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Existence and controllability results for nonlinear fractional integrodifferential equations with infinite delay

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Abstract

This paper deals with the existence of solutions of nonlinear implicit fractional integrodifferential equations with infinite delay. The results are obtained by using the fractional calculus, measure of noncompactness and Darbo's fixed point theorem. To illustrate the theory, controllability problem is studied for the fractional delay equation.

1 Introduction

Many mathematical models in the fields of science and engineering involve fractional derivatives and they are expressed in terms of fractional differential equations. Nowadays the subject has been gaining much importance and attention among researchers. Fractional order models incorporate implicitly memory effects that are difficult to describe using classical calculus. We motivate the paper by providing a model from population dynamics. For the basic theory and applications of fractional differential equations, one can refer [5, 20, 22].

Keywords and phrases: Existence of solutions, controllability, measure of noncompactness, fractional derivative, fixed point theorem.

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The Volterra model for population growth of a species within a closed system is characterized by a nonlinear fractional integrodifferential equation of the form [25]

$$ {}^{C}D^{\alpha}u(t) = au(t) - bu^2(t) - cu(t)\int_0^t u(s)ds, \tag{1.1} $$

$$ u(0) = u_0, \tag{1.2} $$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$, $u(t)$ is the scaled population of identical individuals, t denotes the time, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient. The last integral term represents the effect of toxin accumulation on the species. There are many variations of this model studied by several researchers [29, 30].

Delay is very often encountered in many real world problems and there has been widespread interest in the study of delay differential equations for many years. Several papers have been devoted to the study of existence results for integer order delay differential equations. Differential equations and integrodifferential equations with infinite delays are studied by many authors [3, 4, 6, 13, 14, 19]. In recent years, fractional delay differential and integrodifferential equations begin to arouse the attention of many researchers due to their applications in various fields [12, 26–28]. However very few papers are appeared for fractional integrodifferential equations with infinite delays. These types of equations with integer derivative occur in the study of viscoelastic materials, population dynamics, wave propagation in dissipative materials with memory and many other physical phenomena [14]. So it is natural and interesting to extend the study for the models with fractional derivatives [1, 24]. In this paper we discuss the existence of solutions of implicit fractional integrodifferential equations with infinite delay by using the measure of noncompactness of a set and the Darbo fixed point theorem and apply the method to the controllability problem for the similar class of equations.

2 Preliminaries

Let us recall some basic definitions from fractional calculus [5, 20].

Definition 2.1. *The Riemann-Liouville fractional integral of a function* $f \in L^1([a, b])$ *of order* $\alpha > 0$ *is defined as*

$$ I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \tag{2.1} $$

provided the integral exists.

Definition 2.2. *The Caputo fractional derivative of order* $n-1 < \alpha \leq n$ *is defined as*

$$ {}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \tag{2.2}$$

where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$*. In particular, if* $0 < \alpha \leq 1$,

$$ {}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \tag{2.3}$$

where $f'(s) = Df(s) = \frac{df(s)}{ds}$.

The following property is used in this paper for proving the existence results

$$ I^{\alpha}\, {}^{C}D^{\alpha}f(t) = f(t) - f(a),\ 0 < \alpha < 1. \tag{2.4}$$

Next we introduce the concepts of measure of noncompactness of a set, the modulus of continuity of functions and the Darbo fixed point theorem.

Definition 2.3. *[23] Let* $(X, \|.\|)$ *be a Banach space and* E *be a bounded subset of* X*. Then, the measure of noncompactness of a set* E *is defined by* $\mu(E)$=inf$\{r > 0$; *E can be covered by a finite number of balls whose radii are smaller than* $r\}$.

