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Invariance of minimal prime ideals with applications

Shakir Ali[1], Cihat Abdioglu[*2], Vaishali Varshney[3] and Mohammad Salahuddin Khan[4]

[1,3]Department of Mathematics
Aligarh Muslim University, Aligarh, India
[2]Department of Mathematics and Science Education
Karamanoglu Mehmetbey University
Karaman, Turkey
[4]Department of Applied Mathematics, Z. H. College of Engineering & Technology, Aligarh Muslim University, Aligarh, India
Email: shakir.ali.mm@amu.ac.in, cabdioglu@kmu.edu.tr
vaishalivarshney7625@gmail.com, salahuddinkhan50@gmail.com

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Abstract

The present paper aims to prove the invariance of minimal prime ideals under higher derivations. Later on, with the help of invariance property of minimal prime ideals under higher derivations, we establish the $*$-version of Posner's second theorem for higher derivations in semiprime rings with involution '$*$'.

1 Introduction

Throughout this article, $\mathcal{T}$ will be used to designate a semiprime ring with center $Z(\mathcal{T})$. "Martindale's ring of quotient and extended centroid of $\mathcal{T}$ will be denoted

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[*]**Corresponding Author**

by Q and C, respectively (we refer the reader to [2], for the definitions and related properties of these objects). For any $\vartheta, \ell \in \mathcal{T}$, denote the commutator $\vartheta\ell - \ell\vartheta$ by $[\vartheta, \ell]$. Recall that a ring $\mathcal{T}$ is prime if $\vartheta\mathcal{T}\ell = \{0\}$ implies that $\vartheta = 0$ or $\ell = 0$, and $\mathcal{T}$ is said to be semiprime if $\vartheta\mathcal{T}\vartheta = \{0\}$ implies that $\vartheta = 0$. Recall that an ideal P of $\mathcal{T}$ is said to be prime if $P \neq \mathcal{T}$ and for $\vartheta, \ell \in \mathcal{T}$, $\vartheta\mathcal{T}\ell \subseteq P$ implies that $\vartheta \in P$ or $\ell \in P$. A prime ideal P of $\mathcal{T}$ is minimal if P does not properly include any prime ideals of $\mathcal{T}$. Let $n \geq 2$ be an integer, a ring $\mathcal{T}$ is said to be n-torsion free if $n\vartheta = 0$ implies $\vartheta = 0$ where $\vartheta \in \mathcal{T}$. The right annihilator of a set S in $\mathcal{T}$ is defined as the set $ann_r(S) = \{r \in \mathcal{T} \mid sr = 0 \text{ for all } s \in S\}$. The left annihilator of a set S in $\mathcal{T}$ is defined in a similar manner as the set $ann_l(S) = \{r \in \mathcal{T} \mid rs = 0 \text{ for all } s \in S\}$. If left annihilator and right annihilator coincide, then it is simply called an annihilator of S. A ring $\mathcal{T}$ is of bounded index if there is an integer $n > 1$ such that $\vartheta^n = 0$ whenever ϑ is a nilpotent element of $\mathcal{T}$. The least such positive integer is called the index of $\mathcal{T}$. A ring without nonzero nilpotent elements is merely a ring of bounded index 1. A ring is called a reduced ring if it has no nonzero nilpotent elements. Equivalently, a ring is reduced if it has no nonzero elements with square zero, that is, $\vartheta^2 = 0$ implies $\vartheta = 0$.

An involution is an additive mapping $\vartheta \mapsto \vartheta^*$ of $\mathcal{T}$ such that (i) $(\vartheta\ell)^* = \ell^*\vartheta^*$ and (ii) $(\vartheta^*)^* = \vartheta$ for all $\vartheta, \ell \in \mathcal{T}$. A ring equipped with an involution is referred as ring with involution or a $*$-ring. An element ϑ in a $*$-ring $\mathcal{T}$ is said to be hermitian if $\vartheta^* = \vartheta$ and skew-hermitian if $\vartheta^* = -\vartheta$. $H(\mathcal{T})$ and $S(\mathcal{T})$ refer to the set of all hermitian and skew-hermitian elements of $\mathcal{T}$, respectively. The involution is said to be of the first kind if $Z(\mathcal{T}) \subseteq H(\mathcal{T})$, otherwise it is said to be of the second kind. In the later case, $S(\mathcal{T}) \cap Z(\mathcal{T}) \neq \{0\}$ (see [12] for details). A ring equipped with an involution $*$ is said to be $*$-prime if $\vartheta\mathcal{T}\ell = \vartheta^*\mathcal{T}\ell = \{0\}$ or $\vartheta\mathcal{T}\ell = \vartheta\mathcal{T}\ell^* = \{0\}$ implies that $\vartheta = 0$ or $\ell = 0$. It is easy to verify that every $*$-prime ring is a semiprime ring.

An additive mapping $f : \mathcal{T} \to \mathcal{T}$ is said to be centralizing (resp. commuting) on $\mathcal{T}$ if $[f(\vartheta), \vartheta] \in Z(\mathcal{T})$ (resp. $[f(\vartheta), \vartheta] = 0$) for all $\vartheta \in \mathcal{T}$. A map $d : \mathcal{T} \to \mathcal{T}$ is a derivation of a ring $\mathcal{T}$ if d is additive and satisfies the Leibniz rule; $d(\vartheta\ell) = d(\vartheta)\ell + \vartheta d(\ell)$, for all $\vartheta, \ell \in \mathcal{T}$. An obvious example of a non-trivial derivation is the usual differentiation on the ring $F[\vartheta]$ of polynomials defined over a field F. For a fixed $a \in \mathcal{T}$, define $d : \mathcal{T} \to \mathcal{T}$ by $d(\vartheta) = [a, \vartheta]$ for all $\vartheta \in \mathcal{T}$. The function d so defined can be easily checked to be additive and $d(\vartheta\ell) = [a, \vartheta\ell] = \vartheta[a, \ell] + [a, \vartheta]\ell = \vartheta d(\ell) + d(\vartheta)\ell$ for all $\vartheta, \ell \in \mathcal{T}$. Thus, d is a derivation which is called inner derivation of $\mathcal{T}$ associated with a and is generally denoted by I_a. It is obvious to see that every inner derivation on a ring $\mathcal{T}$ is a derivation. Given a derivation d of $\mathcal{T}$, an ideal I of $\mathcal{T}$ is said to be invariant under d or d-invariant if $d(I) \subseteq I$. Many authors have analyzed the structure of rings and the structure of additive mappings in various ways. They have extended the notion of derivation in numerous directions, such as generalized derivation, Jordan derivation, higher derivation, etc. In 1957, Posner [26] proved that a prime ring must be commutative if it possesses a nonzero centralizing derivation. Vukman [28] extended Posner's second theorem by showing that if d is a derivation of prime ring of characteristic not 2 such that $[[d(\vartheta), \vartheta], \vartheta] = 0$, for all $\vartheta \in \mathcal{T}$, then $d = 0$ or $\mathcal{T}$ is commutative.

