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On sets related to maximizing sequences

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Abstract

If K is a non-empty bounded subset of a metric space (X,d) and $x \in X$, a sequence $\{k_n\}$ in K is called a maximizing sequence for x in K if $\lim_{n\to\infty} d(x,k_n) = \sup\{d(x,k) : k \in K\}$. A point $k_o \in K$ is called a farthest point to x in K if $d(x,k_o) = \sup\{d(x,k) : k \in K\}$. There are some types of sets connected with the concept of maximizing sequences. These sets have played a very significant role in giving some partial affirmative answers to one of the most interesting and hitherto unsolved problem in the theory of farthest points : If every point of a normed linear space X admits a unique farthest point in a given bounded set K in X, then must K be a singleton? Moreover these sets have been very useful in proving many results on the existence and uniqueness of farthest point and in proving the continuity of the farthest point map. This paper deals with some such sets which are related to maximizing sequences, their inter-relationships and their applications in the theory of farthest points. The underlying spaces are metric spaces.

Keywords and phrases: Maximizing sequence, remotal set, uniquely remotal set, nearly compact set.

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1 Introduction

For a non-empty bounded subset K of a metric space (X, d) and $x \in X$, a sequence $\{k_n\}$ in K is called a maximizing sequence for x if $\lim_{n\to\infty} d(x, k_n) = \delta(x, K) \equiv \sup\{d(x, k) : k \in K\}$. A point $k_o \in K$ is called a farthest point to x if $d(x, k_o) = \delta(x, K)$. There are some types of sets connected with the concept of maximizing sequences which are known in the literature. One such type of set is M-compact set introduced in normed linear spaces by Vlasov [25]. M-compact sets have been utilized in giving partial affirmative answers to one of the most interesting and hitherto unsolved problem (see [3], [5]-[7], [12], [15], [16], [18]-[24], [26] and references cited therein) in the theory of farthest points (known as farthest point problem-FPP): If every point of a normed linear space X admits a unique farthest point in a given bounded set K in X then must K be a singleton?

Many researchers have studied M-compact sets and their applications in the theory of farthest points when the underlying spaces are normed linear spaces (see [6], [12], [18], [20] and [25]). Only a few of them have taken up this study in more abstract spaces (see [7], [14]-[16] and [24]). One of the natural ways of development in mathematical research is to refine the framework of existing results and to see which of the results available in normed linear spaces survive in more abstract spaces. Many times, metric is a natural measure of the error while norm is not suitable. With this in mind, we have taken up the study of M-compact sets and their variants in the theory of farthest points in metric spaces. It may be remarked that farthest points have applications in the study of extremal structure of sets, characterization of weakly compact sets, characterization of strictly convex Banach spaces and inner product spaces and in finding the deviation of sets. They are important building blocks of convex sets which are extensively applied in programming (see [12], [23]).

In this paper, we shall study M-compact sets and some variants of M-compact sets, also study their inter-relationships and their applications in the theory of farthest points. The underlying spaces are metric spaces.

2 Definitions, notations and preliminaries

In this section, we give some definitions, notations and related facts. We start with few definitions. The notations and definitions not given here, can be found in the respective cited references.

A non-empty bounded subset K of a metric space (X, d) is said to be:

(i) M-compact [25] or $\underline{\triangle}-compact$ [6] or nearly compact [2] if for any $x \in X$, every maximizing sequence $\{k_n\}$ in K has a convergent subsequence in K,

(ii) nearly Cauchy set [2] if for any $x \in X$, every maximizing sequence $\{k_n\}$ in K is a Cauchy sequence,

(iii) strongly nearly compact [17] if for any $x \in X$, every maximizing sequence $\{k_n\}$ in K is convergent in K,

(iv) remotal (see [12]) if each $x \in X$ has a farthest point in K,

(v) uniquely remotal (see [12]) if each $x \in X$ has a unique farthest point in K.