For the space of continuous functions $C_n[t_0, t_1]$ with norm

$$\|x\| = \max\{|x_i(t)| : i = 1, 2, \ldots, n,\ t \in [t_0, t_1]\},$$

the measure of noncompactness of a set E is given by

$$\mu(E) = \frac{1}{2}\omega_0(E) \doteq \frac{1}{2} \lim_{h \to 0^+} \omega(E, h), \tag{2.5}$$

where $\omega(E, h)$ is the common modulus of continuity of the function which belong to the set E, that is

$$\omega(E, h) = \sup_{x \in E} [\sup |x(t) - x(s)| : |t - s| \leq h] \tag{2.6}$$

and for the space of continuously differentiable functions $C_n^1[t_0, t_1]$ with norm

$$\|x\|_{C_n^1} = \|x\|_{C_n} + \|Dx\|_{C_n},$$

we have

$$\mu(E) = \frac{1}{2}\omega_0(DE),$$

where $DE = \{Dx : x \in E\}$. Introduce the space $C_n^\alpha[t_0, t_1] = \big\{x : {}^CD^\alpha x \in C_n \text{ and } x \in C_n^1\big\}$ with the norm

$$\|x\|_{C_n^\alpha} = \|x\|_{C_n} + \|{}^CD^\alpha x\|_{C_n}.$$

This space is a Banach space under the above norm (see [5]). Then the measure of noncompactness of a set E is given by

$$\mu(E) = \frac{1}{2}\omega_0({}^CD^\alpha E),$$

where ${}^CD^\alpha E = \{{}^CD^\alpha x : x \in E\}$.

Now we state the fixed point theorem due to Darbo [16] as:

Theorem 2.1. *If S is a nonempty bounded closed convex subset of X and $P : S \to S$ is a continuous mapping such that for any set $E \subset S$, we have*

$$\mu(PE) \leq k\mu(E),$$

where k is a constant such that $0 \leq k < 1$, then P has a fixed point.

3 Basic assumptions

Consider the following nonlinear fractional delay integrodifferential equations

$$\begin{aligned} {}^CD^\alpha x(t) &= f(t, x(t), {}^CD^\alpha x(t)) + \int_{-\infty}^t q(t, \tau, x(\tau))d\tau, \qquad t \geq t_0 \quad (3.1)\\ x(t) &= \phi(t), \qquad -\infty < t \leq t_0, \end{aligned}$$

where x, f and q are n-vectors and the initial function $\phi(t)$ is continuous. In the equation (3.1) the fractional derivative occurs implicitly in the nonlinear function

f and so we have to use the method introduced by Dacka in [15] to establish the existence results of (3.1). Let $\Delta = \{(t, s) : -\infty < s \le t \le t_1\}$. Assume the following conditions:

i) The function $q : \Delta \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a constant $N > 0$ such that
$$|q(t, \tau, x(\tau))| \le N.$$

ii) For each $\phi \in C_n(-\infty, t_0]$, the improper Riemann integral
$$Q(t) = \lim_{a \to \infty} \int_{-a}^{t_0} q(t, \tau, \phi(\tau)) d\tau$$
exists and it is continuous on $[t_0, t_1]$. Furthermore there exists a constant $K > 0$ such that
$$|Q(t)| \le K.$$

iii) The function $f : [t_0, t_1] \times R^n \times R^n \to R^n$ is continuous and for every $x, y, \bar{y} \in R^n$ and $t \in [t_0, t_1]$
$$|f(t, x, y) - f(t, x, \bar{y})| \le k|y - \bar{y}|$$
$$\text{and} \quad |f(t, x, y)| \le M,$$
where the constant $M > 0$ and the constant k is such that $0 \le k < 1$.

We shall define the solution of (3.1) with initial function $\phi(t)$ as follows.

Definition 3.1. *Any function $x(t)$ defined in the interval $(-\infty, t_1]$ and satisfy the following conditions is called as the solution.*

a) The function $x(t)$ is continuous in the interval $(-\infty, t_1]$ and of class C_n^α in the interval $[t_0, t_1]$ such that at the point t_0 the right side derivative is taken into account.

b) Equation (3.1) is satisfied by the function $x(t)$ in the interval $[t_0, t_1]$, where as on the interval $(-\infty, t_0]$ the function $x(t) = \phi(t)$.

By the condition (ii) the equation (3.1) takes the following form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= f(t, x(t), {}^{C}D^{\alpha}x(t)) + Q(t) + \int_{t_0}^{t} q(t, \tau, x(\tau))d\tau, \qquad t \geq t_0, \\ x(t) &= \phi(t), \qquad -\infty < t \leq t_0. \end{aligned} \tag{3.2}$$

Applying the property (2.4) to the equation (3.2) we get

$$\begin{aligned} x(t) &= \phi(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s), {}^{C}D^{\alpha}x(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} Q(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{t_0}^{s} q(s, \tau, x(\tau))d\tau\, ds, \;\; t \geq t_0, \\ x(t) &= \phi(t), \text{ on } (-\infty, t_0]. \end{aligned} \tag{3.3}$$

4 Existence theorem

In this section, we prove an existence theorem for the implicit fractional delay integrodifferential equations (3.1).