In [27], Schmidt and Hasse introduced the notion of higher derivation as follows: let $D = (d_i)_{i\in\mathbb{N}}$ be a family of additive mappings $d_i : \mathcal{T} \to \mathcal{T}$. Then D is said to be a higher derivation on $\mathcal{T}$, if $d_0(\vartheta) = \vartheta$ and $d_n(\vartheta\ell) = \sum\limits_{i+j=n} d_i(\vartheta)d_j(\ell)$ for all $\vartheta, \ell \in \mathcal{T}$ and for each $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all non-negative integers. On the similar lines, a higher derivation of rank $n \geq 1$ on $\mathcal{T}$ is defined as a family of additive mappings, say, $(d_0, d_1, \ldots, d_n)$, where d_0 is the identity map on $\mathcal{T}$ and $d_m(\vartheta\ell) = \sum\limits_{i+j=m} d_i(\vartheta)d_j(\ell)$ for all $\vartheta, \ell \in \mathcal{T}$ and $1 \leq m \leq n$. The above expression can be written as $d_m(\vartheta\ell) = d_m(\vartheta)\ell + B_m(\vartheta, \ell) + \vartheta d_m(\ell)$, where $B_m(\vartheta, \ell) = \sum\limits_{\substack{i+j=m\\ i,j\geq 1}} d_i(\vartheta)d_j(\ell)$. This notation has been used more frequently in this article. Observe that for $n = 1$, d_1 is just a derivation of $\mathcal{T}$. If $z \in Z(\mathcal{T})$, then $d_1(z) \in Z(\mathcal{T})$. For $n = 2$, we have

$$d_2(\vartheta z) = d_2(z\vartheta)$$

or

$$d_2(\vartheta)z + d_1(\vartheta)d_1(z) + \vartheta d_2(z) = d_2(z)\vartheta + d_1(z)d_1(\vartheta) + zd_2(\vartheta)$$

for all $\vartheta \in \mathcal{T}$ which gives $d_2(z) \in Z(\mathcal{T})$. Similarly, $d_i(z) \in Z(\mathcal{T})$ for all $i \in \mathbb{N}$.

In [20], Herstein postulated a conjecture stating that in a semiprime ring $\mathcal{T}$, every minimal prime ideal is invariant under any derivation d of $\mathcal{T}$ (cf.; [20]). This classical problem was also brought up by other authors as well. It would be interesting to look at some favourable partial results on this conjecture (see [3,7,10,14] for details). Krempa [14] proved that in a semiprime algebra over a field of characteristic 0, all minimal prime ideals are always invariant under derivation. Propositions 1.1 and 1.3 of [10] provides a much deeper result in this direction. The best result on this conjecture was given by Beidar and Mikhalev [3]. They proved that: let $\mathcal{T}$

be a ring of bounded index m such that the additive order of every nonzero torsion element of $\mathcal{T}$, if any, is strictly larger than m. Then all minimal prime ideals of $\mathcal{T}$ are invariant under any derivation of $\mathcal{T}$. In particular, every minimal prime ideal of a reduced ring is invariant under derivation of the ring. But the above conjecture is not true in general (viz., [6] for counter example). In 2006, Chuang and Lee [7] proved the following: if the semiprime ring $\mathcal{T}$ either satisfies a polynomial identity or has only countably many elements, then there exists a family $\{P_\alpha\}_{\alpha\in A}$ of minimal prime ideals such that $\bigcap_{\alpha\in A} P_\alpha = 0$ and each P_α is d-invariant for any derivation d of $\mathcal{T}$. In this direction, Matczuk [22] proved that every minimal prime ideal P, which has nonzero annihilator of a semiprime ring $\mathcal{T}$ is invariant under any derivation d of $\mathcal{T}$. Very recently, in [19], Lee and Lin generalizes the Matczuk's result for arbitrary rings.

Section 3 investigates the invariance property of prime ideals under higher derivations. In Section 4, we establish $*$-version of Posner's second theorem for higher derivations in semiprime rings with involution. Precisely, we prove the following result: let $\mathcal{T}$ be a semiprime ring with involution $*$ of first kind and L be a $*$-ideal of $\mathcal{T}$. Next, let $(d_i)_{i\in\mathbb{N}}$ be a higher derivation of rank $\mathcal{T}$ on $\mathcal{T}$ such that $[d_n(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in L$, $n \leq r$. Then either $\mathcal{T}$ is a commutative ring or the center of $\mathcal{T}$ is mapped to zero by some linear combination of $(d_i)_{i\in\mathbb{N}}$".

2 Preliminaries

In this section, we compile several familiar facts and outcomes that aid us in establishing our conclusions.

- $Z(Q) \cap \mathcal{T} = Z(\mathcal{T})$ ([15], Proposition 14.17).

- The Martindale ring of quotient Q associated with a semiprime ring $\mathcal{T}$ is itself a semiprime ring ([13], page 65).

- $L, \mathcal{T}$ and Q satisfy the same generalized polynomial identity (GPI) where L is an ideal of $\mathcal{T}$ ([5]).

