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Approximation of functions by the matrix means in Orlics spaces

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Abstract

In this work, the approximation properties of the matrix submethods in Orlicz spaces are investigated . We obtain some results related to trigonometric approximation using matrix submethods of partial sums of Fourier series of functions in Orlicz spaces.The degree of trigonometric approximations by the matrix methods to the functions have been investigated in Orlicz spaces. The error of estimations in this work is obtained in more general terms.

1 Introduction and main results

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$. A convex and continuous function $M:[0,\infty)\to[0,\infty)$ which satisfies the conditions

$$\begin{aligned} M(0) &= 0,\ M(x)>0 \ \text{ for } x>0, \\ \lim_{x\to 0}(M(x)/x) &= 0;\ \ \lim_{x\to\infty}(M(x)/x)=\infty \end{aligned}$$

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is called a *Young function.* We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$ with some constant c, independent of u.

We can consider a right continuous, monotone increasing function $\rho : [0,\infty) \rightarrow [0,\infty)$ with

$$\rho(0) = 0;\ \lim_{t\rightarrow\infty} \rho(t) = \infty \text{ and } \rho(t) > 0 \text{ for } t > 0,$$

then the function defined by

$$N(x) = \int_0^{|x|} \rho(t)\, dt$$

is called $N-$*function*. For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \rightarrow \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\, dx < \infty.$$

The $N-$*function complementary to* M is defined by

$$N(y) := \max_{x\geq 0}(xy - M(x)),\ \text{for } y \geq 0.$$

Let N be the complementary Young function of M. It is well-known [20, p. 69], [33, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{ \int_{\mathbb{T}} |f(x)g(x)|\, dx : g \in \widetilde{L}_N(\mathbb{T}),\ \int_{\mathbb{T}} N(|g(x)|)\, dx \leq 1 \right\},$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \leq 1 \right\},$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [20, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}),\ 1 < p < \infty$. If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coindices with the usual Lebesgue spaces $L_p(\mathbb{T}),\ 1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics (e.g., [2], [4], [26] and [34]). Therefore, investigation of approximation of functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm. The inequalities

$$\|f\|^*_{L_M(\mathbb{T})} \leq \|f\|_{L_M(\mathbb{T})} \leq 2\,\|f\|^*_{L_M(\mathbb{T})}\,, \quad f \in L_M(\mathbb{T})$$

hold [24, p. 80].

If we choose $M(u) = u^p/p,\ 1 < p < \infty$ then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \leq \|u\|_{L_M(\mathbb{T})} \leq q^{1/q}\,\|u\|_{L_p(\mathbb{T})}\,,$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$ space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [24, p. 80]

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \leq \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})}\,,$$

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \leq \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}$$

are satisfied.

If we choose $M(u) = u^p/p \quad (1 < p < \infty)$ then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p}\,|u|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \leq \|u\|_{L_M(\mathbb{T})} \leq q^{1/q}\,\|u\|_{L_p(\mathbb{T})}\,,$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm of the $L_p(\mathbb{T})-$space.

A $N-$function M satisfies the Δ_2-*condition* if

$$\limsup_{x\to\infty} \frac{M\,(2x)}{M\,(x)} < \infty.$$

The Orlicz space $L_M(T)$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [33, p.113].

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* α_M, β_M

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h : (0,\infty) \to (0,\infty], \quad h(t) := \lim_{y\to\infty} \sup \frac{M^{-1}(y)}{M^{-1}(ty)},\ \ t > 0$$

first considered by Matuszewska and Orlicz [24], are called the *Boyd indices* of the Orlicz spaces $L_M(\mathbb{T})$.

It is known that the indices α_M and β_M satisfy $0 \le \alpha_M \le \beta_M \le 1, \alpha_N + \beta_M = 1, \alpha_M + \beta_N = 1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0 < \alpha_M \le \beta_M < 1$. The detailed information about the Boyd indices can be found in [3], [24], [25] and [33].