Theorem 4.1. *Under the assumptions (i) to (iii), the equation* (3.1) *has at least one solution for any initial function* $\phi \in C_n(-\infty, t_0]$.

Proof. Consider the Banach space $C_n^{\alpha}[t_0, t_1]$ and in this space the subset

$$H = \{x : x \in C_n^{\alpha}[t_0, t_1], x(t_0) = \phi(t_0)\}.$$

We now define the mapping $P : C_n^{\alpha}[t_0, t_1] \to C_n^{\alpha}[t_0, t_1]$ by the formula

$$\begin{aligned} Px(t) &= \phi(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s), {}^{C}D^{\alpha}x(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} Q(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{t_0}^{s} q(s, \tau, x(\tau))d\tau ds. \end{aligned} \tag{4.1}$$

Further, consider the closed bounded convex set B in H by

$$B = \{x \in H : \|x\| \leq N_1, \|^C D^\alpha x\| \leq N_2\},$$

where N_1 and N_2 are positive constants defined by

$$N_1 = \frac{(t_1 - t_0)^\alpha}{\Gamma(\alpha + 1)} N_2 + |\phi(t_0)|$$

and

$$N_2 = M + K + (t_1 - t_0)N.$$

Since ϕ, f, Q and q are all continuous functions, it follows that P is continuous and maps B into itself. The functions $Px(t)$ are equicontinuous, since they have uniformly bounded derivatives. Now we shall find an estimate for the modulus of continuity of the functions ${}^C D^\alpha Px(t)$ for $t, s \in [t_0, t_1]$ as in [15].

$$\begin{aligned}
|^C D^\alpha Px(t) -^C D^\alpha Px(s)| \leq\ & |Q(t) - Q(s)| \\
& + \left| \int_{t_0}^{t} q(t, \tau, x(\tau)) d\tau - \int_{t_0}^{s} q(s, \tau, x(\tau)) d\tau \right| \\
& + |f(t, x(t), ^C D^\alpha x(t)) - f(s, x(s), ^C D^\alpha x(s))| \\
\leq\ & |Q(t) - Q(s)| \\
& + \left| \int_{t_0}^{t} q(t, \tau, x(\tau)) d\tau - \int_{t_0}^{s} q(t, \tau, x(\tau)) d\tau \right| \\
& + \left| \int_{t_0}^{s} q(t, \tau, x(\tau)) d\tau - \int_{t_0}^{s} q(s, \tau, x(\tau)) d\tau \right| \\
& + |\ f(t, x(t), ^C D^\alpha x(t)) - f(s, x(s), ^C D^\alpha x(s))\ | \\
\leq\ & |Q(t) - Q(s)| + \int_{s}^{t} |q(t, \tau, x(\tau))| d\tau \\
& + \int_{t_0}^{s} |q(t, \tau, x(\tau)) - q(s, \tau, x(\tau))| d\tau \\
& + |f(t, x(t), ^C D^\alpha x(t)) - f(s, x(s), ^C D^\alpha x(t))| \\
+\ & |f(s, x(s), ^C D^\alpha x(t)) - f(s, x(s), ^C D^\alpha x(s))|.
\end{aligned} \tag{4.2}$$

The first three terms of the right hand side of (4.2) can be estimated from the above by $\beta_0(|t-s|)$, where β_0 is a nonnegative function such that $\lim_{h\to 0^+}\beta_0(h)=0$. The last two terms can be made as $k|^C D^\alpha x(t) -^C D^\alpha x(s)| + \beta_1(|t-s|)$.