- For a prime ring $\mathcal{T}$, its Martindale ring of quotients is itself a prime ring, and the associated extended centroid C is a field. ([12], page 22).

Lemma 2.1. *[9, Proposition 1.1] Let $\mathcal{T}$ be a semiprime ring and let $D = (d_i)_{i\in\mathbb{N}}$ represent a higher derivation on $\mathcal{T}$. Then there exists a unique higher derivation $D^* = (d_i^*)_{i\in\mathbb{N}}$ on Q such that $d_n^*|_{\mathcal{T}} = d_n$ for every $n \in \mathbb{N}$.*

Lemma 2.2. *[17, Main Theorem] Let $\mathcal{T}$ be a semiprime ring, D be a nonzero derivation of $\mathcal{T}$ and L be a nonzero left ideal of $\mathcal{T}$. If for some positive integers $t_0, t_1, \ldots, t_n$ and for all $\vartheta \in L$, the identity $[[\ldots[[D(\vartheta^{t_0}), \vartheta^{t_1}], \vartheta^{t_2}], \ldots], \vartheta^{t_n}] = 0$ holds, then either $D(L) = 0$, or both $D(L)$ and $D(\mathcal{T})L$ are contained within a nonzero central ideal of $\mathcal{T}$. Specifically, if $\mathcal{T}$ is a prime ring, then $\mathcal{T}$ must be commutative.*

Lemma 2.3. *Let $\mathcal{T}$ be a semiprime ring. Let M be a minimal prime ideal of $\mathcal{T}$. Then intersection of the rest of minimal prime ideals of $\mathcal{T}$ is the annihilator of M.*

Proof. Let $\mathcal{A}$ represent the collection of all minimal prime ideals of $\mathcal{T}$. For a minimal prime ideal M of $\mathcal{T}$, $\mathcal{A}_M$ denote the set $\mathcal{A} \setminus \{M\}$. Now for such a ideal M of $\mathcal{T}$, we show that $ann(M) = \bigcap_{\mathcal{P}\in\mathcal{A}_M} \mathcal{P}$. Let $\bigcap_{\mathcal{P}\in\mathcal{A}_M} \mathcal{P} = \mathcal{B}$. Now we have to show $ann(M) = \mathcal{B}$. Consider $M \cap \mathcal{B}$. Any element in $M \cap \mathcal{B}$ is in the intersection of all minimal prime ideals of $\mathcal{T}$. Using the semiprimeness of $\mathcal{T}$, we get $M \cap \mathcal{B} = 0$. Now, $M\mathcal{B} \subseteq M \cap \mathcal{B} = 0$. This gives

$$\mathcal{B} \subseteq ann(M). \tag{2.1}$$

The reverse inclusion is a direct consequence of the inclusion

$$M(ann(M)) = 0 \subseteq \mathcal{P},$$

where $\mathcal{P}$ is a minimal prime ideal of $\mathcal{T}$ different from M. Using primeness of $\mathcal{P}$, for every $\mathcal{P} \in \mathcal{A}_M$ either $M \subseteq \mathcal{P}$ or $ann(M) \subseteq \mathcal{P}$. By the minimality of $\mathcal{P}$, $M \nsubseteq \mathcal{P}$ so we have, $ann(M) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is arbitrary, it follows that

$$ann(M) \subseteq \mathcal{B}. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\mathcal{B} = ann(M) = \bigcap_{\mathcal{P}\in\mathcal{A}_M} \mathcal{P}.$$

This proves the lemma. □

3 Invariance of prime ideals under higher derivations

This section aims to prove the conjecture suggested by Herstein for higher derivations (viz., [20]). A number of authors have proved the Herstein's conjecture in various ways (see [3, 7, 10, 14] for details). Recently, Matczuk [22], proved the following result:

Theorem 3.1. *For a semiprime ring $\mathcal{T}$, any minimal prime ideal $\mathcal{P}$ with a nonzero annihilator is invariant under any derivation d of $\mathcal{T}$.*

Additionally, Lee and Lin [19] extended the previously mentioned result to arbitrary rings. Specifically, they established the following result:

Theorem 3.2. *Let $\mathcal{P}$ be a prime ideal of a ring $\mathcal{T}$ with $l_{\mathcal{T}}(\mathcal{P}) \nsubseteq \mathcal{P}$. Then $\mathcal{P}$ is invariant under any derivation of $\mathcal{T}$.*

A natural question arises: do the above results hold for higher derivations? Theorems 3.3 & 3.4 below give affirmative answers.

Theorem 3.3. *Let $\mathcal{T}$ be a semiprime ring. Then every minimal prime ideal $\mathfrak{K}$ of $\mathcal{T}$ having nonzero annihilator is invariant under the action of the higher derivation $(d_i)_{i\in\mathbb{N}}$ of $\mathcal{T}$.*

Proof. In a semiprime ring $\mathcal{T}$, it is a well-known fact that the right annihilator of an ideal J coincides with its left annihilator which we called the annihilator of J and is denoted by $ann(J)$.
Let $\mathcal{A}$ represent the collection of all minimal prime ideals of the ring $\mathcal{T}$. For a minimal prime ideal $\mathfrak{K}$ of $\mathcal{T}$, $\mathcal{A}_{\mathfrak{K}}$ denote the set $\mathcal{A} \setminus \{\mathfrak{K}\}$. In view of Lemma 2.3, we obtain $ann(\mathfrak{K}) = \bigcap\limits_{\mathcal{P}\in\mathcal{A}_{\mathfrak{K}}} \mathcal{P}$.

