} \right) \mu(0)\Big|_{s=0} \right\|^2 \\
& + \mathbb{E}\left\| \left[\Re(t_2,0) - \Re(t_1,0) \right] \alpha^2 \right\|^2 \\
& + \mathbb{E}\left\| \int_0^{t_1-\epsilon} \Big[\Re(t_2,s) - \Re(t_1,s) \Big] \varrho\big(s,\vartheta(s)\big)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_{t_1-\epsilon}^{t_1} \Big[\Re(t_2,s) - \Re(t_1,s) \Big] \varrho\big(s,\vartheta(s)\big)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_{t_1}^{t_2} \Re(t_2,s)\varrho\big(s,\vartheta(s)\big)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_0^{t_1-\epsilon} \Big[\Re(t_2,s) - \Re(t_1,s) \Big] B\upsilon^\epsilon(s,\vartheta)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_{t_1-\epsilon}^{t_1} \Big[\Re(t_2,s) - \Re(t_1,s) \Big] B\upsilon^\epsilon(s,\vartheta)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_{t_1}^{t_2} \Re(t_2,s)B\upsilon^\epsilon(s,\vartheta)ds \right\|^2 \\
& + \mathbb{E}\left\| \int_0^{t_1-\epsilon} \Big[\Re(t_2,s) - \Re(t_1,s) \Big] \tau\big(s,\vartheta(s)\big)dw(s) \right\|^2 \\
& + \mathbb{E}\left\| \int_{t_1-\epsilon}^{t_1} \Big[\Re(t_2,s) - \Re(t_1,s) \Big] \tau\big(s,\vartheta(s)\big)dw(s) \right\|^2 \\
& + \mathbb{E}\left\| \int_{t_1}^{t_2} \Re(t_2,s)\tau\big(s,\vartheta(s)\big)dw(s) \right\|^2 .
\end{aligned}
$$

Indeed, as $t_2 \to t_1$ and $\epsilon > 0$ sufficiently small, the right-hand side of the above inequality converges to zero independently of $\vartheta \in D(r_0)$. Hence, Φ_2^ϵ maps $D(r_0)$ into an equicontinuous family of functions.

Step 4: Finally, we demonstrate that for fixed $t \in (-\infty, \ell]$, the set $V(t) = \{(\Phi_2^\epsilon \vartheta)(t) : \vartheta \in D(r_0)\}$ is relatively compact.

Clearly, for $t \in (-\infty, 0]$, $(\Phi_2^\epsilon \vartheta)(t) = \mu(t)$ and $V(t) = \{\mu(t)\}$ which is trivially relatively compact. Next, let $t \in [0, \ell]$ be fixed. We have

$$V(t) = -\frac{\partial \Re(t,s)}{\partial s}\mu(0)\Big|_{s=0} + \Re(t,0)\alpha^2 + V_1(t),$$

where $V_1(t)$ is given by

$$\begin{aligned} V_1(t) = \quad & \Big\{u(t) = \int_0^t \Re(t,s)\Big[\varrho\big(s,\vartheta(s)\big) + Bv^\epsilon(s,\vartheta)\Big]ds \\ & + \int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)dw(s) : \vartheta \in D(r_0)\Big\}. \end{aligned}$$

We only need to prove that $V_1(t)$ is relatively compact in H, since $-\frac{\partial \Re(t,s)}{\partial s}\mu(0)\Big|_{s=0} + \Re(t,0)\alpha^2$ is a single point in H. For any $u(t) \in V_1(t)$, we have

$$\begin{aligned} \mathbb{E}\|u(t)\|^2 & \leq \mathbb{E}\left\|\int_0^t \Re(t,s)\Big[\varrho\big(s,\vartheta(s)\big) + Bv^\epsilon(s,\vartheta)\Big]ds\right\|^2 \\ & + \mathbb{E}\left\|\int_0^t \Re(t,s)\tau\big(s,\vartheta(s)\big)dw(s)\right\|^2 \\ & \leq M_1^2\ell\big[g(r) + M_B^2\mathbb{E}\|v\|^2 + Tr(Q)h(r)\big] < \infty, \end{aligned}$$

which implies that $V_1(t)$ is bounded in H. Hence, $V_1(t)$ is relatively compact. Therefore, for each $t \in (-\infty, \ell]$, $(\Phi_2^\epsilon D(r_0))(t)$ is relatively compact in H. Thus, by Arzela–Ascoli theorem, Φ_2^ϵ is a compact operator.

