The Aligarh Bulletin of Mathematics
Volume 43, Number 1 (2024), 57 - 63

ISSN: 0304-9787

Solution of complex differential equations by using variational iteration method

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(Received: May 18, 2024 Accepted: June 30, 2024)

Abstract

This work focuses on a method (Variational Iteration Method) that is based on the Lagrange multiplier to solve complex differential equations. Besides we presented three examples to find out the effectiveness, and accuracy of the proposed technique.

1 Introduction

The Variational Iteration Method (VIM) is one of the estimation methods used to solve both linear and nonlinear equations related to physics and mathematics problems [1–5]. Many authors have used this method (variational iteration method) to solve several problems such as Issa and Düz [6] have employed the variational iteration method to get the Fourier transforms. Xu et al. [7] solved the boundary layer equations of magnetohydrodynamic flow using VIM. Wazwaz [8] has implemented the VIM for solving ODEs with variable coefficients. In [9], Moghimi and Hejazi applied the Variational iteration method to solve Burger, and Fisher equations. The complex differential equations have been solved using several methods [10–12].

Keywords and phrases: Variational Iteration Method, Lagrange multiplier, complex differential equation.

2020 AMS Subject Classification: 34A34.

In this study, we applied the VIM for solving complex differential equations with constant coefficients given by

$$a \cdot \frac{\partial w}{\partial z} + b \cdot \frac{\partial w}{\partial \overline{z}} + c \cdot w = H\left(z, \overline{z}\right). \tag{1.1}$$

Where a, b, c are constant coefficients, and $w\left(z, \overline{z}\right)$ is the unknown function of Eq (1.1).

2 VIM for the Solution of complex differential equations

Theorem 2.1. *Let* $a, b, c \in \mathbb{R}$, $w \in \mathbb{C}$ *and consider the complex differential equations defined by*

$$a \cdot \frac{\partial w}{\partial z} + b \cdot \frac{\partial w}{\partial \overline{z}} + c \cdot w = H\left(z, \overline{z}\right). \tag{2.1}$$

Then the solution of Eq (2.1) *is*

$$\begin{aligned} w_{\alpha+1}(x,y) = w_\alpha(x,y) - \int_0^y e^{\frac{2ic(y-s)}{b-a}} \left(\frac{\partial w_\alpha}{\partial s} + i \left(\frac{a+b}{a-b} \right) \frac{\partial w_\alpha}{\partial x} \right. \\ \left. + \frac{2ic}{a-b} w_\alpha + \frac{2iH_1(x,s)}{b-a} \right) ds, \alpha \geq 0. \end{aligned} \tag{2.2}$$

Proof. By complex derivatives, Eq (2.1) can be written as follows:

$$a \cdot \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) + b \cdot \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) + cw = H_1(x,y) \tag{2.3}$$

$$(a+b)\frac{\partial w}{\partial x} + i(b-a)\frac{\partial w}{\partial y} + 2cw = 2H_1(x,y)$$

$$\frac{\partial w}{\partial y} + \frac{a+b}{i(b-a)}\frac{\partial w}{\partial x} + \frac{2c}{i(b-a)}w = \frac{2H_1}{i(b-a)}. \tag{2.4}$$

Now, by applying the VIM formula of Eq (2.4), we obtain:

$$w_{\alpha+1}(x,y) = w_\alpha(x,y) + \int_0^y \lambda(s,y) \left(\frac{\partial w_\alpha}{\partial s} + i\left(\frac{a+b}{a-b}\right)\frac{\partial w_\alpha}{\partial x} + \frac{2ic}{a-b} w_\alpha + \frac{2iH_1(x,s)}{b-a} \right) ds, \tag{2.5}$$

where $\lambda(s,y)$ is general Lagrange multiplier.

