The Aligarh Bulletin of Mathematics
Volume 43, Number 1 (2024), 65 - 76

ISSN: 0304-9787

Mersenne numbers that are expressible as the summation of two Fibonacci numbers

Ahmet Emin

Department of Mathematics, Faculty of Science
Karabük University, Karabük, Turkey
Email: ahmetemin@karabuk.edu.tr

(Received: May 20, 2024 Accepted: July 25, 2024)

Abstract

This study presents an investigation of the Mersenne numbers that can be written in terms of the summation of two random Fibonacci numbers within the context of linear forms in logarithms of algebraic numbers by using Matveev's theorem and Dujella-Pethö reduction lemma. More precisely, all the solutions to the Diophantine equation $M_k = F_m + F_n$ are presented in this study.

1 Introduction and motivation

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ and the Mersenne sequence $\{M_n\}_{n\geq 0}$ are defined by the initial values $(F_0, F_1) = (0, 1)$ and $(M_0, M_1) = (0, 1)$, respectively, and the recurrence relations $F_n = F_{n-1} + F_{n-2}$ and $M_n = 3M_{n-1} - 2M_{n-2}$, for $n \geq 2$. Furthermore, the Fibonacci sequence can also be generated using Binet's formula, and there is a Binet-like formula for the Mersenne sequence, as shown below:

$$F_n = \frac{\gamma^n - \delta^n}{\sqrt{5}} \text{ and } M_n = 2^n - 1, \tag{1.1}$$

for all non-negative integer, where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$.

Keywords and phrases: Fibonacci number, Mersenne number, Diophantine equation, linear form in logarithm, Matveev's theorem, Dujella-Pethö reduction lemma.
2020 AMS Subject Classification: 11B39, 11D61, 11J86.

Many types of integer sequences have applications in various branches of science, especially in arithmetic and geometry. For this reason, scientists have conducted many studies on different integer sequences. The most studied of these integer sequences are the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Mersenne sequences, among others. In this study, Pell and Mersenne numbers will be discussed, but only in one section, a property of Lucas numbers will be examined.

One can consult [1] to find the properties and examples of Fibonacci, and Lucas numbers.

It has been extensively studied whether the product or sum of the terms of the number sequences mentioned above is equal to the terms of another number sequence. Ddamulira et al. investigated in [2] which of the two Fibonacci numbers the product of their terms equals a Pell number, as well as which of the two Pell numbers the product of is a Fibonacci number. Alekseyev worked to find the common terms of Fibonacci, Pell, Lucas and Pell-Lucas numbers and showed these common terms in [3]. Gaber studied in [4] whether the sum of any two Jacobsthal numbers is a term of the Pell or Pell-Lucas numbers.

In [5], Erduvan and Keskin identified all Fibonacci numbers that result from the product of two Jacobsthal numbers. Alan and Alan researched Mersenne numbers that can be represented as the product of two arbitrary Pell numbers in [6]. There are many similar articles that can be found using integer sequences, but readers may also wish to refer to other similar articles in references [7–10].

The author proved in [11] which Pell numbers can be represented as the sum of two arbitrary Mersenne numbers, as well as the opposite scenario: Mersenne numbers that can be represented as the sum of two arbitrary Pell numbers. However, the current literature does not cover the study of Mersenne numbers that can be expressed as the sum of two random Fibonacci numbers. This deficiency was the motivation behind our study, and we attempted to address it in this research. In this paper, we examine the Diophantine equation

$$M_k = F_m + F_n, \tag{1.2}$$

where $k \geq 1$ and $1 \leq m \leq n$.

2 Preliminaries

This section of the paper provides fundamental definitions, results, and notations from algebraic number theory. One can find the following lemma in the book by Koshy [1].

Lemma 2.1. *[1] For all $n \geq 1$,*

$$\gamma^{n-2} \leq F_n \leq \gamma^{n-1}. \tag{2.1}$$

Lemma 2.2. *For all $n \geq 1$,*

$$2^{n-1} \leq M_n \leq 2^n. \tag{2.2}$$

Proof. The clarity of the proof is attributed to the Binet-like formula of M_n in Equation (1.1). □

Let χ be an algebraic number of degree s and

$$a_0 x^s + a_1 x^{s-1} + \ldots + a_s = \sum_{j=0}^{s} a_j x^{s-j}$$

be its minimal polynomial in $\mathbb{Z}[x]$. The logarithmic height of χ is denoted by $h(\chi)$ and defined by

$$h(\chi) = s^{-1} \left(\log |a_0| + \sum_{i=1}^{s} \log \left(\max \left\{ \left| \chi^{(i)} \right|, 1 \right\} \right) \right), \tag{2.3}$$

where χ^i's are the conjugates of χ.