Let $\mathfrak{K}$ be a minimal prime ideal of $\mathcal{T}$ such that $ann(\mathfrak{K}) = I \neq \{0\}$. It is given that $(d_i)_{i\in\mathbb{N}}$ is higher derivation on $\mathcal{T}$. The proof is carried out using the principle of mathematical induction on i. When $i = 1$, the result is a direct consequence of Theorem 3.1. In the case $i = 2$, for any $q \in \mathfrak{K}$ and $\vartheta \in I$, we have

$$\begin{aligned} 0 &= d_2(q\vartheta) \\ &= d_2(q)\vartheta + d_1(q)d_1(\vartheta) + qd_2(\vartheta). \end{aligned}$$

On multiplying by $t \in I$ from left to the above expression and using the fact that $I\mathfrak{K} = \{0\}$ and $d_1(\mathfrak{K}) \subseteq \mathfrak{K}$, we get $td_2(q)\vartheta = 0$ for all $\vartheta, t \in I$ and $q \in \mathfrak{K}$. This further gives that

$$\{0\} = I\mathcal{T}d_2(\mathfrak{K})\mathcal{T}I \subseteq \mathfrak{K}.$$

Because $\mathfrak{K}$ is a prime ideal and $I \nsubseteq \mathfrak{K}$, we can conclude that $d_2(\mathfrak{K}) \subseteq \mathfrak{K}$. Hence, $\mathfrak{K}$ is invariant under d_2.

Let the result holds for all $i \leq k$, that is, $d_i(\mathfrak{K}) \subseteq \mathfrak{K}$ for all $i \leq k$. Now, for $i = k+1$, and $q \in \mathfrak{K}, \vartheta \in I$, we have

$$\begin{aligned} 0 &= d_{k+1}(q\vartheta) \\ &= d_{k+1}(q)\vartheta + d_k(q)d_1(\vartheta) + \cdots + d_1(q)d_k(\vartheta) + qd_{k+1}(\vartheta). \end{aligned}$$

Again multiply the above relation by $t \in I$ from left and using $I\mathfrak{K} = \{0\}$ and $d_i(\mathfrak{K}) \subseteq \mathfrak{K}$, we conclude that $td_{k+1}(q)\vartheta = 0$ for all $\vartheta, t \in I$ and $q \in \mathfrak{K}$. This implies that

$$\{0\} = I\mathcal{T}d_{k+1}(\mathfrak{K})\mathcal{T}I \subseteq \mathfrak{K}.$$

By the primeness of $\mathfrak{K}$ and $I \nsubseteq \mathfrak{K}$, we obtain $d_{k+1}(\mathfrak{K}) \subseteq \mathfrak{K}$. Hence, $\mathfrak{K}$ is d_i-invariant under higher derivation $(d_i)_{i\in\mathbb{N}}$ of $\mathcal{T}$. □

Let $\mathcal{P}$ be a prime ideal of a semiprime ring $\mathcal{T}$, satisfying the condition $l_{\mathcal{T}}(\mathcal{P}) \neq \{0\}$. Then, it is well-established that the left annihilator of $\mathcal{P}$, denoted by $l_{\mathcal{T}}(\mathcal{P})$, satisfies $l_{\mathcal{T}}(\mathcal{P}) \cap \mathcal{P} = \{0\}$. In particular, we have $l_{\mathcal{T}}(\mathcal{P}) \nsubseteq \mathcal{P}$. Building on this idea, we extend the aforementioned theorem to arbitrary rings in the context of higher derivation.

Theorem 3.4. *Let $\mathcal{T}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{T}$ and $l_{\mathcal{T}}(\mathcal{P})$ be left annihilator of $\mathcal{P}$, with the condition that $l_{\mathcal{T}}(\mathcal{P}) \nsubseteq \mathcal{P}$. Then, $\mathcal{P}$ is d_i-invariant under higher derivation $(d_i)_{i\in\mathbb{N}}$ of $\mathcal{T}$.*

Proof. Suppose $I = l_{\mathcal{T}}(\mathcal{P}) = \{\vartheta \in \mathcal{T} \mid \vartheta p = 0 \text{ for all } p \in \mathcal{P}\}$. Note that I forms an ideal of $\mathcal{T}$. By the definition of I, for any $a \in I$ and $p \in \mathcal{P}$, we have $ap = 0$. We prove the result by induction on i. For $i = 1$, d_1 is a derivation, so the result follows from Theorem 3.2. For $i = 2$, we have

$$\begin{aligned} 0 &= d_2(ap) \\ &= d_2(a)p + d_1(a)d_1(p) + ad_2(p) \end{aligned}$$

for all $a \in I$ and $p \in \mathcal{P}$. This gives that $Iad_2(p) = \{0\}$ for all $a \in I$ and $p \in \mathcal{P}$. Since $a \in I$ is arbitrary, we have $I^2 d_2(p) = \{0\}$. This can be written as

$$I^2 \mathcal{T} d_2(\mathcal{P}) \subseteq \mathcal{P}.$$

Primeness of $\mathcal{P}$ and the condition $I^2 \subseteq I \nsubseteq \mathcal{P}$ yields that $d_2(\mathcal{P}) \subseteq \mathcal{P}$. Now, suppose that the result holds for all $i \leq k$, that is, $d_i(\mathcal{P}) \subseteq \mathcal{P}$ for all $i \leq k$. For $i = k+1$, we have

$$\begin{aligned} 0 &= d_{k+1}(ap) \\ &= d_{k+1}(a)p + d_k(a)d_1(p) + \cdots + d_1(a)d_k(p) + ad_{k+1}(p) \end{aligned}$$

for all $a \in I$ and $p \in \mathcal{P}$. This implies

$$Iad_{k+1}(p) = 0$$

for all $a \in I$ and $p \in \mathcal{P}$, which further gives

$$I^2 \mathcal{T} d_{k+1}(\mathcal{P}) \subseteq \mathcal{P}.$$

Again by using the primeness of $\mathcal{P}$ and the condition $I \nsubseteq \mathcal{P}$, we conclude that $d_{k+1}(\mathcal{P}) \subseteq \mathcal{P}$. Hence, $\mathcal{P}$ is invariant under higher derivation $(d_i)_{i\in\mathbb{N}}$ of $\mathcal{T}$. This proves the theorem completely. □

Using the analogous argument, we may come up with the following:

Corollary 3.1. *Let $\mathcal{T}$ be a ring and $\mathcal{P}$ a prime ideal of $\mathcal{T}$ with $r_{\mathcal{T}}(\mathcal{P}) \nsubseteq \mathcal{P}$, where $r_{\mathcal{T}}(\mathcal{P})$ is a right annihilator of $\mathcal{P}$. Then, $\mathcal{P}$ is d_i-invariant under higher derivations $(d_i)_{i\in\mathbb{N}}$ of $\mathcal{T}$.*

It is worth noting that in the hypotheses of Theorem 3.4, the condition of $l_{\mathcal{T}}(\mathcal{P}) \nsubseteq \mathcal{P}$ is necessary. This is shown by the following example.