Let $L_M(\mathbb{T})$ be a Orlicz space. For $f \in L_M(\mathbb{T})$ we set

$$(\nu_h f)(x) := \frac{1}{h}\int_{-h}^{h} f(x+t)\,dt,\ 0 < h < \pi,\ x \in \mathbb{T}.$$

By reference [14, Lemma 1], the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T})$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T})} \le c\,\|f\|_{L_M(\mathbb{T})}.$$

The function

$$\Omega_M(\delta,\ f) := \sup_{0<h\le\delta} \|f(\cdot) - (\nu_h f)\|_{L_M(\mathbb{T})},\ \delta > 0$$

is called the *modulus of continuity* of $f \in L_M(\mathbb{T})$.

It can easily be shown that $\Omega_M(\cdot,\ f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega_M(\delta,\ f) = 0,\ \ \Omega_M(\delta,\ f+g) \le \Omega_M(\delta,\ f) + \Omega_M(\delta, g)$$

for $f,\ g\in L_M(\mathbb{T})$.

We will use the relation $f=O(g)$ which means that $f\leq cg$ for a constant c independent of f and g.

Let $f\in L_M(\mathbb{T})$. We define the following class of functions:

$$\Omega_M(f,\delta)=O\left(\omega(\delta)\right),\ \ \delta>0$$

where ω is a functions of modulus of continuity type on interval $[0,2\pi]$ .That ω is a nondecreasing continuous function having the following properties : $\omega(0)=0,\ \omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$ for any $0<\delta_1\leq\delta_2\leq\delta_1+\delta_2\leq 2\pi$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\left(f\right)\cos kx+b_k\left(f\right)\sin kx\right) \tag{1.1}$$

be the Fourier series of the function $f\in L^1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ the Fourier coefficients of the function f. The *n-th partial sum* of series (1.1) is defined, as

$$\begin{aligned} S_n(x,f) &:= \frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k(f)\cos kx+b_k(f)\sin kx\right), \\ &= \frac{a_0}{2}+\sum_{k=1}^{n}B_k(x,f)=\sum_{k=0}^{n}B_k(x,f)\ ,\ B_0(x,f):=\frac{a_0}{2}\ , \\ B_k(x,f) &:= \left(a_k(f)\cos kx+b_k(f)\sin kx\right). \end{aligned}$$

If $A:=(a_{n,k})_{0\leq n,k<\infty}$ be an infinite matrix of real numbers such that

$$a_{n,k}\geq 0 \text{ when } k,n=0,1,2,\ldots,\ \lim_{n\to\infty}a_{n,k}=0 \text{ and } \sum_{k=0}^{\infty}a_{n,k}=1$$

or $A_0:=(a_{n,k})_{0\leq k\leq n<\infty}$ where

$$a_{n,k}=0 \text{ when } k>n.$$

We define the means of the series (1.1), as

$$T_n^{(A)}(x,f):=\sum_{k=0}^{\infty}a_{n,k}S_k(x,f)$$

$$T_n^{(A_0)}(x,f) := \sum_{k=0}^{n} a_{n,k} S_k(x,f).$$

We will use the relation $f = O(g)$ which means that $f \leq cg$ for a constant c independent of f and g.

Note that the results obtained in [10], [13], [21] and [36] have been generalized and improved by Łenski and Szal [23]. The results in these studies are obtained in variable exponent Lebesgue space $L_{2\pi}^{p(x)},\ p(x) \geq 1$. The general methods of summability of Fourier series of functions from variable exponent Lebesgue space $L_{2\pi}^{p(x)},\ p(x) \geq 1$ have been investigated in study [23]. For estimate of the error of approximation of functions by the matrix means a modulus of continuity constructed by the Steklov functions of the increments of considered functions without of absolute values is used. In the present paper, we study the degree of approximation by the matrix submethods $T_n^{(\lambda)}(\cdot,f)$ of the partial sums of their Fourier series of functions in Orlicz spaces. The results obtained in Orlicz spaces in this study are analogue to the results obtained in [23] for variable exponent Lebesgue spaces $L_{2\pi}^{p(x)},\ p(x) \geq 1$. In addition, in this study we used the proof method in [23]. Similar problems about approximation properties of the different sums, constructed according to the Fourier series of given functions in the different spaces have been investigated by several authors (see, for example, [1], [5–19], [21-23], [27-32], [35] and [36]).

