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On primary decomposition of modules and associated primes

Mohamamd Irshad[1], **Chintamani Tiwari**[1] **and Dy Outdom**[2]

[1]Department of Mathematics, Maharishi University of Information Technology, Lucknow-226013, India
[2]Department of Mathematics
Royal University of Phnom Penh, Cambodia
Email: irshad.muit@gmail.com, chintamani.tiwari@muit.in, dyoutdom926@gmail.com

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Abstract

Basic principles of primary decomposition of modules are outlined here. Let R be a commutative ring with unity graded over Abelian group G, which is finitely generated and N be a submodule of an R-module M. In usual, for G-graded modules (abbreviated as G gr-modules) there don't exist primary decomposition which are graded ourselves but graded primary decomposition exist for grading by torsion free group. In this paper, we explore the special features of primary decomposition and their associated primes in the case of graded modules.

1 Introduction

In the reference of algebra of geometry, the notion of homogenous coordinate ring for toric varieties gave new direction for theory of rings which have gradings over finitely generated (abbreviated as f.g.) Abelian groups. In modern research, general primary decomposition is encounter with graded primary decomposition of

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gr-modules. In algebra, the concept of grading, specifically graded rings (abbreviated as gr-rings) and graded modules, is fundamental in the theory of ring's homological aspects. As a result of the rising importance of rings with a group graded structure in these years, the primary focus of gr-ring theory research is on the graded prime ideals. It may be impossible to summarize the structure of graded ideals in graded commutative rings, but it is possible to get specific details on the ring's structure and size by focusing on the graded prime ideals, graded commutative rings, can be evaluated using the chain of graded prime ideals and the number in the long chain. Also known as the graded ring's Krull dimension, it is critical for graded commutable rings. The concept of graded prime ideals encompasses a wide range of theories, including unique factorization, the process of localization, and algebraic structure. Graduate algebra and contemporary research themes are well-served by these courses Secondary module theory had been researched extensively by a various number of authors. Because of this, analogue of many ideas has been produced in latest studies. A G gr-rings (graded rings) and gr-submodules (graded submodules) of gr-modules (graded modules) are the subject of this article. We discuss some special properties of primary decomposition over gr-modules in this research; also, we determine the relationships between the primary decomposition of modules and their associated primes.

2 The results

Throughout this paper, let G be a f.g. (finitely generated) Abelian group, R be a commutative ring with unity and M be a f.g. G gr-R-module (G graded-R-module). Then

1. A submodule N of an R-module M is said to be a primary submodule of an R-module M if N is a proper submodule, and map $f_a : M/N \rightarrow M/N$ is either injective or nilpotent for every $a \in R$.

2. A submodule N of module M is said to have a primary decomposition if N can be expressed as $N = \bigcap_{r=1}^{n} N_r$ where N_r are P_r- primary submodule.

3. A prime ideal P of an R-module M is said to be an associated prime of M or P associated M if P annihilated some non-zero $a \in M$. The collection of such associated primes of M is denoted by $Ass(M)$ or $AP(M)$.

4. $Ass^G M$ denoted the set of all G-associated ideals of an R-module M.

5. If for some G-prime ideal P, $Ass^G M = \{P\}$, then R-module M is G-coprimary.

6. A G gr-Submodule N of M is called G-primary if M/N is G-coprimary.

7. A G gr-R module M is said to be G-secondary if $M \neq 0$ and the endomorphism $\lambda_{a,M}$, $\forall a \in h(r)$ is surjective or nilpotent.