Example 3.1. *Let $\mathcal{T} = \mathbb{Q}[X]$ be the ring of polynomials over $\mathbb{Q}$. Consider $\mathcal{P} =< x^2+1 >$ to be the prime ideal of $\mathcal{T}$. Define $d : \mathcal{T} \longrightarrow \mathcal{T}$ by $d(f) = f'$ for all $f \in \mathcal{T}$. Then, it is straightforward to check that d is a derivation on $\mathcal{T}$. Moreover, if we put $d_0 = id_{\mathcal{T}}$ (the identity map on $\mathcal{T}$) and $d_n(f) = \frac{f^{(n)}}{n!}$ for $n \geq 1$, $f \in \mathcal{T}$,* *then $D = (d_n)_{n\in\mathbb{N}}$ is a higher derivation on $\mathcal{T}$. Further, it is easy to see that $l_{\mathcal{T}}(\mathcal{P}) \subseteq \mathcal{P}$. However, $d_i(\mathcal{P}) \nsubseteq \mathcal{P}$ for some values of $\mathcal{P}$.*

4 Applications

This section focuses on examining the applications of the results established in the preceding section, including connections to a renowned result established by Posner [26]. In [26], Posner made groundbreaking contributions to the study of centralizing and commuting mappings. He proved a fundamental theorem stating that the presence of a nonzero centralizing derivation in a prime ring ensures the ring's commutativity. Further, Lanski [16] used differential identities to extend Posner's work in more general setting for Lie ideals. Furthermore, Posner's second theorem has also been generalized in a number of ways and several important outcomes have already been derived (see for example, [4, 8, 18, 25] for details).

On the other hand, recently Ali and Dar [1] demonstrated the $*$-version of result established by Posner. Indeed, they proved that if $\mathcal{T}$ is a prime ring that is 2-torsion free and possesses an involution $*$ and d is a nonzero derivation of $\mathcal{T}$ such that $[d(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in \mathcal{T}$, along with the condition $d(S(\mathcal{T}) \cap Z(\mathcal{T})) \neq \{0\}$, then $\mathcal{T}$ is commutative. Further, this result improved in [24, Theorem 3.7] and they removed the condition $d(S(\mathcal{T}) \cap Z(\mathcal{T})) \neq \{0\}$. Our next theorem is the $*$-version of Posner's second theorem for higher derivations in rings with involution.

Theorem 4.1. *Let $\mathcal{T}$ be a semiprime ring equipped with an involution $*$ of the first kind and L be a $*$-ideal of $\mathcal{T}$. Next, let $(d_i)_{i\in\mathbb{N}}$ be a higher derivation of rank r on $\mathcal{T}$ such that $[d_n(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in L$, $n \leq r$. Then, either $\mathcal{T}$ is a commutative ring or there exists a linear combination of $(d_i)_{i\in\mathbb{N}}$ that maps the center of $\mathcal{T}$ to zero.*

Proof. Under the stated assumption, it holds that

$$[d_n(\vartheta), \vartheta^*] \in Z(\mathcal{T}) \tag{4.1}$$

$\forall\, \vartheta \in L$. By applying a linearization to (4.1), it follows that

$$[d_n(\vartheta), \ell^*] + [d_n(\ell), \vartheta^*] \in Z(\mathcal{T}) \tag{4.2}$$

$\forall\, \vartheta, \ell \in L$. Substituting ℓ with ℓh in (4.2), where $h \in H(\mathcal{T}) \cap Z(\mathcal{T})$, results in

$$[B_n(\ell, h) + \ell d_n(h), \vartheta^*] \in Z(\mathcal{T})$$

$\forall\, \vartheta, \ell \in L$, which infer that

$$[[B_n(\ell, h) + \ell d_n(h), \vartheta], t] = 0 \tag{4.3}$$

$\forall\, \vartheta, \ell, t \in L$. Since L and Q both satisfy the same generalized polynomial identity, and as stated in Lemma 2.1, any higher derivation defined on $\mathcal{T}$ can be extended uniquely to a higher derivation on Q. Thus, we can conclude that

$$[[B_n(\ell, h) + \ell d_n(h), \vartheta], t] = 0 \tag{4.4}$$

for all $\vartheta, \ell, t \in Q$. Let $\mathcal{P}$ be a prime ideal of Q with $l_Q(\mathcal{P}) \neq \{0\}$. Then $\bar{Q} = Q/\mathcal{P}$ is a prime ring. Utilizing the invariance of prime ideals under higher derivation, we introduce a family of additive mappings $(\bar{d}_i)_{i\in\mathbb{N}} : \bar{Q} \to \bar{Q}$ such that $\bar{d}_i(q + \mathcal{P}) = d_i(q) + \mathcal{P}$ for all $q \in Q$. It is easy to check that $(\bar{d}_i)_{i\in\mathbb{N}}$ is a higher derivation on $\bar{Q}$. From (4.4), we conclude that

$$[[B_n(\bar{\ell}, \bar{h}) + \bar{\ell}\bar{d}_n(\bar{h}), \bar{\vartheta}], \bar{t}] = \bar{0}$$