Our main results are the followings:

Theorem 1.1. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and the following conditions hold:*

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = O\left(\frac{1}{n+1}\right),\ \beta \geq 0 \tag{1.2}$$

and

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = O(n+1)\,. \tag{1.3}$$

Then the estimate

$$\left\| T_n^{(A)}(\cdot,f) - f \right\|_{L_M(\mathbb{T})} = O\left(\Omega_M\left(f, \frac{1}{n+1}\right)\right) + \sum_{k=0}^{n} a_{n,k} \Omega_M\left(f, \frac{1}{n+1}\right)$$

holds.

Theorem 1.2. *Let $L_M\ (\mathbb{T})$ be a reflexive Orlicz space and the following conditions hold:*

$$\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|=O\left(a_{n,n}\right),\ \beta\geq 0 \tag{1.4}$$

and

$$(n+1)\,a_{n,n}=O\left(1\right). \tag{1.5}$$

Then the estimate

$$\left\|T_n^{(A_0)}(\cdot,f)\ -f\right\|_{L_M(\mathbb{T})}=O\left(\sum_{k=0}^{n}a_{n,k}\Omega_M\left(f,\frac{1}{k+1}\right)\right)$$

holds.

Theorem 1.3. *Let $f\in Lip(\omega,M)$ and the conditions (1.2) and (1.3) hold. Then, the estimate*

$$\left\|T_n^{(A)}(\cdot,f)\ -f\right\|_{L_M(\mathbb{T})}=O\left(\omega\left(\frac{1}{n+1}\right)\right)$$

holds.

Theorem 1.4. *Let $f\in Lip(\omega,M)$. If the conditions (1.4) and (1.5) hold, then the estimate*

$$\left\|T_n^{(A_0)}(\cdot,f)\ -f\right\|_{L_M(\mathbb{T})}=O\left(\omega\left(\frac{1}{n+1}\right)\right)$$

holds.

2 Auxiliary results

In the proof of the main result we need the following Lemmas:

Lemma 2.1. [23] *Let $\beta\geq 0$ and $0<t\leq\pi$. Then the inequality*

$$\left|\sum_{k=0}^{\infty}(k+1)^{\beta}\frac{\sin\frac{(2l-1)t}{2}}{2\sin\frac{t}{2}}\right|\leq\pi^2\frac{(n+1)^{\beta}}{t^2}$$

holds.

Lemma 2.2.[23] *If the conditions (1.2) and (1.3) hold, then the estimate*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\sum_{\nu=0}^{k}l_{\nu}\cos\nu t\right|dt = O(1),$$

holds, where $l_{\nu}=\begin{cases}1, & when\ \nu=0,\\ \frac{\pi}{4\sin\frac{\pi}{8}}, & when\ \nu>0.\end{cases}$

Lemma 2.3. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space . Then, for an arbitrary measurable function $f(x,y)$ defined on $\mathbb{T}\times\mathbb{T}$, the following inequality holds:*

$$\left\|\int_{\mathbb{T}} f(\cdot,y)\right\|_{L_M(\mathbb{T})} \le 2\int_{\mathbb{T}}\|f(\cdot,y)\|_{L_M(\mathbb{T})}\,dy.$$

Proof. It is clear that according to [20, page-80] the Hölder inequalities

$$\left|\int_{\mathbb{T}} f(x)\,g(x)\,dx\right| \le \int_{\mathbb{T}}|f(x)\,g(x)|\,dx \le \|f\|_{L_M(\mathbb{T})}\|g\|^{*}_{L_N(\mathbb{T})},$$

$$\left|\int_{\mathbb{T}} f(x)\,g(x)\,dx\right| \le \int_{\mathbb{T}}|f(x)\,g(x)|\,dx \le \|f\|^{*}_{L_M(\mathbb{T})}\|g\|_{L_N(\mathbb{T})},$$

hold for every $f\in L_M(\mathbb{T})$ and $g\in L_N(\mathbb{T})$. On the other hand the inequalities

$$\|f\|^{*}_{L_M(\mathbb{T})} \le \|f\|_{L_M(\mathbb{T})} \le 2\|f\|^{*}_{L_M(\mathbb{T})} \tag{2.1}$$

hold. By the definition of the norm and the Fubini theorem we have

$$\left\|\int_{\mathbb{T}} f(\cdot,y)\right\|^{*}_{L_M(\mathbb{T})} \le \int_{\mathbb{T}}\|f(\cdot,y)\|^{*}_{L_M(\mathbb{T})}\,dy. \tag{2.2}$$