There are also numerous properties related to logarithmic height, as follows:

$$h(\chi_1 + \chi_2) \leq h(\chi_1) + h(\chi_2) + \log 2, \tag{2.4}$$

$$h\left(\chi_1 \chi_2^{\pm 1}\right) \leq h(\chi_1) + h(\chi_2), \tag{2.5}$$

$$h(\chi^r) = |r| \, h(\chi). \tag{2.6}$$

Let $\chi_1, \chi_2, \ldots, \chi_r$ be nonzero real algebraic numbers in a number field $\mathbb{T}$ of degree $d_{\mathbb{T}}$, and let $t_1, t_2, \ldots, t_r$ be nonzero rational numbers. Also

$$\Lambda = {\chi_1}^{t_1} {\chi_2}^{t_2} \ldots {\chi_r}^{t_r} - 1 \text{ and } B \geq \max\{|t_1|, |t_2|, \ldots, |t_r|\}.$$

Let $A_1, A_2, \ldots, A_r$ be the positive real numbers such that

$$A_j \geq \max\{d_{\mathbb{T}} h(\chi_j), |\log \chi_j|, 0.16\} \text{ for all } j = 1, 2, \ldots, r. \tag{2.7}$$

Based on the notations mentioned above, an important theorem established by Matveev in [13], will be presented as follows:

Theorem 2.1. *[13] If $\Lambda \neq 0$ and $\mathbb{T}$ is real algebraic number field of degree $d_{\mathbb{T}}$, then,*

$$\log(|\Lambda|) > -1.4 \times 30^{r+3} \times r^{4.5} \times d_{\mathbb{T}}^2 \times (1 + \log d_{\mathbb{T}}) \times (1 + \log B) \times A_1 \times A_2 \times \cdots \times A_r.$$

To reduce the bounds from applying Theorem 2.1, the following Lemma was developed by Dujella and Pethö (Lemma 5(a)) in [14].

Lemma 2.3. *[14] Let M be a positive integer, p/q be a convergent of the continued fraction expansion of the irrational τ such that $q > 6M$, and let X, Y, μ be some real numbers with $X > 0$ and $Y > 1$. Let $\varepsilon =: \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (k, n, ν) of inequality*

$$0 < k\tau - n + \mu < XY^{-\nu}$$

with

$$k \leq M \text{ and } \nu \geq \frac{\log(Xq/\varepsilon)}{\log Y}.$$

3 The results

The main result of the paper is given below.

Theorem 3.1. *Let k, m, and n be a positive integers. Then, Equation* (1.2) *holds only for the triples of*

$$(k, m, n) \in \{(2, 1, 3), (2, 2, 3), (3, 3, 5), (4, 3, 7), (6, 6, 10)\}. \qquad (3.1)$$

Proof. Assume that Equation (1.2) holds. When $m = n$, Equation (1.2) is reduced to $M_k = 2^k - 1 = 2F_n$, which is a contradiction. Since the right-hand side of the equation is an even number, the left-hand side is not. Therefore, for the rest of the paper, we assume that $m < n$.

If $n \leq 400$, a brute-force search using Mathematica for $1 \leq m < n \leq 400$, yields the solutions $(k, m, n) \in \{(2, 1, 3), (2, 2, 3), (3, 3, 5), (4, 3, 7), (6, 6, 10)\}$. Henceforth, we will consider $n > 400$ for the remainder of the paper.

If m and n are two consecutive integers, we can consider the equation $M_k = 2^k - 1 = F_m + F_n = F_{n-1} + F_n = F_{n+1}$. Therefore, we have $F_{n+1} = 2^k - 1$. According to Theorem 1.2. in [12], the only solution is $(k, m, n) \in \{(2, 2, 3)\}$. Therefore, we can even suppose that $n - m > 1$ and, specifically, $n - m \geq 2$.