$\forall\, \bar{\vartheta}, \bar{r} \in \bar{Q}$. This can be reformulated as

$$[\delta(\bar{\vartheta}), \bar{t}] = \bar{0}$$

$\forall\, \bar{\vartheta}, \bar{t} \in \bar{Q}$ and $\bar{h} \in Z(\bar{Q})$, where $\delta(\bar{\vartheta}) = [B_n(\bar{\ell}, \bar{h}) + \bar{\ell}\bar{d}_n(\bar{h}), \bar{\vartheta}]$ is an inner derivation for fix $\bar{\ell}$ and $\bar{h}$. Replace $\bar{r}$ by $\bar{\vartheta}$, we get

$$[\delta(\bar{\vartheta}), \bar{\vartheta}] = \bar{0}. \tag{4.5}$$

Thus, in view of Lemma 2.2, we establish that $\delta = \bar{0}$ or $\bar{Q}$ is commutative.
If $\bar{Q}$ is commutative, then $[\bar{\vartheta}, \bar{\ell}] = \bar{0}\ \forall\, \bar{\vartheta}, \bar{\ell} \in \bar{Q}$. This implies $[\vartheta, \ell] \in \mathcal{P}$ for all $\vartheta, \ell \in Q$. Since $\mathcal{P}$ represents an arbitrary prime ideal of Q and $\bigcap\{\mathcal{P} \mid \mathcal{P}$ is a prime ideal of $Q\} = \{0\}$, it follows that $[\vartheta, \ell] = 0\ \forall\, \vartheta, \ell \in Q$. This shows that Q is commutative, leading to the conclusion that $\mathcal{T}$ is commutative. On the other hand,

if $\delta = \bar{0}$, then

$$[B_n(\bar{\ell}, \bar{h}) + \bar{\ell}\bar{d}_n(\bar{h}), \bar{\vartheta}] = \bar{0} \tag{4.6}$$

for all $\bar{\vartheta} \in \bar{Q}$. Substituting the expression for $B_n(\bar{\ell}, \bar{h})$ into the preceding equation, we get

$$[\bar{d}_{n-1}(\bar{\ell})\bar{d}_1(\bar{h}) + \bar{d}_{n-2}(\bar{\ell})\bar{d}_2(\bar{h}) + \cdots + \bar{d}_1(\bar{\ell})\bar{d}_{n-1}(\bar{h}) + \bar{\ell}\bar{d}_n(\bar{h}), \bar{\vartheta}] = \bar{0}$$

$\forall\, \bar{\vartheta} \in \bar{Q}$. Using the fact $\bar{d}_i(\bar{h}) \in Z(\bar{Q})$ for $\bar{h} \in Z(\bar{Q})$, the above equation reduces to

$$\begin{aligned}&[\bar{d}_{n-1}(\bar{\ell}), \bar{\vartheta}]\bar{d}_1(\bar{h}) + [\bar{d}_{n-2}(\bar{\ell}), \bar{\vartheta}]\bar{d}_2(\bar{h}) + \cdots + [\bar{d}_1(\bar{\ell}), \bar{\vartheta}]\bar{d}_{n-1}(\bar{h}) \\ &+ [\bar{\ell}, \bar{\vartheta}]\bar{d}_n(\bar{h}) = \bar{0}\end{aligned} \tag{4.7}$$

for all $\bar{\vartheta} \in \bar{Q}$. Replacing $\bar{\vartheta}$ by $\bar{\vartheta}\bar{r}$ (where $\bar{r} \in \bar{Q}$) in (4.7), we obtain

$$\begin{aligned}&[\bar{d}_{n-1}(\bar{\ell}), \bar{\vartheta}]\bar{r}\bar{d}_1(\bar{h}) + [\bar{d}_{n-2}(\bar{\ell}), \bar{\vartheta}]\bar{r}\bar{d}_2(\bar{h}) + \cdots + [\bar{d}_1(\bar{\ell}), \bar{\vartheta}]\bar{r}\bar{d}_{n-1}(\bar{h}) \\ &+ [\bar{d}_0(\bar{\ell}), \bar{\vartheta}]\bar{r}\bar{d}_n(\bar{h}) = \bar{0}\end{aligned}$$

$\forall\, \bar{\vartheta}, \bar{r} \in \bar{Q}$. By [21, Corollary, page 444], we conclude that either the set $\{[\bar{d}_0(\bar{\ell}), \bar{v}], [\bar{d}_1(\bar{\ell}), \bar{v}], \ldots, [\bar{d}_{n-1}(\bar{\ell}), \bar{v}]\}$ is linearly dependent or $\{\bar{d}_1(\bar{h}), \bar{d}_2(\bar{h}), \ldots, \bar{d}_n(\bar{h})\}$ is linearly dependent over the center $\bar{C}$ of $\bar{Q}$.

First, we consider the set $\{\bar{d}_1(\bar{h}),\ \bar{d}_2(\bar{h}), \ldots, \bar{d}_n(\bar{h})\}$ is linearly dependent over $\bar{C}$, then there exist scalars $\bar{\varphi}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_n \in \bar{C}$, not all zero such that $\sum\limits_{i=1}^{n} \bar{\gamma}_i\bar{d}_i(\bar{h}) = \bar{0}$ or $\sum\limits_{i=1}^{n} \gamma_i d_i(h) \in \mathcal{P}$. Since $\mathcal{P}$ is an arbitrary prime ideal of Q and $\bigcap\{\mathcal{P} \mid \mathcal{P}$ is a prime ideal of $Q\} = \{0\}$, so we have $\sum\limits_{i=1}^{n} \gamma_i d_i(h) = 0$ for all $h \in Z(\mathcal{T})$. That is, $\sum\limits_{i=1}^{n} \gamma_i d_i(Z(\mathcal{T})) = \{0\}$, which is the required result.