Using (2.1) and (2.2), we arrive at the following Minkowski inequality for the Orlicz space $L_M(\mathbb{T})$:

$$\left\|\int_{\mathbb{T}} f(\cdot,y)\right\|_{L_M(\mathbb{T})} \le \int_{\mathbb{T}}\|f(\cdot,y)\|_{L_M(\mathbb{T})}\,dy.$$

The proof of Lemma 2.3 is completed.

Lemma 2.4. *Let $f\in L_M(\mathbb{T})$. Then, the estimate*

$$\left\| f_{\frac{1}{2\lambda}}\left(\cdot,+\tau\right)\right\|_{L_M(\mathbb{T})} = O(1), \left\| f\left(\cdot\right)\right\|_{L_M(\mathbb{T})}$$

holds, for every $\tau \in \mathbb{R}$, *where* $f_{\frac{1}{2\lambda}}\left(\tau\right) = \lambda \int_{-\frac{1}{2\lambda}+\tau}^{\frac{1}{2\lambda}+\tau} f\left(t\right) dt$ *with* $\lambda > 1$.

Proof. Let $\|f\|_{L_M(\mathbb{T})} \leq 1$ and $\tau \in \mathbb{R}$. There exists an integer m such that $m\pi h \leq \tau \leq (m+2)\,\pi h$. Now, we use the definition of the norm, apply Jensen's inequality, and take the supremum into the integral we obtain

$$\left\| f_{\frac{1}{2\lambda}}\left(\cdot,+\tau\right)\right\|_{L_M(\mathbb{T})} = \sup \int_{-\pi}^{\pi} \left| f_{\frac{1}{2\lambda}}\left(x+\tau\right)\right| \left|g\left(x\right)\right| dx$$

$$= \sup \int_{-\pi}^{\pi} \left| \lambda \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} f\left(t\right) dt \right| \left|g\left(x\right)\right| dx$$

$$= \sup \int_{-\pi-(m+1)\pi h}^{\pi-(m+1)\pi h} \sup \left| \lambda \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} f\left(t\right) dt \right| \left|g\left(x\right)\right| dx$$

$$= O\left(1\right) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \sup \left\{ \left| \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} f\left(t\right) dt \right| \left|g\left(x\right)\right| dx \right\} dt$$

$$= O\left(1\right) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \sup \left\{ \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} \left|f\left(t\right)\right| dt \left|g\left(x\right)\right| dx \right\} dt$$

$$= O\left(1\right) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \sup \left\{ \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} \left|f\left(t\right)\right| dt \left|g\left(x\right)\right| dx \right\} dt$$

$$= O\left(1\right) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \sup \left\{ \int_{-\pi+x+\tau}^{\pi+x+\tau} \left|f\left(t\right)\right| dt \left|g\left(x\right)\right| dx \right\} dt$$

$$= O\left(1\right) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \sup \left\{ \int_{-\pi}^{\pi} \left|f\left(x\right) g\left(x\right)\right| dx \right\} dt$$

$$= O\left(1\right) \|f\|_{L_M\ (\mathbb{T})},$$

where all the supremum above are taken over all functions $g \in L_N\ (\mathbb{T})$ with $\rho\left(g, N\right) \leq 1$.

The proof of Lemma 2.4 is completed.

Lemma 2.5. *Let $f \in L_M\ (\mathbb{T})$ and T_n be a trigonometric polynomial of the degree at most n, such that $\|f - T_n\| = O(1)\,\Omega\left(f, \frac{1}{n+1}\right)_M$. If the conditions (1.2) and (1.3) hold, then the estimate*

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(\cdot, f - T_n\right) \right\|_{L_M(\mathbb{T})} = O\left(\Omega_M\left(f, \frac{1}{n+1}\right)\right)$$

holds.