Proposition 2.1. *If N is a Submodule of M, then $Ass(M) \subseteq Ass(N) \cup Ass(M/N)$.*

Proof. The map $g : R/A \to M$ be an intjective homomorphism for each $A \in Ass(M)$, then $G = g(R/A)$ and $H = G \cap N$ if $g(a + A) \to g(a + A) \in N$ is injective, if $g(a + A) \in N$ it necessarily belong to $G \cap N = 0$. Thus, there is isomorphism between G and submodule of M/N, so by definition of G, there is map from R/A to M/N is an injective, thus $A \in Ass(M/N)$ if $G \neq 0$ and $x \neq 0 \in G$, then x necessarily belong to both G, N and G is isomorphic to R/A by map g. Thus $x \in N$ and the annihilator of some non-zero element of R/A correspond with the annihilator of x, so $Ann(x) = A$ and $A \in Ass(N)$. □

Corollary 2.1. $Ass(\oplus_{i \in I} M_i) = \cup_{i \in I} Ass(M_i)$

$$
\begin{aligned}
&= Ass(M_1) \cup Ass(M_2 \oplus M_3) \\
&= Ass(M_1) \cup Ass(M_2) \cup (M_3),
\end{aligned}
$$

in general

$$\implies Ass(\oplus_{i \in I} M_i) = \cup_{i \in I} Ass(M_i).$$

Proposition 2.2. *P-secondary modules are G gr-P-secondary.*

Proof. To prove that the result it is sufficient we take two G gr-P-secondary modules. Let L and M be any two G gr-P-secondary modules with $P = G_r(Ann(L)) = G_r(Ann(M))$. Let $a \in h(R)$ and $\lambda_{a,M}$ is not surjective,then $a(L \oplus M)$ which implies that either $aLf = L$ or $aMf = M$, suppose $aLf = L$ then $\exists K > 0$ such that $a^k L = 0$. which implise that $a^k \in Ann(L) \implies a \in P = G_r(Ann(L))$. Hence, by induction the finite direct sum of G gr-P-secondary modules is G gr-P-secondary.

Let us consider M is a G gr-p-secondary module, so that $P = G_r(Ann(M))$. Let $\lambda : M \to M' = \dfrac{M}{N}$ be the usual projection from M to a non-zero element of M', let $a \in h(R)$ with $\lambda_{a,M}$ surjective then $\exists \quad K > 0$ such that $a^k M = 0 \implies a^k M' + N = N$ as $a^k M = 0$ $a^k \in Ann(M/N) \implies r \in G_r(Ann(M/N))$, which show that M' is G gr-p-secondary.

Proposition 2.3. *Let M be a gr-secondary module over a graded ring R. The set of ideals $Ann(x)$ of R, contains each element as a maximal element where $x \neq 0$ run over the set of elements of M, belongs to $Ass(M)$.*

Proof. Let $a = (Ann(x), 0 \neq x \in M)$ be such a minimal element. To show that a is prime, it is sufficient. As $x \neq 0$, $a \neq R$. Let two element e, f of R such that $ef \in a \implies (ef)^n x = 0$ for some $n > 0$, if $f \notin a \implies \lambda_{f,x}$ is surjective, that is $f^n x = x$, then $e^n x = e^n(f^n x) = (ef)^n x = 0 \implies a$ is nilpotent. Therefore, $x \neq 0, e \in Ann(fx)$ and $a \subset Ann(fx)$ as a is maximal element, $Ann(fx) = a$, hence $e \in a$, so that a is prime.

Proposition 2.4. *Let R be a G gr-ring with commutativity and M be a gr-secondary R-module and N be a gr-secondary submodule of M. Then*

$$Ass(N) \subset Ass(M) \subset Ass(N) \cup Ass(M/N).$$

Proof. The inclusion $Ass(N) \subset Ass(M)$ is natural. Let $p \in Ass(M)$, E be the gr-secondary submodule of M is isomorphic to R/P and $F = E \cap N$. If $F = 0$, then E is isomorphic to gr-secondary submodule of M/N, whence $p \in Ass(M/N)$. If $F \neq 0$, $p = Ann(F)$ where each element $\neq 0$ (by Preposition 2.3) $p \in Ass(F) \subset Ass(N)$.

Corollary 2.2. *If a gr-secondary R-module M is the direct sum of family of submodule $(M)_{i \in I}$, then $Ass^G M = \cup_{i \in I} Ass M_i$.*

Proposition 2.5. *Let decomposition $G = G_1 \times G_2$, where G_1 be free and G_2 finite, and M be a G gr-secondary module which is f, g. Assume that the projection of G on to the corresponding factors induces by the G_1 and G_2 grading of M. Then,*

1. *Every prime ideal of G is G_2-prime.*
2. *$Ass^{G_1} M = Ass M$ and $Ass^G M = Ass^{G_2} M$.*

Proof. (1) The proof is stated forward.