Now we get a relation between n and a. Considering Lemma 2.1 and Lemma 2.2, we can write

$$2^{k-1} \leq M_k = F_n + F_m \leq \gamma^{n-1} + \gamma^{m-1} < \gamma^{n+m}. \tag{3.2}$$

From Equation (3.2), we conclude that

$$(k-1)\log 2 < (n+m)\log\gamma \Rightarrow k < 1 + (n+m)\frac{\log\gamma}{\log 2} < 2n + 1$$

which satisfies $k < 2n + 1$. Applying the Binet's formulas in Equation (1.1) to Equation (1.2) yields

$$M_k = F_n + F_m \Rightarrow 2^k - 1 = \frac{\gamma^n - \delta^n}{\sqrt{5}} + \frac{\gamma^m - \delta^m}{\sqrt{5}} \tag{3.3}$$

and from this, we get

$$\sqrt{5} \cdot 2^k - \gamma^n = \gamma^m - \delta^n - \delta^m + \sqrt{5}.$$

Dividing both sides of the last equation by γ^n and taking absolute values we get

$$\left|\frac{2^k.\sqrt{5}}{\gamma^n} - 1\right| = \left|\frac{\gamma^m}{\gamma^n} - \frac{\delta^n}{\gamma^n} - \frac{\delta^m}{\gamma^n} + \frac{\sqrt{5}}{\gamma^n}\right| < \frac{1}{\gamma^{n-m}} + \frac{|\delta|^n}{\gamma^n} + \frac{|\delta|^m}{\gamma^n} + \frac{\sqrt{5}}{\gamma^n} < \frac{7}{\gamma^{n-m}}.$$

As a result, we have

$$|\Lambda_1| < \frac{7}{\gamma^{n-m}}, \quad \Lambda_1 := 2^k \cdot \gamma^{-n} \cdot \sqrt{5} - 1. \tag{3.4}$$

According to Theorem 2.1, we get $r = 3,\ \chi_1 = 2,\ \chi_2 = \gamma,\ \chi_3 = \sqrt{5},\ t_1 = k,\ t_2 = -n,$ and $t_3 = 1$. Because of $\chi_1, \chi_2, \chi_3 \in \mathbb{Q}\left(\sqrt{5}\right)$, we should consider $\mathbb{T} = \mathbb{Q}\left(\sqrt{5}\right)$ of degree $d_{\mathbb{T}} = 2$. It is clear that $\Lambda_1 \neq 0$. Indeed, if $\Lambda_1 = 0$, then we obtain $\gamma^n = \sqrt{5} \cdot 2^k$. If we compute the square of both sides of this equation, we obtain $\gamma^{2n} = 5 \cdot 4^k$, which leads to a contradiction because the left-hand side, $\gamma^{2n} \in \mathbb{Z}$, is impossible. So, $\Lambda_1 \neq 0$. From Equations (2.3) and (2.7), we can compute

$$h(\chi_1) = \log 2,\ h(\chi_2) = \frac{1}{2}\log\gamma,\ h(\chi_3) = \frac{1}{2}\log 5,$$
$$A_1 = 2\log 2,\ A_2 = \log\gamma, \text{ and } A_3 = \log 5.$$

Besides, for $B = 2n+1$, $B \geq \max\{k, |-n|, 1\}$, since $k < 2n+1$. As a result, based on Theorem 2.1, with certain mathematical simplifications, we obtain

$$\log(|\Lambda_1|) > -1.1 \times 10^{12}(1 + \log(2n+1))$$

and with certain mathematical simplifications of the above inequality, we obtain,

$$\log(|\Lambda_1|) > -4.5 \times 10^{12}\log n \tag{3.5}$$

where we used the fact that $1 + \log(2n+1) < 4\log n$, for $n \geq 2$. From Equation (3.4), we have

$$\log(|\Lambda_1|) < \log 7 - (n-m)\log\gamma. \tag{3.6}$$

From Equations (3.5) and (3.6), we get that

$$(n-m)\log\gamma < 4.6 \times 10^{12}\log n. \tag{3.7}$$

Furthermore, if we rearrange Equation (1.2) as follows:

$$M_k = F_m + F_n \Rightarrow 2^k - 1 = \frac{\gamma^m - \delta^m}{\sqrt{5}} + \frac{\gamma^n - \delta^n}{\sqrt{5}}$$
$$\Rightarrow 2^k - \frac{\gamma^n}{\sqrt{5}}\left(1 + \gamma^{m-n}\right) = 1 - \frac{\delta^n}{\sqrt{5}} - \frac{\delta^m}{\sqrt{5}}.$$