Proof. We denote

$$f_h\left(t\right) := \frac{1}{2h} \int_{-h}^{h} f\left(x + t\right) dx.$$

According to [23, Lemma 5] the following equations hold:

$$a_0\left(f\right) = a_0\left(f_h\right), \quad a_l\left(f\right) = \frac{lh}{\sin lh} a_l\left(f_h\right), \ (l = 1, 2, 3, ...) \tag{2.3}$$

and

$$b_l\left(f\right) = \frac{lh}{\sin lh} b_l\left(f_h\right), \ (l = 1, 2, 3, ...). \tag{2.4}$$

Using (2.3) and (2.4) we have [23]

$$
\begin{aligned}
S_k(x,f) &= \frac{a_0(f)}{2}+\sum_{l=1}^{k}\left(a_l(f)\cos lx+b_l(f)\sin lx\right)\\
&= \frac{a_0(f_h)}{2}+\sum_{l=1}^{k}\left(\frac{lh}{\sin lh}a_l(f_h)\cos lx+\frac{lh}{\sin lh}b_l(f)\sin lx\right)\\
&= \frac{1}{2\pi}\int_{-h}^{h} f_h(t)\,dt+\sum_{l=1}^{k}\frac{lh}{\sin lh}\frac{1}{\pi}\int_{-\pi}^{\pi} f_h(t)\cos ltdt\cos lx\\
&\quad +\sum_{l=1}^{k}\frac{lh}{\sin lh}\frac{1}{\pi}\int_{-\pi}^{\pi} f_h(t)\sin ltdt\sin lx\\
&= \frac{1}{2\pi}\int_{-h}^{h} f_h(t)\,dt+\sum_{l=1}^{k}\frac{lh}{\sin lh}\frac{1}{\pi}\int_{-\pi}^{\pi} f_h(t)\cos l(t-x)\,dt\\
&= \frac{1}{2\pi}\int_{-h}^{h} f_h(t)\,dt+\frac{1}{\pi}\sum_{l=1}^{k}\frac{lh}{\sin lh}\int_{-\pi}^{\pi} f_h(t+x)\cos ltdt\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f_h(t+x)\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt. \qquad (2.5)
\end{aligned}
$$

If we show

$$
T_{n,h}(t)=\frac{1}{2h}\int_{-h}^{h} T_n(x+t)\,dx.
$$

then according to (2.5) we have

$$
\begin{aligned}
S_k(x,f-T_n) &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f-T_n)_h(x+t)\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f_h(t+x)-T_{n,h}(t+x)\right)\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt. \qquad (2.6)
\end{aligned}
$$

Using (2.6) we obtain [23]

$$\begin{aligned}&\sum_{k=0}^{\infty}a_{n,k}S_k\left(x,f-T_n\right)\\ =\;&\sum_{k=0}^{\infty}a_{n,k}\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f_h\left(t+x\right)-T_{n,h}\left(t+x\right)\right)\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt\\ =\;&\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f_h\left(t+x\right)-T_{n,h}\left(t+x\right)\right)\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt\end{aligned} \tag{2.7}$$

Let 0$<h<\frac{1}{2}$ and $|t|\leq\pi$.Then using (2.7), Lemma 2.3 and Lemma 2.4 we reach

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k\left(\cdot,f-T_n\right)\right\|_{L_M(\mathbb{T})}\\ \leq\;&\frac{2}{\pi}\int_{-\pi}^{\pi}\left\|f_h\left(t+\cdot\right)-T_{n,h}\left(t+\cdot\right)\right\|_{L_M(\mathbb{T})}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt\right|\\ =\;&O\left(1\right)\frac{1}{\pi}\int_{-\pi}^{\pi}\left\|f-T_n\right\|_{L_M(\mathbb{T})}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{l=1}^{k}\frac{lh}{\sin lh}\cos lt\right)dt\right|.\end{aligned} \tag{2.8}$$

If $h=\frac{\pi}{8l}<\frac{1}{2}$ is taken into account in inequality (2.8) we have

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k\left(\cdot,f-T_n\right)\right\|_{L_M(\mathbb{T})}\\ =\;&O\left(1\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{l=1}^{k}\cos lt\right)dt\right|\left\|f-T_n\right\|_{L_M(\mathbb{T})}\end{aligned} \tag{2.9}$$

In the other hand according to Lemma 2.2 the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{l=1}^{k}\cos lt\right)dt\right|=O\left(1\right) \tag{2.10}$$

holds.