For a remotal subset K of X, the multivalued mapping F_K taking each $x \in X$ to its set of farthest points $F_K(x) \equiv \{k \in K : d(x,k) = \delta(x,K)\}$ is called farthest point map or metric anti projection (see [12]). The map is single-valued for uniquely remotal sets. The number $\delta(x,K)$ is called the deviation of K from x. It is the farthest distance of x from K. This farthest distance of x from K may or may not be attained by an element of K. If the distance is attained, then the collection of all such points of K is the set $F_K(x)$. The collection of all points of K for which the farthest distance of x from K is attained for some $x \in X$ is denoted by $Far(K)$, i.e., $Far(K) = \underset{x \in X}{\cup} F_K(x)$.

A center (or Chebyshev center) of a bounded non-empty set K in a metric space (X,d) is an element $x_o \in X$ for which $\sup\limits_{y \in K} d(x_o, y) = \inf\limits_{x \in X} \sup\limits_{y \in K} d(x,y) \equiv r(K)$. The collection of all centers of K is denoted by $E(K)$. The space X is said to admit centers (see [12]) if for every bounded non-empty set K in X, the set $E(K)$ is nonempty. The set K is said to be centerable if $r(K) = \frac{1}{2}\, diam\, K$.

A metric space (X,d) is said to be:

(i) externally convex (see [24]) if for all distinct points $x, y \in X$ such that $d(x,y) = \lambda$ and $r > \lambda$, there exists a unique $z \in X$ such that $d(x,y) + d(y,z) = d(x,z) = r$,

(ii) a linear metric space (see [24]) if (a) X is a linear space (b) addition and scalar multiplication in X are continuous and (c) d is translation invariant, i.e., $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$. A complete locally convex linear metric space is called a Frechet space (see [24]).

Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y)$$

holds for all $u \in X$. A metric space (X, d) together with a convex structure W is called a convex metric space(see [23]).

A normed linear space X is said to be strictly convex (see [16]) if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in X$, $\|x\| = 1$, $\|y\| = 1$ and $x \neq y$.

Throughout this article, $\overline{K}$ or $cl(K)$ denotes closure of the set K and $conv(K)$ denotes convex hull of K.

Remark 2.1. *In normed linear spaces, the notion of $M-$compact sets was introduced by Vlasov in [25] and was called $\triangle-compact$ by Blatter [6]. They used this concept in proving some results on the existence and uniqueness of farthest points, in proving continuity of the farthest point map and in giving some partial answers to the FPP in the theory of farthest points. This notion was extended to metric spaces under the name 'nearly compact' in [2] wherein the existence of farthest points was discussed. $M-$compact sets are important in the theory of farthest points as such sets are always remotal. (Theorem 3.1) All compact sets are M-compact but the converse is not true. An M-compact set need not even be closed. The study of remotal and uniquely remotal sets has attracted many researchers due to their connection to the geometry of Banach spaces (see [19]).*

Example 2.1. *Let $X = \{(x, y) : x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$, $K = X \setminus \{(0, 0)\}$. Then, K is M-compact but not compact (see [2]).*

Example 2.2. *[18] In the 2-dimensional Euclidean space $\mathbb{R}^2$, the set K consisting of open unit square together with its corners is remotal. This set is not compact. It is not even closed.*

Whereas every singleton set is uniquely remotal, the following example shows that converse is not true.

Example 2.3. *[15] Let $X = \mathbb{R} \setminus \{0\}$ with the usual metric d defined by $d(x, y) = |x - y|$ and $K = [-1, 1] \setminus \{0\}$. Then, the set K is a uniquely remotal set but is not a singleton.*

A remotal set need not be uniquely remotal, as shown by the following example.