These arguments lead us to conclude that $\Phi^\epsilon = \Phi_1^\epsilon + \Phi_2^\epsilon$ is a condensing map on $D(r_0)$. By Krasnoselskii's fixed point theorem, there exists a fixed point $\vartheta(\cdot)$ for Φ^ϵ on $D(r_0)$, which is a mild solution for the system (1.1). $\square$

In the next theorem, we will prove the approximate controllability of the system (1.1).

Theorem 3.2. *If all the assumptions of Theorem 3.1 hold, and the functions φ, ϱ and τ are uniformly bounded. Then the system (1.1) is approximately controllable on* $[0, \ell]$.

Proof. Let $\vartheta^\ell \in H$, $\epsilon \in (0,1)$, and $\vartheta^\epsilon(\cdot)$ be a fixed point of Φ^ϵ in $D(r_0)$. According to Theorem 3.1, $\vartheta^\epsilon(\cdot)$ is the mild solution of system (1.1) under the control function provided by equation (3.1). Consequently, $\vartheta^\epsilon(\cdot)$ satisfies the following:

$$\begin{aligned}
\vartheta^\epsilon(\ell) &= -\frac{\partial \Re(\ell,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0} + \Re(t,0)[\alpha^1+\alpha^2] - \varphi(\ell,\vartheta^\epsilon_\ell) \\
&\quad + \int_0^\ell \Re(\ell,s)\varrho\big(s,\vartheta^\epsilon(s)\big)ds + \int_0^\ell \Re(\ell,s)Bv^\epsilon(s,\vartheta^\epsilon)ds \\
&\quad + \int_0^\ell \Re(\ell,s)\tau\big(s,\vartheta^\epsilon(s)\big)dw(s) \\
&= -\frac{\partial \Re(\ell,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0} + \Re(\ell,0)[\alpha^1+\alpha^2] - \varphi(\ell,\vartheta^\epsilon_\ell) \\
&\quad + \int_0^\ell \Re(\ell,s)\varrho\big(s,\vartheta^\epsilon(s)\big)ds + \int_0^\ell \Re(\ell,s)BB^*\Re^*(\ell,\ell)\mathcal{V}(\epsilon,\Gamma_0^\ell)p(\vartheta^\epsilon)ds \\
&\quad + \int_0^\ell \Re(\ell,s)\tau\big(s,\vartheta^\epsilon(s)\big)dw(s) \\
&= \mathbb{E}\vartheta_\ell + \int_0^\ell \varpi(s)dw(s) - \epsilon\mathcal{V}(\epsilon,\Gamma_0^\ell)p(\vartheta^\epsilon). \qquad (3.9)
\end{aligned}$$

The uniform boundedness of the functions ϱ and τ and the compactness of $\Re(t,s)$ imply that there are subsequences of the terms $\int_0^\ell \Re(\ell,s)\varrho\big(s,\vartheta_\epsilon(s)\big)ds$, and $\int_0^\ell \Re(\ell,s)$ $\tau\big(s,\vartheta_\epsilon(s)\big)ds$, denoted by themselves respectively, that converge to $\hat{\varrho}$ and $\hat{\tau}$, respectively.