Consequently, if the $\|f-T_n\|_{L_M(\mathbb{T})}=O\left(1\right)\Omega_M\left(f,\frac{1}{n+1}\right)$ condition given in the Lemma 2.5 and the relations (2.9) and (2.10) are used, we find that

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k\left(\cdot,f-T_n\right)\right\|\\ = \;& O\left(1\right)\|f-T_n\|_{L_M(\mathbb{T})}=O\left(1\right)\Omega_M\left(f,\frac{1}{n+1}\right).\end{aligned}$$

Thus, Lemma 2.5 is proved.

3 Proofs of the main results

Proof of Theorem 1.1. Let T_n be the polynomial that satisfies the condition

$$\|f-T_n\|_{L_M(\mathbb{T})}=O\left(\Omega_M\left(f,\frac{1}{n+1}\right)\right). \tag{3.1}$$

Its known that the following equality holds:

$$S_k\left(x,f-T_n\right)=\begin{cases}S_k\left(x,f\right)-T_k\left(x\right), & \text{for } k\leq n\\ S_k\left(x,f\right)-T_n\left(x\right), & \text{for } k\geq n\end{cases}. \tag{3.2}$$

Using (3.1), (3.2) and Lemma 2.5 we find that

$$
\begin{aligned}
& \left\|T_n^{(A)}(\cdot, f)-f\right\|_{L_M(\mathbb{T})} \\
= & \left\|T_n^{(A)}(\cdot, f)-\sum_{k=0}^n a_{n,k} T_k-\sum_{k=n+1}^{\infty} a_{n,k} T_n\right. \\
& \left.+\sum_{k=0}^n a_{n,k} T_k+\sum_{k=n+1}^{\infty} a_{n,k} T_n-f\right\|_{L_M(\mathbb{T})} \\
\leq & \left\|T_n^{(A)}(\cdot, f)-\sum_{k=0}^n a_{n,k} T_k-\sum_{k=n+1}^{\infty} a_{n,k} T_n\right\|_{L_M(\mathbb{T})} \\
& +\left\|\sum_{k=0}^n a_{n,k} T_k+\sum_{k=n+1}^{\infty} a_{n,k} T_n-f\right\|_{L_M(\mathbb{T})} \\
= & \left\|\sum_{k=0}^n a_{n,k}\left\{S_k(\cdot, f)-T_k\right\}+\sum_{k=n+1}^{\infty} a_{n,k}\left\{S_k(\cdot, f)-T_n\right\}\right\|_{L_M(\mathbb{T})} \\
& +\left\|\sum_{k=0}^n a_{n,k}\left(f-T_k\right)+\sum_{k=n+1}^{\infty} a_{n,k}\left(f-T_n\right)\right\|_{L_M(\mathbb{T})} \\
\leq & \left\|\sum_{k=0}^n a_{n,k} S_k\left(\cdot, f-T_n\right)\right\|_{L_M(\mathbb{T})}+\left\|\sum_{k=0}^n a_{n,k}\left\{f-T_n\right\}\right\|_{L_M(\mathbb{T})} \\
& +\left\|\sum_{k=n+1}^{\infty} a_{n,k}\left\{f-T_n\right\}\right\|_{L_M(\mathbb{T})} \\
= & \left\|\sum_{k=0}^n a_{n,k} S_k\left(\cdot, f-T_n\right)\right\|_{L_M(\mathbb{T})}+O(1) \sum_{k=0}^n a_{n,k} \Omega\left(f, \frac{1}{k+1}\right)_M \\
& +\sum_{k=n+1}^{\infty} a_{n,k} O\left(\Omega\left(f, \frac{1}{n+1}\right)_M\right) \\
= & O\left(\Omega\left(f, \frac{1}{n+1}\right)_M\right)+\sum_{k=0}^n a_{n,k} \Omega\left(f, \frac{1}{k+1}\right)_M .
\end{aligned}
$$

Hence, theorem is proved.