Example 2.4. *[22] Let $X = \mathbb{R}^2$ with the usual metric and $K = \{(x, 0) : -1 \leq x \leq 0\}$. Here $F_K(p) = \{(-1, 0), (0, 0)\}$, where $p = (-\frac{1}{2}, y), y \in \mathbb{R}$. It is easy to see that each point of $\mathbb{R}^2 \setminus \{(-\frac{1}{2}, y) : y \in \mathbb{R}\}$ has unique farthest point in K. The set K is remotal but not uniquely remotal.*

M-compact sets have also been used in the study of Chebyshev centers. It is easy to see [18] that if A is M-compact, then $\overline{A}$ is M-compact but converse is not true. It is known (see [21], [24]) that converse is true if the space is a strictly convex normed linear space (or an externally convex metric space) and A is also uniquely remotal. It is proved in [24] that if A is uniquely remotal M-compact subset of a metric space (X, d), then $\overline{A}$ is uniquely remotal. Moreover, (see [21], [24]) if A is uniquely remotal M-compact subset of a normed linear space (or a convex metric space) X, then the derived set Á is either empty or $\acute{A}$ is uniquely remotal and M-compact. Further, if A is uniquely remotal and M-compact subset of a Banach space X and $\acute{A}$ is compact, then A is a singleton. Using the result (see [20]) that in a Banach space X if A is bounded centerable M-compact subset containing its Chebyshev center, then A attains its diameter. The following characterization of strictly convex Banach spaces was proved in [20]:

A Banach space X is strictly convex if and only if every bounded centerable M-compact subset A of X which contains its Chebyshev center is CCNF (Chebyshev center not in Far(A)).

Remark 2.2. *The notion of nearly compact sets is analogous to the notion of approximatively compact sets, introduced by Efimov and Steckin [9] while dealing with two outstanding and hitherto unsolved problems in the theory of best approximation, viz., proving the convexity of Chebyshev sets in a Hilbert space, and of characterizing those Banach spaces in which every Chebyshev set is convex (see [13]). Approximatively compact sets played a vital role in proving many results in the theory of best approximation and in giving partial affirmative answers to the problem of convexity of Chebyshev sets (which is so closely related to the FPP that in a Hilbert space, the solution of one will lead to a solution of the other- see [11], [13]).*

Remark 2.3. *The notion of nearly Cauchy set (which is analogous to the notion of approximatively Cauchy set introduced in [1]) was introduced in [2] wherein relationships between nearly compact sets and nearly Cauchy sets were discussed, and existence of farthest points was proved for complete nearly Cauchy sets.*

Remark 2.4. *The notion of strongly nearly compact set is analogous to the notion of strongly approximative compact sets. Strongly approximative compact sets were introduced and discussed in Banach spaces by Bandyopadhyay et al. [4], and in metric spaces by Narang and Gupta [17], wherein a variety of results were proved* *in the theory of nearest points. It is easy to see that, a complete nearly Cauchy set is strongly nearly compact.*

3 Main results

Before proving our main results, we briefly survey some known results related to the definitions of Section 2. The following theorem on the existence of farthest points was proved for metric spaces in [2]. For Banach spaces, this result was proved by Blatter [6].

Theorem 3.1. *A nearly compact set in a metric space is remotal.*

Since a complete nearly Cauchy set is nearly compact, we have the following corollary:

Corollary 3.1. *[2] A complete nearly Cauchy set in a metric space is remotal.*

Since a closed subset of a complete metric space is complete, we have the following corollary:

Corollary 3.2. *[2] A closed nearly Cauchy set in a complete metric space is remotal.*

Blatter [6] proved that in a Banach space, a nearly compact uniquely remotal set supports a continuous farthest point map. This result was extended to metric spaces in [14].