To find the Lagrange multiplier, we use the stationary condition

$$\delta w_{\alpha+1}(x,y) = \delta w_\alpha(x,y) + \delta \int_0^y \lambda(s,y) \left(\frac{\partial w_\alpha}{\partial s} + i\left(\frac{a+b}{a-b}\right)\frac{\partial w_\alpha}{\partial x} + \frac{2ic}{a-b} w_\alpha + \frac{2iH_1(x,s)}{b-a} \right) ds \tag{2.6}$$

$$\begin{aligned}&\delta w_{\alpha+1}(x,y)\\ &= \delta w_\alpha(x,y) + \delta \left[\lambda(s,y)\, w_\alpha(x,s) \updownarrow_{s=y} - \int_0^y w_\alpha(x,s) \frac{\partial \lambda(s,y)}{\partial s} \right]\\ &- i\frac{a+b}{b-a}\delta \int_0^y \frac{\partial w_\alpha}{\partial x} ds - \frac{2ic}{b-a}\delta \int_0^y \lambda(s,y)\, w_\alpha(x,s) ds + \delta \frac{2i}{b-a} \int_0^y H_1(x,s) ds\end{aligned} \tag{2.7}$$

$$1 + \lambda(y,y) = 0, \; -\frac{\partial \lambda(s,y)}{\partial s} - \frac{2ic\lambda(s,y)}{b-a} = 0$$

$$\lambda(s,y) = -e^{\frac{2ic(y-s)}{b-a}}. \tag{2.8}$$

Set Eq (2.8) into Eq (2.5), we obtain

$$w_{\alpha+1}(x,y) = w_\alpha(x,y) - \int_0^y e^{\frac{2ic(y-s)}{b-a}} \left(\frac{\partial w_\alpha}{\partial s} + i\left(\frac{a+b}{a-b}\right)\frac{\partial w_\alpha}{\partial x} + \frac{2ic}{a-b} w_\alpha + \frac{2iH_1(x,s)}{b-a} \right) ds \tag{2.9}$$

□

3 Examples of Applying Variational Iteration Method on Complex Differential Equations

Example 3.1. *Consider the following equation*

$$\begin{aligned} \frac{\partial w}{\partial z}+2\frac{\partial w}{\partial \overline{z}} &= 3z^2+2, \qquad (3.1)\\ w(x,0) &= x^3+x, \qquad (3.2) \end{aligned}$$

$a=1, b=2, c=0, H(z,\overline{z})=3z^2+2, H_1(x,y)=3(x+iy)^2+2=3x^2-3y^2+2+6ixy, \lambda(s,y)=-e^{\frac{2ic(y-s)}{b-a}}=-1.$

By applying Theorem 2.1 of Eq (3.1)*, we obtain*

$$\begin{aligned} w_{\alpha+1}(x,y) &= w_\alpha(x,y)-\int_0^y\left(\frac{\partial w_\alpha}{\partial s}+i\left(\frac{a+b}{a-b}\right)\frac{\partial w_\alpha}{\partial x}+\frac{2ic}{a-b}w_\alpha+\frac{2iH_1(x,s)}{b-a}\right)ds\\ &= w_\alpha(x,y)-\int_0^y\left(\frac{\partial w_\alpha}{\partial s}-3i\frac{\partial w_\alpha}{\partial x}+2i\left(3x^2-3s^2+2+6ixs\right)\right)ds. \end{aligned} \qquad (3.3)$$

From condition $w_0(x,y)=x^3+x$,

$$\begin{aligned} w_1(x,y) &= w_0(x,y)-\int_0^y\left(-3i\frac{\partial w_0}{\partial x}+2i(3x^2+6ixs-3s^2+2)\right)ds\\ &= x^3+x-\int_0^y(-3i(3x^2+1)+2i(3x^2+6ixs-3s^2+2))ds\\ &= x^3+x+6xy^2+i\left(3x^2y-y+2y^3\right) \end{aligned} \qquad (3.4)$$

$$\begin{aligned} w_2(x,y) &= x^3+x+6xy^2+i\left(3x^2y-y+2y^3\right)-\int_0^y(18xs-18is^2)ds\\ &= x^3+x-3xy^2+i(3x^2y-y+8y^3) \end{aligned} \qquad (3.5)$$

$$\begin{aligned} w_3(x,y) &= x^3+x-3xy^2+i(3x^2y-y+8y^3)-\int_0^y 27is^2ds\\ &= x^3+x-3xy^2+i(3x^2y-y-y^3) \end{aligned} \qquad (3.6)$$

$$w_\alpha(x,y)=z^3+\overline{z}.$$

Example 3.2. *Consider the following equation*

$$\begin{aligned} \frac{\partial w}{\partial z} - \frac{\partial w}{\partial \overline{z}} - w &= 0, && (3.7)\\ w(x,0) &= e^{3x}, && (3.8) \end{aligned}$$

$a = 1, b = -1, c = -1, H(z, \overline{z}) = 0, H_1(x, y) = 0, \lambda(s, y) = -e^{\frac{2ic(y-s)}{b-a}} = -e^{i(y-s)}$.