Proof of Theorem 1.2. We can note that the assumptions on entries of A_0 yield

$$\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|$$

$$=\sum_{k=0}^{n-1}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|+a_{n,n}$$

$$=O\left(a_{n,n}\right)+a_{n,n}=O\left(a_{n,n}\right)=O\left(\frac{1}{n+1}\right)$$

and

$$\sum_{k=0}^{\infty}(k+1)\,a_{n,k}=\sum_{k=0}^{n}(k+1)\,a_{n,k}\leq(n+1)\sum_{k=0}^{n}a_{n,k}=n+1.$$

Considering the monotonicity of $\Omega\left(f,\delta\right)_M$ with respect to $\delta>0$ and using Theorem. 1.1, we conclude that

$$\sum_{k=0}^{n}a_{n,k}\Omega_M\left(f,\frac{1}{k+1}\right)\geq\Omega_M\left(f,\frac{1}{n+1}\right)\sum_{k=0}^{n}a_{n,k}=\Omega_M\left(f,\frac{1}{n+1}\right)$$

which completes the proof.

Proof of Theorem 1.3. Let ω be a function of mudulus of continuity type. For the function ω the inequality

$$\omega\left(n\delta\right)\leq n\omega\left(\delta\right),\ n\in\mathbb{N}_0. \tag{3.3}$$

holds. Then from we can write the inequality

$$\omega\left(\lambda\delta\right)\leq\left(\lambda+1\right)\omega\left(\delta\right),\ \lambda\geq0. \tag{3.4}$$

Using (3.4), for $0<\delta_1\leq\delta_2$ gives us

$$\begin{aligned}\omega\left(\delta_2\right) &= \omega\left(\frac{\delta_1}{\delta_1}\delta_2\right) \le \left(\frac{\delta_2}{\delta_1}+1\right)\omega\left(\delta_1\right) \\ &\le \left(\frac{\delta_2}{\delta_1}+\frac{\delta_1}{\delta_1}\right)\omega\left(\delta_1\right) \le \left(\frac{\delta_2}{\delta_1}+\frac{\delta_2}{\delta_1}\right) = 2\frac{\delta_2}{\delta_1}\omega\left(\delta_1\right). \qquad (3.5)\end{aligned}$$

Taking into account of (3.5) we obtain inequality,

$$\frac{\omega\left(\delta_2\right)}{\delta_2} \le 2\frac{\omega\left(\delta_1\right)}{\delta_1}..$$

Therefore, using relation (1.2) for $\beta > 0$, we conclude that

$$\sum_{k=0}^{n} a_{n,k}\omega\left(\frac{1}{k+1}\right) = \sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\frac{\omega\left(\frac{1}{k+1}\right)}{\frac{1}{k+1}}$$

$$\le 2\left(n+1\right)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}$$

$$= 2\left(n+1\right)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}\left(k+1\right)^{\beta-1}\frac{a_{n,k}}{\left(k+1\right)^{\beta}}$$

$$\le 2\left(n+1\right)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}\left|\frac{a_{n,k}}{\left(k+1\right)^{\beta}}-\frac{a_{n,k+1}}{\left(k+2\right)^{\beta}}\right|\sum_{p=0}^{k}\left(p+1\right)^{\beta-1}$$

$$= O\left(n+1\right)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}\left(k+1\right)^{\beta}\left|\frac{a_{n,k}}{\left(k+1\right)^{\beta}}-\frac{a_{n,k+1}}{\left(k+2\right)^{\beta}}\right|$$

$$= \left(n+1\right)\omega\left(\frac{1}{n+1}\right)O\left(\frac{1}{n+1}\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right).$$

The proof of Theorem 1.3 is completed.

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