(2) The first claim follows by mean of the second, where G_2 is trivial by (1) it is sufficient to show that $Ass^{G_2}M \subset Ass^G M$. Let$p \in Ass^{G_2}M, m \in h(M)$ homogeneous elements of degree t such that $p = ann(m)$ and $r \in p$ homogeneous of G_2 degree s. It can be expressed as $r = \sum_{g_1 \in G_1} f_{(g_1,s)}$, $m = \sum_{g \in G_1} m_{(g,t)}$ and $f^{g_1}m = 0, f^{g_1} \in ann(m) = P$, which prove that $Ass^G \subset p$ as $M \neq \phi$ and $Ass^G M = P$.

Theorem 2.1. *Let us consider M be a G gr-secondary module, and N be a gr-submodule of M. Then*

1. $Ass^G N \subset Ass^G M \subset Ass^G N \cup Ass^G M/N$.

2. *Let $\{N_i\}_{i \in I}$ is a finite chain of G gr-submodules of M such that $\cap_{i \in I} N_i = \{0\}$ then $Ass^G M \subset \cup_{i \in I} Ass^G M/N_i$.*

3. *If $I \subset Ass^G M$ then $\exists$ a submodule N of M such that $Ass^G N = Ass^G M/I$ and $Ass^G M/N = I$.*

Proof. Let $\{M_j\}_{j \in J}$ be a chain of gr-secondary submodules of M, then $\cup_{j \in J} M_j$ is a gr-secondary module too. If it occur with M, then M is gr-secondary itself and $Ass^G M = \cup_{j \in J} Ass^G M_j$.

1. Now, it is apparenrt that $Ass^G N \subset Ass^G M$. Let $p \in Ass^G M \quad \exists$ a submodule $T \cong R/P$. Here, we take is secondary $Ass^G F \subset Ass^G N$. If $F = 0$, then $Ass^G F = \phi$. But, then $T \cong M/N$ and thus $p \in Ass^G M/N$. If $F \neq 0$, then $Ass^G F = p$.

2. We provide the same in direct sum $\oplus_{i \in I} M/N_i$ (see Corollary 2.1).

3. Let E be denoted set of gr-secondary submodule P of M such that $Ass^G(P) \subset Ass^G(M/I)$ by $Ass^G M = \cup_{j \in J} Ass^G M_j$, the set is non-empty and $0 \in E$ and $E \neq I$. Suppose that N be a maximal element of E then $Ass^G(N) \subset Ass^G(M/I)$. We shall see that $Ass^G(M/N) \subset I$. Let $p \in Ass^G(M/N)$, then (M/N) contain a submodule $F/N \cong R/P$ by (1), $Ass^G F \subset Ass^G N \cup \{p\}$. But $F \notin E$ as N is maximal and thus $P \in I$.

Proposition 2.6. *Let M be a gr-secondaey R-module over a G gr-ring R and N be a gr-P-secondary R module of M.*

1. *If S is a gr-primary submodule of M, then $N \cap S$ is gr-P-secondary.*

2. *If S is a gr-prime submodule of M, then $N \cap S$ is gr-P-secondary.*

Proof. Suppose that $x \in h(R)$ and let $x \in P$. Then $x^m(N \cap S) \subseteq x^m N = 0$ for few m, so x is nilpotent on $N \cap S$. Let $x \notin P$; we show that x divides $N \cap S$. It is sufficient to prove that $N \cap S \subseteq x(N \cap S)$. If $y \in N \cap S$, then $y = xm$ for some $m = \sum_{i=1}^{n} m_{g_i} \in N$ with $m_{g_i} \neq 0$. Then for each $i = 1, \ldots, s, am_{g_i} \in S$ since S is a gr-submodule of M. It follows that $m_{g_i} \in S$ for every i (otherwise, if $m_{g_i} \notin S$ for some j and $x^s \in (S :_R M)$ for some s, then $m_{g_j} \in N = x^s N \subseteq x^s M \subseteq S$, this is a contradiction, so $m \in S$; hence $y \in x(N \cap S)$ and complete the proof.