Let us take

$$\chi = \mathbb{E}\vartheta_\ell + \frac{\partial \Re(\ell,s)[\mu(0)+\varphi(0,\mu)]}{\partial s}\Big|_{s=0} + \int_0^\ell \varpi(s)dw(s) - \Re(\ell,0)\big[\alpha^1+\alpha^2\big] - \hat{\varrho} - \hat{\tau}.$$

Then, we have

$$\begin{aligned}\mathbb{E}\big\|p(\vartheta^{\epsilon})-\chi\big\|^{2} &\leq 2\mathbb{E}\bigg\|\int_{0}^{\ell}\Re(\ell,s)\varrho\big(s,\vartheta^{\epsilon}(s)\big)ds-\hat{\varrho}\bigg\|^{2}\\ &+ 2\mathbb{E}\bigg\|\int_{0}^{\ell}\Re(\ell,s)\tau\big(s,\vartheta^{\epsilon}(s)\big)dw(s)-\hat{\tau}\bigg\|^{2}\\ &\to 0 \quad \text{as} \quad \epsilon\to 0.\end{aligned}$$

By utilizing (3.9) and (3.10) along with assumption (H0), we obtain

$$\begin{aligned}&\mathbb{E}\left\|\vartheta^{\epsilon}(\ell)-\mathbb{E}\vartheta_{\ell}-\int_{0}^{\ell}\varpi(s)dw(s)\right\|^{2}\\ \leq\ & 2\mathbb{E}\big\|\epsilon\mathcal{V}(\epsilon,\Gamma_{0}^{\ell})(\chi)\big\|^{2}+2\mathbb{E}\big\|\epsilon\mathcal{V}(\epsilon,\Gamma_{0}^{\ell})\big\|^{2}\mathbb{E}\big\|p(\vartheta^{\epsilon})-\chi\big\|^{2}\\ \to\ & 0 \text{ as } \epsilon\to 0.\end{aligned}$$

This completes the proof. □

4 Application

Consider the following second order stochastic wave equation:

$$\begin{cases}\dfrac{\partial^{2}}{\partial t^{2}}\left[\vartheta(t,y)+\displaystyle\int_{-\infty}^{0}\alpha_{1}(s)\varphi\left(t+s,\vartheta(t+s,y)\right)ds\right]\\ \quad=\dfrac{\partial^{2}}{\partial y^{2}}\left[\vartheta(t,y)+\displaystyle\int_{-\infty}^{0}\alpha_{1}(s)\varphi\left(t+s,\vartheta(t+s,y)\right)ds\right]\\ \quad\quad+\mathcal{F}(t)\left[\vartheta(t,y)+\displaystyle\int_{-\infty}^{0}\alpha_{1}(s)\varphi\left(t+s,\vartheta(t+s,y)\right)ds\right]\\ \quad\quad+\displaystyle\int_{0}^{t}\hbar(t-s)\dfrac{\partial^{2}}{\partial y^{2}}\left[\vartheta(t,y)+\displaystyle\int_{-\infty}^{0}\alpha_{1}(p)\varphi\left(t+p,\vartheta(t+p,y)\right)dp\right]ds\\ \quad\quad+B\upsilon(t,y)+\varrho\big(t,\vartheta(t,y)\big)+\tau\big(t,\vartheta(t,y)\big)\dfrac{dw(t)}{dt},\quad t\in[0,\ell],\quad y\in[0,\pi],\end{cases}\tag{4.1}$$

with the Dirichlet boundary conditions

$$\vartheta(t,0)=\vartheta(t,\pi)=0,\quad t\in[0,\ell],$$

and the initial conditions

$$\vartheta(t,y)=\mu(t,y),\quad t\in(-\infty,0],\quad y\in[0,\pi],\quad \frac{\partial}{\partial t}\vartheta(0,y)=b_1(y).$$

Here, $\mathcal{F}$ and $\hbar$ are continuous functions mapping from $[0,\ell]$ to $\mathbb{R}$, while α_1 is a continuous function defined on $(-\infty,0]$ with values in $\mathbb{R}$. The functions $\mu:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ and $\hbar:[0,\pi]\to\mathbb{R}$ satisfy appropriate conditions. Additionally, $w(t)$ represents a Wiener process defined on the probability space $(\Omega,\Upsilon,\mathbb{P})$.