Setting $\beta = \beta_0 + \beta_1$, we finally obtain

$$|^C D^\alpha Px(t) -^C D^\alpha Px(s)| \leqslant k|^C D^\alpha x(t) -^C D^\alpha x(s)| + \beta(|t-s|).$$

Hence, we infer that

$$\omega(^C D^\alpha Px, h) \leqslant k\omega(^C D^\alpha x, h) + \beta(h).$$

Thus, for any bounded set $E \subset B \subset H$, we have

$$\mu(PE) \leqslant k\mu(E).$$

Consequently by the Darbo fixed point theorem, the mapping P has a fixed point $x \in C_n^\alpha[t_0, t_1]$ such that

$$x(t) = Px(t).$$

Clearly the extension of this function to the interval $(-\infty, t_0]$ by means of the function $\phi(t)$ is a solution of the following equation

$$\begin{aligned} x(t) &= \phi(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f(s, x(s), ^C D^\alpha x(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} Q(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\int_{t_0}^s q(s,\tau,x(\tau))d\tau ds, t \geq t_0, \\ x(t) &= \phi(t), \text{ on } (-\infty, t_0]. \end{aligned}$$

Hence, the proof is complete. □

Remark 4.1. *It should be observed that if we assume that the function f and q satisfy also the Lipschitz condition with respect to x, then the uniqueness of the solution of equation* (3.1) *can be established by standard technique.*

Remark 4.2. *The existence problem for the case $\alpha = 1$ is studied in [3] and the* *nonlinear function f is independent of $\dot{x}$ is thoroughly discussed in [13].*

Example 4.1. *Consider the scalar fractional delay integrodifferential equation*

$$\begin{aligned} {}^C D^\alpha x(t) &= \sin {}^C D^\alpha x(t) + \int_{-\infty}^{t} e^{-(t-s)} \frac{1}{1+x^2(s)} ds, \qquad t \geq 0, \\ x(t) &= 1, \qquad -\infty < t \leq 0. \end{aligned} \tag{4.3}$$

Here $\phi(t) = 1,\ f(t, x(t), {}^C D^\alpha x(t)) = \sin {}^C D^\alpha x(t)$ and $q(t, s, x(s)) = e^{-(t-s)} \frac{1}{1+x^2(s)}$. Note that $Q(t) = \lim_{a\to\infty} \int_{-a}^{0} \frac{e^{-(t-s)}}{2} ds = \frac{e^{-t}}{2}$ and so $|Q(t)| < \frac{1}{2}$. Since all the conditions of the above theorem are satisfied and hence the equation (4.3) *has a solution.*

5 Controllability result

The problem of controllability of linear and nonlinear dynamical systems including delay systems in finite dimensional spaces is well established [2, 7, 8, 21]. Recently this problem has been extended to fractional dynamical systems by many authors [5, 9–11, 17, 18]. Here as an application of the above theory we discuss the controllability of fractional delay integrodifferential systems by introducing a control variable in the equation (3.1).

Consider the fractional delay integrodifferential control system of the form

$$\begin{aligned} {}^C D^\alpha x(t) &= B(t, x(t))u(t) + f(t, x(t), {}^C D^\alpha x(t)) \\ &\quad + \int_{-\infty}^{t} q(t, \tau, x(\tau)) d\tau,\ t_0 \leq t \leq t_1, \\ x(t) &= \phi(t), \qquad -\infty < t \leq t_0, \end{aligned} \tag{5.1}$$

where B is an $n \times m$ continuous matrix valued function on $[t_0, t_1] \times R^n$ and $u \in R^m$.

Definition 5.1. *[18] The system* (5.1) *is said to be controllable on $[t_0, t_1]$, if for every $\phi \in C_n(-\infty, t_0]$ and every $x_1 \in R^n$ there exists a control function $u(t)$ defined on $[t_0, t_1]$ such that the solution of* (5.1) *satisfies $x(t_1) = x_1$.*

Define the controllability matrix W by

$$W(t, x) = \int_{t_0}^{t} B(s, x) B^*(s, x) ds,$$

where the star denotes the matrix transpose. Assume that

(iv) there exists a constant $L > 0$ such that

$$\|B(t,x)\| \le L \text{ for all } (t,x) \in [t_0, t_1] \times R^n,$$

and the norm of a continuous matrix valued function is taken as in [2]. Put

$$p(t,x) = \phi(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big(Q(s) + f(s, x(s), {}^C D^\alpha x(s)) + \int_{t_0}^{s} q(s,\tau,x(\tau)) d\tau \Big) ds$$

Theorem 5.1. *Given the system* (5.1) *with conditions (i) to (iv) and assume that*

$$\inf_{x \in C_n^\alpha} det\, W(t,x) > 0.$$

Then, the system (5.1) *is controllable on* $[t_0, t_1]$.