Theorem 3.2. *If K is nearly compact and uniquely remotal subset of a metric space (X, d), then the farthest point map F_K is continuous.*

Remark 3.1. *If K is nearly compact subset of a metric space (X, d), then the set-valued farthest point map F_K is upper semi continuous (see [14]).*

Remark 3.2. *Blatter [6] has shown that if K is nearly compact uniquely remotal subset of a Banach space X, then K is singleton. This result is not true in a metric space as shown by the following example. (The result is not true even in a linear metric space-see[7]).*

Example 3.1. *[16] Let $X = \mathbb{R}\backslash\{0\}$ with the usual metric. Take $K = [-1, 1]\backslash\{0\}$. Then, K is nearly compact uniquely remotal set in X but K is not a singleton.*

Infact, Blatter [6] proved the following:

Theorem 3.3. *Let K be a non-empty bounded subset of a Banach space X. Then the following are equivalent:*

(i) *K is nearly compact uniquely remotal subset of X,*

(ii) *K is uniquely remotal and the farthest point map is continuous,*

(iii) *K is remotal and the farthest point map is lower semi-continuous,*

(iv) *K is a singleton.*

The following partial affirmative answer to the FPP was given in [15] for nearly compact sets:

Theorem 3.4. *If K is a totally bounded nearly compact set in a Frechet space (X, d) and each point of cl(conv K) admits a unique farthest point in K, then K is a singleton.*

Remark 3.3. *For compact subsets of Banach Spaces, this result was proved by Klee [11]. Can we extend the result to metric spaces?*

Using the idea of Chebyshev centers, Panda and Kapoor [18] proved the following:

Theorem 3.5. *Let X be a normed linear space admitting centers and K be a nearly compact uniquely remotal subset of X. Then, K is a singleton.*

Remark 3.4. *Whether we can prove a similar result in metric spaces remains to be investigated.*

The following affirmative answer to the FPP was given by Astaneh [3]:

Theorem 3.6. *Let K be a nearly compact subset of a Hilbert space H. If cl(K) admits a unique farthest point to the Chebyshev center of K, then K is a singleton.*

Now, we discuss our main results. The following theorem gives relationship among nearly compact, nearly Cauchy and strongly nearly compact sets.

Theorem 3.7.

(i) Every complete nearly Cauchy set K in a metric space (X, d) is strongly nearly compact,

(ii) The set K is strongly nearly compact if and only if K is nearly compact and nearly Cauchy set.

Proof. (i) Let $x \in X$ and $\{k_n\}$ be a maximizing sequence for x in K, i.e., $\lim_{n\to\infty} d(x, k_n) = \delta(x, K)$. Since, K is nearly Cauchy, $\{k_n\}$ is a Cauchy sequence in K. Since K is complete, $\{k_n\}$ converges to some $k_o \in K$, and hence K is strongly nearly compact.

(ii) Suppose K is strongly nearly compact. Then, K is nearly compact and nearly Cauchy set by the definitions. Conversely, suppose K is nearly Cauchy and nearly compact. Let $x \in X$ and $\{k_n\}$ be a maximizing sequence for x in K i.e. $\lim_{n\to\infty} d(x, k_n) = \delta(x, K)$. Since K is nearly Cauchy, $\{k_n\}$ is a Cauchy sequence in K. Since, K is nearly compact, $\{k_n\}$ has subsequence $\{k_{n_i}\}$ such that $\{k_{n_i}\} \to k_o \in K$. Now, using the well known result that if a Cauchy sequence has a convergent subsequence, then the Cauchy sequence itself is convergent, we obtain $\{k_n\} \to k_o \in K$. Hence, K is strongly nearly compact.

□

Since a closed subset of a complete metric space is complete, we obtain the following:

Corollary 3.3. *A closed nearly Cauchy set in a complete metric space is strongly nearly compact.*

For nearly Cauchy sets, we have the following theorem:

Theorem 3.8. *A closed (complete) nearly Cauchy set in a complete metric space (metric space) is uniquely remotal.*

Proof. Let K be a closed nearly Cauchy set in a complete metric space (X, d). Then, by Corollary 3.2, K is remotal. Suppose, for some $x \in X$, there exists $y_1, y_2 \in K$ such that $y_1 \neq y_2$ and $d(x, y_1) = \delta(x, K) = d(x, y_2)$. Consider the sequence $\{k_n\}$ such that $k_{2n} = y_1$ and $k_{2n+1} = y_2$. Then, $d(x, k_n) = \delta(x, K)$ for all n and so $\lim_{n\to\infty} d(x, k_n) = \delta(x, K)$ i.e. $\{k_n\}$ is a maximizing sequence for x in K. Since $y_1 \neq y_2$, $\{k_n\}$ is not a Cauchy sequence. But this contradicts that K is nearly Cauchy. Hence $y_1 = y_2$ and consequently, K is uniquely remotal. □

