The Aligarh Bulletin of Mathematics
Volume 43, Number 1 (2024), 49 - 56

ISSN: 0304-9787

Solution of Airy equation in terms of Dirac delta distribution

Murat Düz and Ahmad Issa

Department of Mathematics, Faculty of Science
Karabuk University, Karabuk, Turkey
Email: mduz@karabuk.edu.tr, ahmad93.issa18@gmail.com

(Received: April 30, 2024 Accepted: July 31, 2024)

Abstract

In this paper, we used the Fourier transform method to solve the Airy equation. The solution of this equation depends on Dirac delta distribution which is very important for the properties of Fourier transforms.

1 Introduction

There are many methods used for solving linear differential equations. One of these methods is integral transformations. Well-known integral transforms are Laplace and Fourier transforms. The Fourier transform is an integral transform used in many fields of mathematics and engineering [1–5]. In this study, we use the Fourier transform to solve an Airy equation

$$y'' - ty = 0, \tag{1.1}$$

which is a special kind of linear differential equation

$$y^{(m)} - ty = 0. \tag{1.2}$$

Keywords and phrases: Airy equation, Fourier transform, Dirac delta.
2020 AMS Subject Classification: 34A30, 42A38.

2 Preliminaries

Definition 2.1. *[6] The Fourier transform of function $f(t)$ is defined as*

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt = F(w). \tag{2.1}$$

Definition 2.2. *[6] The inverse Fourier transform of $F(w)$ is given by:*

$$\mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}dw = f(t). \tag{2.2}$$

Theorem 2.1. *[7] Let $c_1, c_2 \in \mathbb{R}$, then $\mathcal{F}[c_1f_1(t) + c_2f_2(t)] = c_1\mathcal{F}[f_1(t)] + c_2\mathcal{F}[f_2(t)]$. That means the Fourier transform is a linear.*

Definition 2.3. *[7] The Dirac delta distribution can be defined as limit for $\varepsilon \to 0$ of following function*

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\epsilon} & , \quad 0 < t < \epsilon \\ 0 & , \quad t < 0 \\ 0 & , \quad t > \epsilon \end{cases}.$$

The Dirac delta distribution has the following properties [7, 8].

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \tag{2.3}$$

$$t\delta(t) = 0, \tag{2.4}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0), \tag{2.5}$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t - t_0)dt = (-1)^n f^{(n)}(t_0), \tag{2.6}$$

$$(w - w_0)^n\delta^{(n)}(w - w_0) = n!(-1)^n\delta(w - w_0), \tag{2.7}$$

$$\int_{-\infty}^{\infty} \frac{\delta(w - w_0)f(w)}{(w - w_0)^n}dw = \frac{1}{n!}\frac{d^nf(w)}{dw^n}|_{w=w_0}. \tag{2.8}$$

Where $\delta(w - w_0)$ is defined as following

$$\delta(w - w_0) = \begin{cases} 0, \text{if } w \neq w_0 \\ \infty, w = w_0 \end{cases} .$$

Theorem 2.2. *[10] (i) The Fourier transform of the Dirac delta distribution is 1. (ii) if $\mathcal{F}[y(t)] = Y(w)$, Then, $\mathcal{F}[y^{(n)}(t)] = (iw)^n\, Y(w)$.*

3 FTM for Airy equation

3.1 Solution of Airy equation

Let we apply Fourier transform to Equation (1.1), we obtain

$$\mathcal{F}[y'' - ty] = \mathcal{F}[0].$$

Suppose that $\mathcal{F}[y] = Y$

$$(iw)^2 Y - i\frac{dY}{dw} = 0$$

$$\frac{dY}{Y} = iw^2 dw$$

$$Y = e^{\dfrac{iw^3}{3}}.$$

If we have taken inverse Fourier, then we have obtained that

$$y = \mathcal{F}^{-1}[Y] = \mathcal{F}^{-1}\left[e^{\dfrac{iw^3}{3}}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\dfrac{iw^3}{3}} e^{iwt} dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\dfrac{w^3}{3} + wt\right)} dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\cos\left(\frac{w^3}{3} + wt\right) + i\sin\left(\frac{w^3}{3} + wt\right)\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(\frac{w^3}{3} + wt\right)dw + i\frac{1}{2\pi}\int_{-\infty}^{\infty}\sin\left(\frac{w^3}{3} + wt\right)dw.$$

But $\sin\left(\frac{w^3}{3}+wt\right)$ is an odd function in according to w. So,

$$\int_{-\infty}^{\infty}\sin\left(\frac{w^3}{3}+wt\right)dw=0.$$

Therefore, the solution of Airy equation have been obtained as

$$y=\frac{1}{\pi}\int_{0}^{\infty}\cos\left(\frac{w^3}{3}+wt\right)dw.$$

This solution is called Airy function of first kind and is shown as $A_i(t)$. The Airy equation is second order. Therefore it has two linear independent solution. Second solution has been shown with $B_i(t)$ and defined as

$$B_i(t)=\frac{1}{\pi}\int_{0}^{\infty}\left(e^{\left(\frac{-w^3}{3}+wt\right)}+\sin\left(\frac{w^3}{3}+wt\right)\right)dw.$$