Proof. Let $\phi \in C_n(-\infty, t_0]$ be an arbitrary initial function. Define the nonlinear mapping $T : C_n^\alpha[t_0,t_1] \to C_n^\alpha[t_0,t_1]$ by

$$\begin{aligned} Tx(t) &= \phi(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} [B(s,x(s))u(s) + Q(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left(f(s, x(s), {}^C D^\alpha x(s)) + \int_{t_0}^{s} q(s,\tau,x(\tau)) d\tau \right) ds, \end{aligned} \tag{5.2}$$

where the control u is given by

$$u(t) = (t_1 - t)^{1-\alpha} B^*(t,x) W^{-1}(t_1, x)[x_1 - p(t_1, x)]. \tag{5.3}$$

Substituting (5.3) into (5.2), we get

$$\begin{aligned}
&Tx(t)\\
&= \phi(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}(t_1-s)^{1-\alpha}B(s,x(s))\\
&\qquad B^*(s,x)W^{-1}(t_1,x)[x_1-p(t_1,x)]ds \qquad (5.4)\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(Q(s)+f(s,x(s),{}^{C}D^{\alpha}x(s))+\int_{t_0}^{s}q(s,\tau,x(\tau))d\tau\right)ds.
\end{aligned}$$

Since all the functions involved in the definition of the operator T are continuous, it is easy to show that T is continuous and maps $C_n^{\alpha}[t_0,t_1]$ into itself. Consider the closed convex subset of $C_n^{\alpha}[t_0,t_1]$:

$$H = \{x : x \in C_n^{\alpha}[t_0,t_1], \|x\| \le M_1, \|{}^{C}D^{\alpha}x(t)\| \le M_2\},$$

where M_1 and M_2 are positive constants defined by

$$\begin{aligned}
M_1 &= \frac{(t_1-t_0)^{\alpha}}{\Gamma(\alpha+1)}(M_3+K)+M_4;\ \ M_2 = L(t_1-t_0)^{1-\alpha}M_3+M_4;\\
M_3 &= L\sup_{x\in C_n^{\alpha}}\|W^{-1}(t_1,x)\|[|x_1|+M_4];\\
M_4 &= |\phi(t_0)|+\frac{(t_1-t_0)^{\alpha}}{\Gamma(\alpha+1)}[K+M+N(t_1-t_0)].
\end{aligned}$$

The operator T maps H into itself. It is easy to see that all the functions Tx with $x \in H$ are equicontinuous, since they have uniformly bounded derivative. By a similar argument as in the above theorem we can show that

$$|{}^{C}D^{\alpha}Tx(t)-{}^{C}D^{\alpha}Tx(s)| \leqslant k|{}^{C}D^{\alpha}x(t)-{}^{C}D^{\alpha}x(s)|+\beta(|t-s|).$$

Hence, for any set $E \subset H$, we have $\mu(TE) \leqslant k\mu(E)$. Consequently, by the Darbo fixed point theorem, there exists a function $x \in C_n^{\alpha}[t_0,t_1]$ such that $x(t) = Tx(t)$. Writing this explicitly and extending the function by the function ϕ on $(-\infty,t_0]$, we get

$$\begin{aligned} x(t) &= \phi(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} [B(s, x(s))u(s) + Q(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left(f(s, x(s), {}^C D^\alpha x(s)) + \int_{t_0}^{s} q(s, \tau, x(\tau)) d\tau \right) ds \end{aligned} \tag{5.5}$$

on $[t_0, t_1]$ and $x(t) = \phi(t)$ on $(-\infty, t_0]$. Substituting (5.3) into (5.5), it can be easily shown that $x(t_1) = x_1$. Hence, the system (5.1) is controllable on $[t_0, t_1]$. □

Remark 5.1. *The controllability problem for the case $\alpha = 1$ is studied in [2].*

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