For nearly compact uniquely remotal sets, we have the following:

Theorem 3.9. *A nearly compact uniquely remotal subset of a metric space is strongly nearly compact.*

Proof. Let K be a nearly compact uniquely remotal subset of a metric space (X, d), $x \in X$ and $\{k_n\}$ be a maximizing sequence for x in K, i.e., $\lim_{n\to\infty} d(x, k_n) = \delta(x, K)$. Since, K is nearly compact $\{k_n\}$ has a subsequence $\{k_{n_i}\} \to k_o \in K$. Then, $d(x, k_o) = \delta(x, K)$ i.e. $k_o \in K$ is a farthest point to x in K. We claim that every subsequence of $\{k_n\}$ also converges to k_o. Suppose $\{k_n\}$ has a subsequence $\{k_{n_l}\}$ such that $\{k_{n_l}\} \to k^* \in K$. Then, $d(x, k^*) = \lim_{n\to\infty} d(x, k_{n_l}) = \delta(x, K)$, i.e., k^* is also a farthest point to x in K. Since, K is uniquely remotal, $k^* = k_o$. Therefore, every subsequence of $\{k_n\}$ converges to k_o and so $\{k_n\} \to k_o \in K$. Hence, K is strongly nearly compact. □

For strongly nearly compact sets, we have the following:

Theorem 3.10. *Every strongly nearly compact set in a metric space is uniquely remotal.*

Proof. Let K be a strongly nearly compact subset of a metric space (X, d). Then, K is nearly compact, and hence remotal by Theorem 3.1. Suppose for some $x \in X$, there are two different farthest points to x in K, say k_o, k^* i.e. $d(x, k_o) = \delta(x, K) = d(x, k^*)$. Consider the sequence $\{k_n\}$ in K such that $k_{2n} = k_o$, $k_{2n+1} = k^*$. Then, $\{k_n\}$ is maximizing sequence for x in K. Since $k_o \neq k^*$, $\{k_n\}$ is not convergent in K as it is not a Cauchy sequence, which is a contradiction. Therefore, $k_o = k^*$, and so K is uniquely remotal. □

From Theorems 3.9 and 3.10, we obtain that strongly nearly compact sets are precisely those which are nearly compact and uniquely remotal.

Applying Theorem 3.7(i), we have the following:

Corollary 3.4. *Every complete nearly Cauchy set in a metric space is uniquely remotal.*

Applying Theorem 3.2, we obtain the following:

Corollary 3.5. *If K is strongly nearly compact subset of a metric space (X, d), then the farthest point map F_K is continuous.*

For Banach spaces, Corollary 3.5 was proved in [10]. Whether its converse holds is not known. The answer appears to be in negative even for Banach spaces (see [10]). The converse holds for the Banach spaces whose norms are sufficiently well behaved, i.e., those Banach spaces for which the norms of X and of the conjugate space X^* are both Frechet differentiable (see Corollary 3.5 [10]).

Applying Theorem 3.7(i), we obtain the following corollary:

Corollary 3.6. *If K is complete nearly Cauchy set in a metric space (X, d), then the farthest point map F_K is continuous.*

For complete metric spaces, we have

Theorem 3.11. *Let (X, d) be a complete metric space. Then, the following are equivalent:*

(i) Every bounded closed subset K of X is a nearly compact uniquely remotal set.

(ii) For every bounded closed subset K of X and $x \in X$, every maximizing sequence $\{k_n\}$ in K is a Cauchy sequence.