Theorem 2.2. *Let M be a gr-secondary R-module over a G gr-ring R and $N \neq 0$ a gr-P- prime R-submodule of M. Then N is gr-P-secondary.*

Proof. Let us suppose that M is a gr-Q-secondary R module and let $x \in h(R)$. If $x \in Q$, then for some s, $x^s N \subset x^s M$ then x is nilpotent on N. Assume that $x \neq Q$. We prove that x devides N so we let $y \in N$. Then $\exists\, z = \sum_{i=1}^{t} z_{g_i} \in M$ with $z_{g_i} \neq 0$ such that $y = xz$, as N is graded, $xz_{g_i} \in N$ for every $i = 1, \ldots, t$, so for each i, N graded prime gives $z_{g_i} \in N$; hence $z \in N$. Which shows that x divides N, so N is a gr-Q-secondary R-module.

Now we have to prove that $P = Q$. Since $P \subseteq Q$ is trivial under the inclusion. We will show the reverse inclusion. Assume that $c = \sum_{i=1}^{t} c_{h_i} Q$ with $c_{h_i} \neq 0$. Then there are integers m_i such that $c_{h_i}^{m_i} M = 0$ for $i = 1, \ldots, n$ as Q is graded and M is gr-Q-secondary. Since $M \neq N$, the element $x = x_{g_1} + \cdots + x_{g_n} \in M$ with $x_{g_i} \neq 0$ such that $x_{g_i} \notin N$ for some i. Therefore, for each $i = 1, \cdots, n, c_{h_i}^{m_i} x_{g_w} = 0 \in N$, so N graded Prime assign $c_{h_i} \in P$; Therefore $c \in P$, hence proof the theorem.

Theorem 2.3. *Let M be a G gr-module. Then,*

1. *Every G-associated G-prime ideal with M, annihilates the homogeneous element of M.*

2. *The set $Ass^G M$ contains all maximal element of all annihilator ideals of set $ann(x)$, where x is homogeneous element belong to M.*

3. $Ass^G M = \phi$ *iff* $M = \{0\}$.

Proof. (1) Let us consider G-prime ideal P associated with M. Then for some x of $M, P = ann(x)$. Write m as finite sum $\sum_{a \in G} x_a$, where $x_a \in M_a$ for each $a \in G$. Thus for all homogeneous element $h \in P$, we get $kx = \sum_{a \in G} kx_a = 0$, for each $a \in G$, the summands kx_a get pairwise distinct degree and it must be zero identically, i.e., $kx_a = 0$ for each $a \in G$. Therefore, $P \subset ann(x_a)$ for each $a \in G$. Additionally, $\cap_{a \in G} ann(x_a) \subset P$ since P is G-prime and $ann(xa)$ are homogeneous, there exist some $y \in G$ such that $ann(x_y) \subset P$, and therefore $ann(x_y) = P$.

(2) Let maximal annihilator ideal for $0 \neq x$ homogeneous be $I = ann(x)$ and let us consider homogeneous $h, k \in G$ such that $hk \in I$ and $k \in I$. Thus, $h \in ann(kx)$ and moreover $I \subset ann(kx)$. Since, I is maximal ideal and $kx \neq 0$, which shows that $I = ann(kx)$ and $h \in I$. Hence, I is G-prime.

(3) If $M = 0$, which show that $Ass^G M = \phi$. If $M \neq 0$, by (2) for some $x \in M$, there exist a maximal ideal of the form $ann(x)$, which are belongs to $Ass^G M$. □

3 Conclusion

In this paper, we discussed, the primary decomposition of modules and gr-modules are quite different. Moreover, we dissertated whether for some G gr-modules M and N, where N is a submodule of M, there exist a decomposition $N = \bigcap_{r=1}^{n} N_r$ where N_r are P_r-primary submodule, $ann(M/N_r)$ are irreducible in an appropriate sense.

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