Taking absolute values after dividing both sides of the last equation by $\frac{\gamma^n}{\sqrt{5}}(1 + \gamma^{m-n})$, we get

$$\left|\frac{2^k \cdot \sqrt{5}}{\gamma^n \cdot (1 + \gamma^{m-n})} - 1\right| < \frac{6}{\gamma^n}$$

and

$$|\Lambda_2| < \frac{6}{\gamma^n}, \quad \Lambda_2 := 2^k \gamma^{-n} \frac{\sqrt{5}}{1+\gamma^{m-n}} - 1. \tag{3.8}$$

To apply the Matveev theorem into Equation (3.8), we can consider that case where $r = 3$, $\chi_1 = 2$, $\chi_2 = \gamma$, $\chi_3 = \sqrt{5}/\left(1+\gamma^{m-n}\right)$, $t_1 = k$, $t_2 = -n$, and $t_3 = 1$. Since $\chi_1, \chi_2, \chi_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ we can take $\mathbb{T} = \mathbb{Q}\left(\sqrt{5}\right)$ of degree $d_{\mathbb{T}} = 2$. As can be seen, since $2^k \cdot \sqrt{5} = \gamma^n + \gamma^m$ is never satisfied, $\Lambda_2 \neq 0$. Besides, if we take $B = 2n+1$, then $B \geq \max\{k, |-n|, 1\}$, since $k < 2n+1$. In this case, we can compute the followings:

$$h(\chi_1) = \log 2,\ h(\chi_2) = \frac{1}{2}\log\gamma, A_1 = 2\log 2,\ \text{and}\ A_2 = \log\gamma.$$

From (2.4), (2.5), (2.6), and (2.7) we get

$$h(\chi_3) \leq 1.5 + (n-m)\log\gamma.$$

Therefore, we can take

$$A_3 = 3 + 2(n-m)\log\gamma = d_{\mathbb{T}}(1.5 + (n-m)\log\gamma) \geq d_{\mathbb{T}} h(\chi_3).$$

In this case, according to Matveev's theorem, we can write

$$\log(|\Lambda_2|) > -6.5 \times 10^{11} \times (3 + 2(n-m)\log\gamma). \tag{3.9}$$

From the right-hand side of the Equation (3.8) we get

$$\log(|\Lambda_2|) < \log 6 - n\log\gamma. \tag{3.10}$$

Considering the Equations (3.7), (3.9), and (3.10), we deduce that

$$n < 3.81 \times 10^{26}. \tag{3.11}$$

Thus, we can summarize the results mentioned above with a lemma as follows:

Lemma 3.1. *All possible solutions of Equation* (1.2) *are over the ranges* $1 \leq m < n, n > 400$*, and* $k < 2n + 1 < 7.7 \times 10^{26}$.

3.1 Reducing the bounds on n

Now we need to reduce the bound. To achieve this, we will repeatedly use the following result, which is a slight variation of a result originally developed by Dujella and Pethö in [14].

We first consider the notation

$$\Gamma_1 := k \log 2 - n \log \gamma + \log \sqrt{5}. \tag{3.12}$$

Then, from inequality (3.4), we have

$$|\Lambda_1| = \left| e^{\Gamma_1} - 1 \right| < \frac{7}{\gamma^{n-m}}. \tag{3.13}$$

Now, by using the Equations (1.1) and (1.2), we can write

$$\frac{\gamma^n}{\sqrt{5}} = F_n + \frac{\delta^n}{\sqrt{5}} < F_n + \frac{1}{2} < F_n + F_m = M_k = 2^k - 1 < 2^k$$

for $m \geq 1$. Thus $1 < 2^k \cdot \gamma^{-n} \cdot \sqrt{5}$, and so, $\Gamma_1 > 0$. Considering this inequality with (3.13), we get that

$$0 < \Gamma_1 \leq e^{\Gamma_1} - 1 < \frac{7}{\gamma^{n-m}},$$

where we used the fact that $l \leq e^l - 1$ for all $l \in \mathbb{R}$. Using the formula (3.12) in the above inequality, we find that

$$\Gamma_1 := k \log 2 - n \log \gamma + \log \sqrt{5} < \frac{7}{\gamma^{n-m}}. \tag{3.14}$$