Proof. $(i) \Rightarrow (ii)$ Let $x \in X$ and $\{k_n\}$ be a maximizing sequence for x in K, i.e., $\lim\limits_{n\to\infty} d(x, k_n) = \delta(x, K)$. Since K is nearly compact, $\{k_n\}$ has a subsequence $\{k_{n_i}\} \to k_o \in K$. Then, $d(x, k_o) = \delta(x, K)$, and so $k_o \in F_K(x)$. We claim that, every subsequence of $\{k_n\}$ also converges to k_o. Suppose $\{k_{n_l}\}$ is a subsequence of $\{k_n\}$ such that $k_{n_l} \to k_o' \in K$, $k_o \neq k_o'$. Then $d(x, k_o') = \lim d(x, k_{n_l}) = \delta(x, K)$ and so $k_o' \in F_K(x)$. Since, K is uniquely remotal, $k_o' = k_o$, a contradiction. Therefore, every subsequence of $\{k_n\}$ converges to $k_o \in K$, and hence $\{k_n\} \to k_o \in K$ i.e. $\{k_n\}$ is a convergent sequence, and so is a Cauchy sequence.

$(ii) \Rightarrow (i)$ Let $x \in X$ and $\{k_n\}$ be a maximizing sequence for x in K. Then, by the hypothesis, $\{k_n\}$ is a Cauchy sequence. Since, K being closed, is complete, $\{k_n\} \to k_o \in K$, and therefore, K is nearly compact. Now we show that K is uniquely remotal. Suppose for some, $x \in X$, there exist $k_1, k_2 \in F_K(x)$, $k_1 \neq k_2$.

Then, $d(x, k_1) = \delta(x, K) = d(x, k_2)$. Consider the sequence $\{k_n\}$ in K such that $k_{2n} = k_1$ and $k_{2n+1} = k_2$. Then, $\{k_n\}$ is a maximizing sequence for x in K. Since $k_1 \neq k_2$, $\{k_n\}$ is not a Cauchy sequence, a contradiction. Therefore, $k_1 = k_2$, and hence K is uniquely remotal. $\square$

Theorem 3.11 can be restated as:

Theorem 3.12. *Let (X, d) be a complete metric space. Then, the following are equivalent:*
(i) *Every bounded closed subset of X, is nearly compact uniquely remotal set.*
(ii) *Every bounded closed subset of X is nearly Cauchy set.*

Since strongly nearly compact set is nearly compact and uniquely remotal, from the proof of above theorem we obtain:

Corollary 3.7. *Let (X, d) be a complete metric space. Then, the following are equivalent:*

(i) Every bounded closed subset of X is strongly nearly compact uniquely remotal set.

(ii) Every bounded closed subset of X is nearly Cauchy set.

Remark 3.5. *From the proof of the Theorem 3.11 $(i) \Rightarrow (ii)$ part, we observe that a nearly compact uniquely remotal set is strongly nearly compact.*

Remark 3.6. *For Banach spaces, Theorem 3.11 was proved by Fitzpatrick [10].*

Remark 3.7. *For a bounded subset K of X and $x \in X$, we say that, the problem of farthest points $\max(x, K)$ is well posed if it has a unique solution $k_o(x) \in K$ and every maximizing sequence $\{k_n\}$ in K converges to $k_o(x)$ (see [8]). Some results on the well posedness of the problem $\max(x, K)$ are known in the literature (see [8], [23]). It follows from the above discussion that, the problem $\max(x, K)$ is well posed if K is strongly nearly compact or K is complete nearly Cauchy set.*

Conclusion: Most of the literature available in the theory of farthest points is either in Hilbert spaces or in normed linear spaces. The construction of the farthest point theory in more abstract spaces is challenging. Although, some attempts have been made in this direction by few researchers (viz.; [2], [12], [14]-[16], [22]-[24] and references cited therein), but much remains to be done.

The aim of this paper is twofold, first to discuss M-compact sets and some of its variants which are related to maximizing sequences, their inter relationships and applications to the farthest point theory, secondly, to see which of the results available in normed linear spaces on these topics survive in metric spaces.

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