Now, if the set $\{[\bar{d}_0(\bar{\ell}), \bar{\vartheta}], [\bar{d}_1(\bar{\ell}), \bar{\vartheta}], \ldots, [\bar{d}_{n-1}(\bar{\ell}), \bar{\vartheta}]\}$ is linearly dependent over $\bar{C}$, then there exist scalars $\bar{\varphi}_0, \bar{\varphi}_1, \bar{\varphi}_2, \ldots, \bar{\varphi}_{n-1} \in \bar{C}$, not all zero, such that

$$\bar{\varphi}_0[\bar{d}_0(\bar{\ell}), \bar{\vartheta}] + \bar{\varphi}_1[\bar{d}_1(\bar{\ell}), \bar{\vartheta}] + \cdots + \bar{\varphi}_{n-1}[\bar{d}_{n-1}(\bar{\ell}), \bar{\vartheta}] = \bar{0}.$$

Assume k to be the highest value of the index such that $\bar{\varphi}_k \neq 0$. In this way, the concluding expression takes the form

$$\sum_{i=0}^{k} \bar{\varphi}_i[\bar{d}_i(\bar{\ell}), \bar{\vartheta}] = \sum_{i=0}^{k} \bar{\varphi}_i \delta^i_{\bar{d}_i(\bar{\ell})}(\bar{\vartheta}) = \bar{0} \ \forall \ \bar{\vartheta} \in \bar{Q}, \tag{4.8}$$

where $\delta^i_{\bar{d}_i(\bar{\ell})}(\bar{\vartheta}) = [\bar{d}_i(\bar{\ell}), \bar{\vartheta}]$ for $i = 0, 1, \ldots, k$. Based on equation (4.8), we observe that the set of derivations $\left\{\delta^0_{\bar{d}_0}(\bar{\ell}), \delta^1_{\bar{d}_1}(\bar{\ell}), \ldots, \delta^k_{\bar{d}_k}(\bar{\ell})\right\}$ satisfies a linear relation over $\bar{Q}$ with coefficients in $\bar{C}$ of length $k+1$. According to [9, Corollary 1.4], there are $\bar{q_0} = \bar{1}, \bar{q_1}, \ldots, \bar{q_k} \in \bar{Q}$ such that

$$\sum_{i=0}^{k} \bar{q}_{k-i} \delta^i_{\bar{d}_i(\bar{\ell})} = \bar{0}. \tag{4.9}$$

Furthermore,

$$\delta^0_{\bar{d}_0(\bar{\ell})} = \delta_{\bar{q_1}} \ \text{ and } \ \delta^s_{\bar{d}_s(\bar{\ell})}(\bar{t}) = \delta_{\bar{q_s}}(\bar{t}) - \sum_{i=1}^{s-1} \bar{q}_i \delta^{s-i}_{\bar{d}_{s-i}(\bar{\ell})}(\bar{t}) \tag{4.10}$$

$\forall \ \bar{t} \in \bar{Q}, \ 2 \leq s \leq k$. If $\delta^0_{\bar{d}_0(\bar{\ell})} = \delta_{\bar{q_1}} = [\bar{d}_0(\bar{\ell}), \bar{t}] = [\bar{\ell}, \bar{t}] = [\bar{q}_1, \bar{t}]$ for all $\bar{t}, \bar{\ell} \in \bar{Q}$. Since $\bar{\ell}$ is arbitrarily fixed, so we replace $\bar{\ell}$ by $\bar{t}$, to get $[\bar{q}_1, \bar{t}] = \bar{0}$ for all $\bar{t} \in \bar{Q}$. This gives $\bar{q}_1 \in \bar{C}$. The second expression of equation (4.10) can also be written as

$$[\bar{d}_s(\bar{\ell}), \bar{t}] = [\bar{q_s}, \bar{t}] - \sum_{i=1}^{s-1} \bar{q}_i [\bar{d}_{s-i}(\bar{\ell}), \bar{t}] \ \forall \ \bar{t}, \bar{\ell} \in \bar{Q}. \tag{4.11}$$

In particular, for $\bar{\ell} = \bar{1}$ and using $\bar{d}_k(\bar{1}) = \bar{0}$ for all $k \geq 1$, we have $[\bar{q_s}, \bar{t}] = \bar{0}$ for all $\bar{t} \in \bar{Q}$. Thus $\bar{q_s} \in \bar{C} \ \ \forall \ \ 2 \leq s \leq k$. Hence, when $s = k$, the second term in (4.10) simplifies to

$$\delta^k_{\bar{d}_k(\bar{\ell})}(\bar{t}) + \sum_{i=1}^{k-1} \bar{q}_i \delta^{k-i}_{\bar{d}_{k-i}(\bar{\ell})}(\bar{t}) = \bar{0}. \tag{4.12}$$

Combining (4.9) and (4.12), we find that

$$\bar{q_k} \delta^0_{\bar{d}_0(\bar{\ell})}(\bar{t}) = \bar{0}$$

for all $\bar{t}, \bar{\ell} \in \bar{Q}$. That is,

$$\bar{q_k}[\bar{d_0}(\bar{\ell}), \bar{t}] = \bar{0}$$

for all $\bar{t}, \bar{\ell} \in \bar{Q}$. As $\bar{q_k} \in \bar{C}$, it follows that

$$\begin{aligned} \bar{0} &= [\bar{d_0}(\bar{\ell}), \bar{t}] \\ &= [\bar{\ell}, \bar{t}] \end{aligned}$$

for all $\bar{t}, \bar{\ell} \in \bar{Q}$. That is, $[\ell, t] \in \mathcal{P}$ for all $\ell, t \in Q$. Since $\mathcal{P}$ is an arbitrary prime ideal of Q and $\bigcap\{\mathcal{P} \mid \mathcal{P}$ is a prime ideal of $Q\} = \{0\}$, it follows that $[\ell, t] = 0$ for all $t, \ell \in Q$. Thus, Q is commutative and hence $\mathcal{T}$ is commutative. This completes the proof of the theorem. □

The necessity of the semiprimeness condition in Theorem 4.1 is established through the following example.