To represent the system (4.1) in the abstract form of (1.1), we choose the space $H=U=L^2[0,\pi]$. We define the operator $A_0:D(A_0)\subseteq H\to H$ as $A_0z=z''$, with the domain $D(A_0)=\{z\in H: z,z'$ are absolutely continuous $, z''\in H, z(0)=z(\pi)=0\}$. Here, A_0 is the infinitesimal generator of a cosine family $C(t)$ associated with the sine function $S(t)$ on H. Furthermore, A_0 has a discrete spectrum with the eigenvalues $-n^2,\ n\in\mathbb{N}$, with the corresponding eigenvectors $e_n(x)$ given by

$$e_n(x)=\frac{1}{\sqrt{2\pi}}e^{inx},\ n\in\mathbb{N}.$$

The set $\{e_n:n\in\mathbb{N}\}$ constitutes an orthonormal basis of H. Utilizing this notion, the operator A_0 can be represented as

$$A_0z=\sum_{n\in\mathbb{N}}-n^2\langle z,e_n\rangle e_n,\ z\in D(A),$$

where the cosine function is defined as

$$C(t)z=\sum_{n\in\mathbb{N}}\cos(nt)\langle z,e_n\rangle e_n,\quad t\in\mathbb{R},$$

and the sine function is given by

$$S(t)z=\sum_{n\in\mathbb{N}}\frac{\sin(nt)}{n}\langle z,e_n\rangle e_n,\quad t\in\mathbb{R}.$$

From these representations, it is evident that $\|C(t)\|\le 1$, and $S(t)$ is compact for all $t\in\mathbb{R}$.

Let us define $A(t)z=A_0z+\mathcal{F}(t)z$. Clearly, $A(t)$ is a closed linear operator.

Consider $\vartheta(t)(y)=\vartheta(t,y)$ for $t\in[0,\ell]$ and $y\in[0,\pi]$. We define the bounded linear operator $B : U \to H$ as $Bv(t)(y) = Bv(t, y)$. Here, the functions φ : $[0,\ell] \times \wp \to H$, $\varrho : [0,\ell] \times H \to H$, and $\tau : [0,\ell] \times H \to L_0^2$ are given by

$$\begin{aligned} \varphi(t, \vartheta_t)(y) &= \int_{-\infty}^{0} \alpha_1(s)\varphi\left(t+s, \vartheta(t+s, y)\right) ds, \\ \varrho(t, \vartheta(t))(y) &= \varrho(t, \vartheta(t, y)), \\ \tau(t, \vartheta(t))(y) &= \tau(t, \vartheta(t, y)). \end{aligned}$$

Additionally, we assume that these functions are chosen in a manner that they satisfies the assumptions of the theorem. By defining $\zeta(t, s) = \hbar(t - s)A_0$ for $0 \leq s \leq t \leq \ell$ on $D(A)$ and collecting these definitions, we can rewrite (4.1) in an abstract form (1.1).

Since the conditions (B1)-(B3) from Section 2 are satisfied, it follows that there exists a resolvent operator $(\Re(t, s))_{t\geq s}$ associated with the system (4.1). Furthermore, from (2.3), it can be deduced that the operator $\Re(t, s)$ is compact. Therefore, all conditions of Theorem 3.2 are met. Consequently, utilizing Theorem 3.2, the differential equation (4.1) is approximately controllable.

Conclusion:

This study focuses on establishing the existence, uniqueness, and controllability of a second-order non-autonomous neutral stochastic differential equation. Our approach involves resolvent operator techniques along with stochastic analysis theory to derive these results. Additionally, we employ the semigroup theory of bounded linear operators, the Arzela-Ascoli theorem, and Krasnoselskii's fixed point theorem to support our analysis. Furthermore, we provide an example to demonstrate the abstract findings.

Acknowledgment:

The author is thankful to the editor(s) and referee(s) for their valuable suggenstions to improve the manuscript.

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