By applying Theorem 2.1 of Eq (3.7)*, we obtain*

$$w_{n+1}(x,y) = w_n(x,y) - \int_0^y e^{i(y-s)} \left(\frac{\partial w_n}{\partial s} - i w_n \right) ds.$$

From condition $w_0(x, y) = e^{3x}$,

$$w_1(x,y) = e^{3x} - \int_0^y e^{i(y-s)}(-ie^{3x})ds = e^{3x} + ie^{3x}\frac{e^{i(y-s)}}{-i} \updownarrow_0^y = e^{3x+iy}$$

$$w_2(x,y) = e^{3x+iy} - \int_0^y e^{i(y-s)}(ie^{3x+is} - ie^{3x+is})ds = e^{3x+iy} = e^{2z+\overline{z}}.$$

Example 3.3. *Consider the following equation*

$$\begin{aligned} 2\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \overline{z}} &= 4z + 1, && (3.9)\\ w(x,0) &= x^2 + 5x, && (3.10) \end{aligned}$$

$a = 2, b = -1, c = 0, H(z, \overline{z}) = 4z + 1, H_1(x, y) = 4x + 1 + 4iy, \lambda(s, y) = -e^{\frac{2ic(y-s)}{b-a}} = -1$,

$$w_{n+1}(x,y) = w_n(x,y) - \int_0^y \left(\frac{\partial w_n}{\partial s} + i\left(\frac{a+b}{a-b}\right)\frac{\partial w_n}{\partial x} + \frac{2ic}{a-b}w_n + \frac{2iF_1(x,s)}{b-a} \right) ds.$$

By applying Theorem 2.1 of Eq (3.9)*, we obtain*

$$w_{n+1}(x,y) = w_n(x,y) - \int_0^y \left(\frac{\partial w_n}{\partial s} + \frac{i}{3}\frac{\partial w_n}{\partial x} - \frac{2i}{3}(4x + 1 + 4is) \right) ds.$$

From condition $w_0(x, y) = x^2 + 5x$,

$$
\begin{aligned}
w_1(x,y) &= x^2+5x-\int_0^y \left(\tfrac{i}{3}(2x+5)-\tfrac{2i}{3}(4x+1+4is)\right) ds \\
&= x^2+5x-\tfrac{i}{3}(2x+5)s+\tfrac{2i}{3}(4xs+s+2is^2) \updownarrow_0^y \\
&= x^2+5x-\tfrac{i}{3}(2x+5)y+\tfrac{2i}{3}(4xy+y+2iy^2) \\
&= x^2+5x-\tfrac{4y^2}{3}+\tfrac{i}{3}(6xy-3y),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&w_2(x,y) = x^2+5x-\frac{4y^2}{3}+\frac{i}{3}(6xy-3y) \\
&-\int_0^y \left[\left(-\frac{8s}{3}+2ix-i\right)+\frac{i}{3}\left(2x+5+2is\right)-\frac{2i}{3}(4x+1+4is)\right] ds \\
&= x^2+5x-\frac{4y^2}{3}+\frac{i}{3}(6xy-3y) \\
&-\int_0^y \left[-\frac{8s}{3}-\frac{2s}{3}+\frac{8s}{3}+i\left(2x-1+\frac{2x}{3}+\frac{5}{3}-\frac{8x}{3}-\frac{2}{3}\right)\right] ds \\
&= x^2+5x-\frac{4y^2}{3}+\frac{i}{3}(6xy-3y)+\frac{y^2}{3} = x^2+5x-y^2+i(2xy-y) \\
&= z^2+3\overline{z}+2z.
\end{aligned}
\tag{3.12}
$$

4 Conclusion

In this work, we have used the method (Variational Iteration Method) to solve complex differential equations. Moreover, the results showed that the proposed method is an accurate and suitable scientific method for dealing with complex differential equations.

Acknowledgement
The authors are grateful to the learned refree for his careful reading and useful suggestions on the manuscripts.

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