Example 4.1. *Let us consider $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$, where $\mathcal{T}_1 = \left\{\begin{bmatrix}\alpha & \beta \\ 0 & \gamma\end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{Q}\right\}$ and $\mathcal{T}_2 = \mathbb{Q}[X]$ and let $L = L_1 \times L_2$, where $L_1 = \left\{\begin{bmatrix}0 & \beta \\ 0 & 0\end{bmatrix} \mid \beta \in \mathbb{Q}\right\}$ and $L_2 = \mathbb{Q}[X]$ are ideals of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The involution Φ on $\mathcal{T}_1$ is defined by $\Phi(N) = \begin{bmatrix}\gamma & \beta \\ 0 & \alpha\end{bmatrix}$ for all $N \in \mathcal{T}_1$ and the involution Ψ on $\mathcal{T}_2$ is defined by $\Psi(f) = f$ for all $f \in \mathcal{T}_2$. Let us define involution $*$ on $\mathcal{T}$ by setting $(N, f)^* = (\Phi(M), \Psi(f))$ for all $(N, f) \in \mathcal{T}$. It can be readily verified that $\mathcal{T}$ is not a semiprime ring equipped with an involution $*$ of the first kind. Define $d_n : \mathcal{T}_1 \to \mathcal{T}_1$ by $d_0 = id_{\mathcal{T}_1}$ and $d_n\left(\begin{bmatrix}\alpha & \beta \\ 0 & \gamma\end{bmatrix}\right) = \begin{bmatrix}0 & \frac{\alpha-\gamma}{(n-1)!} + \frac{\beta}{n!} \\ 0 & 0\end{bmatrix}$, for all $\alpha, \beta, \gamma \in \mathbb{Q}$ and $g_n : \mathcal{T}_2 \to \mathcal{T}_2$ by $g_0 = id_{\mathcal{T}_2}$ and $g_n(f) = \frac{f^{(n)}}{n!}$, for all $f \in \mathcal{T}_2$ and $f^{(n)}$ denotes the n^{th} derivative of f. It is easy to see that $(d_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are higher derivations of $\mathcal{T}_1$ and $\mathcal{T}_2$. Next, define $D_n : \mathcal{T} \to \mathcal{T}$ by $D_n(N, f) = (d_n(N), g_n(f))$, where $D_0 = id_{\mathcal{T}}$ for all $(N, f) \in \mathcal{T}$. We can easily check that $(D_n)_{n\in\mathbb{N}}$ is a higher derivation of $\mathcal{T}$. For any element $(N, f) \in L$, the condition of Theorem 4.1, i.e., $[D_n(N, f), (N, f)^*] \in Z(\mathcal{T})$ is satisfied. However, $\mathcal{T}$ is a noncommutative ring and $\sum_{i=1}^{n} \lambda_i D_i(Z(\mathcal{T})) \neq \{0\}$.*

Theorem 4.1 has several fascinating corollaries, some of which are interesting by themselves.

Corollary 4.1. *Assume that $\mathcal{T}$ is a $*$-prime ring with involution $*$ of the first kind, and let L be a $*$-ideal of $\mathcal{T}$. Next, let $(d_i)_{i\in\mathbb{N}}$ be a higher derivation of rank r on $\mathcal{T}$ such that $[d_n(\vartheta), \vartheta^*] \in Z(\mathcal{T})\ \forall\ \vartheta \in L$, $n \leq r$. Then one of the following must hold: either $\mathcal{T}$ is a commutative ring, or there exists a linear combination of $(d_i)_{i\in\mathbb{N}}$ that maps the center of $\mathcal{T}$ to zero.*

Proof. It is a well-established fact that every $*$-prime ring is also a semiprime ring. Therefore, the conclusion directly follows from Theorem 4.1. □

Corollary 4.2. *Let $\mathcal{T}$ be a prime ring with involution $*$ of the first kind and L be a $*$-ideal of $\mathcal{T}$. Next, let $(d_i)_{i\in\mathbb{N}}$ be a higher derivation of rank $\mathcal{T}$ on $\mathcal{T}$ such that $[d_n(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in L$, $n \leq r$. Then, either $\mathcal{T}$ is a commutative ring or the center of $\mathcal{T}$ is mapped to zero by some linear combination of $(d_i)_{i\in\mathbb{N}}$.*

Corollary 4.3. *Let $\mathcal{T}$ be a semiprime ring with involution $*$ of the first kind. Next, let d be a derivation on $\mathcal{T}$ such that $[d(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in L$, where L be a $*$-ideal of $\mathcal{T}$. Then, either $\mathcal{T}$ is a commutative ring or $d(Z(\mathcal{T})) = \{0\}$.*

Proof. Put $n = 1$ and $d_1 = d$ in previous theorem, we get either $\mathcal{T}$ is commutative or $\{d(h)\}$, where $h \in Z(\mathcal{T}) \subseteq Z(Q) = C$ is linearly dependent over C which implies $d(h) = 0$. Since $h \in Z(\mathcal{T})$ was arbitrary, we get $d(Z(\mathcal{T})) = \{0\}$. □

Corollary 4.4. *Let $\mathcal{T}$ be a $*$-prime ring with involution $*$ of the first kind. Next, let d be a derivation on $\mathcal{T}$ such that $[d(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in L$, where L be a $*$-ideal of $\mathcal{T}$. Then, either $\mathcal{T}$ is a commutative ring or $d(Z(\mathcal{T})) = \{0\}$.*

Corollary 4.5. *Let $\mathcal{T}$ be a prime ring with involution $*$ of the first kind. Next, let d be a derivation on $\mathcal{T}$ such that $[d(\vartheta), \vartheta^*] \in Z(\mathcal{T})$ for all $\vartheta \in L$, where L be a $*$-ideal of $\mathcal{T}$. Then, either $\mathcal{T}$ is a commutative ring or $d(Z(\mathcal{T})) = \{0\}$.*

5 Conclusion

In conclusion, the present work has successfully demonstrated the invariance property of minimal prime ideals under higher derivations. Building upon this crucial result, we have extended our investigation to establish the $*$-version of Posner's second theorem for higher derivations in semiprime rings equipped with an involution $*$. This achievement not only advances our comprehension of the intricate relationship between minimal prime ideals and higher derivations but also provides valuable perspectives to the domain of ring theory.

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