Dividing both sides of the above inequality by $\log \gamma$, we obtain

$$0 < k \frac{\log 2}{\log \gamma} - n + \frac{\log \sqrt{5}}{\log \gamma} < \frac{7}{\gamma^{n-m} \log \gamma} < \frac{15}{\gamma^{n-m}}.$$

Then, accordingly Lemma 2.3, for $M = 7.7 \times 10^{26}$ $(M > 2n+1 > k)$ and $\tau = \frac{\log 2}{\log \gamma}$, $67th$ convergent of the continued fraction expansion of τ is

$$\frac{p_{67}}{q_{67}} = \frac{7297782051934206759257011802 16}{50664261769939766769526399782 1}$$

and so, $6M < q_{67} = 506642617699397667695263997821$. Therefore, we have

$$\varepsilon = \|\mu q_{67}\| - M \|\tau q_{67}\|, \; \varepsilon > 0.01, \; \mu = \frac{\log \sqrt{5}}{\log \gamma}.$$

So, taking $X := 15$, $Y := \gamma$, and $\nu := n - m$ into account, since from Lemma 2.3 the inequality

$$n - m > 157 > \frac{\log (X q_{67}/\varepsilon)}{\log Y}$$

has no solution, we deduce that $n - m \leq 157$.

Now we consider the notation

$$\Gamma_2 := k \log 2 - n \log \gamma + \log \left(\frac{\sqrt{5}}{1 + \gamma^{m-n}} \right). \tag{3.15}$$

Then, from inequality (3.8), we have

$$|\Lambda_2| = \left| e^{\Gamma_2} - 1 \right| < \frac{6}{\gamma^n}. \tag{3.16}$$

Since $\Lambda_2 \neq 0$, we observe that $\Gamma_2 \neq 0$. Therefore, we consider the following cases:

- If $\Gamma_2 > 0$, then $e^{\Gamma_2} - 1 > 0$. Thus, using Equation (3.16) and the fact that $l \leq e^l - 1$ for all $l \in \mathbb{R}$, we find that

$$0 < \Gamma_2 \leq e^{\Gamma_2} - 1 < \frac{6}{\gamma^n}.$$

- If $\Gamma_2 < 0$, it is clear that $\frac{6}{\gamma^n} < \frac{1}{2}$ for $n > 400$. Thus, based on (3.16), which implies that $\left| e^{\Gamma 2} - 1 \right| < \frac{1}{2}$, we have $e^{|\Gamma 2|} < 2$. Since $\Gamma_2 < 0$, so we can conclude that

$$0 < |\Gamma_2| < e^{|\Gamma_2|} - 1 = e^{|\Gamma_2|} \left|e^{\Gamma_2} - 1\right| < \frac{12}{\gamma^n}.$$

In either case, we find that the inequality

$$0 < |\Gamma_2| < \frac{12}{\gamma^n}$$

holds for all $n > 400$. Substituting the formula for Γ_2 into the previous inequality and following the argument in Equation (3.15), then dividing both sides of the inequality by $\log \gamma$, we obtain

$$0 < \left| k\frac{\log 2}{\log \gamma} - n + \frac{\log\left(\sqrt{5}/\left(1+\gamma^{m-n}\right)\right)}{\log \gamma} \right| < \frac{12}{\gamma^n \log \gamma} < \frac{25}{\gamma^n}. \tag{3.17}$$

As mentioned above, based on the Lemma 2.3, τ is taken as $\frac{\log 2}{\log \gamma}$ again. If the $67th$ convergent of the continued fraction expansion of this τ is taken, then we have that

$$\varepsilon = \|\mu q_{67}\| - M\|\tau q_{67}\|, \ \varepsilon > 0.001.$$

Where $M = 7.7{\times}10^{26}$ $(M > 2n+1 > k)$, $6M < q_{67}$ and $\mu = \frac{\log\left(\sqrt{5}/\left(1+\gamma^{m-n}\right)\right)}{\log \gamma}$, for all $n - m \in [1, 157]$ except when $n - m = 2$.

As a result, taking $X := 25$, $Y := \gamma$, and $\nu := n$ into account, we obtain that $n \leq 163$. This is a contradiction because we assumed that $n > 400$.

3.2 Examining the special case for $n - m = 2$

Finally, we will examine the special case for $n - m = 2$. If $n - m = 2$, then $\mu = \frac{\log\left(\sqrt{5}/\left(1+\gamma^{m-n}\right)\right)}{\log \gamma} = 1$. In this case, Lemma 2.3 cannot be applied because the inequality (3.17) has the form $0 < \left|k\frac{\log 2}{\log \gamma} - n + 1\right| < 25 \cdot \gamma^{-n}$. Therefore, the Dujella and Pethö reduction method is not applicable here, as the ϵ value in Lemma 2.3 would always be negative.

Let us try to find an upper bound for the value of n using the well-known identity between Lucas and Fibonacci numbers, which is $L_n = F_{n-1} + F_{n+1}$ for all $n \geq 1$, see [1]. Since $n - m = 2$, we can express this identity as $F_{n-2} + F_n = 2^k - 1$, where $n = m + 2 > 400$. From here, the problem can be transformed into a simpler form by converting it to the equation $L_{n-1} = 2^k - 1$. According to Theorem 1 in [15], the only solution is $(k, m, n) \in \{(3, 3, 5)\}$, implying that there is no solution for $n > 400$ and only a solution for $n = 5$. This completes the investigation for the special case of $n - m = 2$, indicating the completion of the proof of Theorem 3.1. □

Acknowledgement

The author is thankful to the editor(s) and referee(s) for their valuable suggestions to improve the manuscript.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Second Edition*, John Wiley & Sons, New York, 2019.

[2] M. Ddamulira, F. Luca, M. Rakotomalala, Fibonacci Numbers which are Products of Two Pell Numbers. *Fibonacci Quart.*, **54(1)**(2016), 11-18.

[3] M. A. Alekseyev, On the Intersections of Fibonacci, Pell, and Lucas Numbers, *Integers*, **11(A)**(2011), 1–18.

[4] A. Gaber, Intersections of Pell, Pell-Lucas Numbers and Sums of Two Jacobsthal Numbers. *Punjab Univ. J. Math. (Lahore)*, **55(5-6)**(2023), 241-252. https://doi.org/10.52280/pujm.2023.55(5-6)05

[5] F. Erduvan, R. Keskin, Fibonacci numbers which are products of two Jacobsthal numbers. *Tbilisi Math. J.*, **14(2)**(2021). https://doi.org/105-116. 10.32513/tmj/19322008126

[6] M. Alan, K. S. Alan, Mersenne numbers which are products of two Pell numbers, *Bol. Soc. Mat. Mex.*, **28(38)** (2022). https://doi.org/10.1007/s40590-022-00430-x

[7] B. Edjeou, A. Tall, On Diophantine equations involving difference of Lucas Numbers and powers of 2, *Aligarh Bull. Math.*, **42(2)**(2023), 95-125.

[8] F. Luca, A. Togbé, On the x-coordinates of Pell equations which are Fibonacci numbers, *Math. Scand.*, **122(1)**(2018), 18–30. https://doi.org/10.7146/math.scand.a-97271

[9] J. J. Bravo, F. Luca, Coincidences in generalized Fibonacci sequences, *J. Number Theory*, **133(6)**(2013), 2121-2137. https://doi.org/10.1016/j.jnt.2012.11.006

[10] A. Emin, On the Diophantine Equation $L_m^2 + L_n^2 = 2^a$, *Comptes Rendus Acad. Bulg. Sci.*, **77(8)**(2024), 1128-1137. https://doi.org/10.7546/CRABS.2024.08.02

[11] A. Emin, Pell Numbers that can be Written as the Sum of Two Mersenne Numbers, *Bull. Int. Math. Virtual Inst.*, **14(1)**(2024), 129-137.

[12] J. J. Bravo, C. A. Gómez, Mersenne k-Fibonacci Numbers, *Glasnik Mat. Ser.*, **51(2)** (2016), 307-319. https://doi.org/10.3336/gm.51.2.02

[13] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II, *Izv. Ross. Akad. Nauk Ser. Math.*, **64(6)**(2000), 125-180.

[14] A. Dujella, A. Pethö, A generalization of a theorem of Baker and Davenport, *The Quart. J. Math. Oxford Ser.(2)*, **(49)195** (1998), 291-–306. https://doi.org/10.1093/qmathj/49.3.291

[15] A. Altasan, M. Alan, Mersenne Numbers in Generalized Lucas Sequences, *C. R. Acad. Bulg. Sci.*, **77(1)**(2024), 3-10. https://doi.org/10.7546/CRABS.2024.01.01