Also, the general solution of the Airy equation can be found easily with power series by

$$f(t)=c_1f_1(t)+c_2f_2(t),$$

where

$$f_1(t)=1+\sum_{k=1}^{\infty}\frac{t^{3k}}{(2.3)(5.6)\dots(3k-1)(3k)},$$

and

$$f_2(t)=t+\sum_{k=1}^{\infty}\frac{t^{3k+1}}{(3.4)(6.7)\dots(3k)(3k+1)}.$$

There are some equalities with Airy functions between $f_1(t)$ and $f_2(t)$.

$$A_i(t)=c_1f_1(t)-c_2f_2(t),$$

$$B_i(t)=\sqrt{3}[c_1f_1(t)+c_2f_2(t)],$$

$$A_i(0)=c_1, B_i(0)=\sqrt{3}c_1,$$

$$A_i^{'}(0)=-c_2, B_i^{'}(0)=\sqrt{3}c_2.$$

More information about the Airy functions and Airy equation can been found in [9–12]. Now, let us obtain the solution of the Airy equation in terms of the Dirac delta distribution by using Fourier transform. To do this, we used the series expansion of the Fourier transform of the solution.

3.2 An alternative representation of solution of Airy equation

Theorem 3.1. *A solution of differential equation* $y^{(m)} - ty = 0$ *is*

$$y = c\left[\delta(t) - \frac{\delta^{(m+1)}}{m+1} + \frac{\delta^{(2m+2)}}{2!(m+1)^2} - \frac{\delta^{(3m+3)}}{3!(m+1)^3} + \cdots\right].$$

Proof. Using the Fourier transform of Equation (1.2), we get

$$\mathcal{F}[y^{(m)} - ty] = \mathcal{F}[0].$$

Suppose that $\mathcal{F}[y] = Y$,

$$(iw)^m Y - i\frac{dY}{dw} = 0$$

$$\frac{dY}{Y} = iw^m dw$$

$$Y = ce^{\dfrac{-(iw)^{m+1}}{m+1}} = c\left[1 - \frac{(iw)^{m+1}}{m+1} + \frac{(iw)^{2m+2}}{2!(m+1)^2} - \frac{(iw)^{3m+3}}{3!(m+1)^3} + \cdots\right]. \tag{3.1}$$

If we take inverse Fourier transform of (3.1), we obtain a solution of Equation (1.2)

$$y = c\left[\delta(t) - \frac{\delta^{(m+1)}}{m+1} + \frac{\delta^{(2m+2)}}{2!(m+1)^2} - \frac{\delta^{(3m+3)}}{3!(m+1)^3}\cdots\right]$$

□

Now, we will solve the Airy equation using the proposed method

$$\mathcal{F}[y'' - ty] = \mathcal{F}[0].$$

Suppose that $\mathcal{F}[y] = Y$,

$$(iw)^2 Y - i\frac{dY}{dw} = 0$$

$$\frac{dY}{Y} = iw^2 dw$$

$$Y = ce^{\frac{iw^3}{3}} = c\left[1 + \frac{iw^3}{3} - \frac{w^6}{2!9} - \frac{iw^9}{3!27} + \frac{w^{12}}{4!81} + \cdots\right]$$

$$y = \mathcal{F}^{-1}[Y] = c\mathcal{F}^{-1}\left[1 + \frac{iw^3}{3} - \frac{w^6}{2!9} - \frac{iw^9}{3!27} + \frac{w^{12}}{4!81} + \cdots\right]$$

$$= c\left[\delta(t) + \frac{i}{3}\frac{\delta^{(3)}}{i^3} - \frac{1}{2!9}\frac{\delta^{(6)}}{i^6} - \frac{i}{3!27}\frac{\delta^{(9)}}{i^9} + \cdots\right]$$

$$= c\left[\delta(t) + \frac{3!}{3}\frac{\delta(t)}{t^3} + \frac{6!}{2!3^2}\frac{\delta(t)}{t^6} + \frac{9!}{3!3^3}\frac{\delta(t)}{t^9} + \frac{12!}{4!3^4}\frac{\delta(t)}{t^{12}} + \ldots\right]$$

$$= c\delta(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n}.$$

Besides, we will check the solution of equation (1.1)

$$y = c\delta(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n},$$

$$y' = c\delta^{'}(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n} + c\delta(t)\sum_{n=0}^{\infty}\frac{(-3n)(3n)!}{n!3^n}t^{-3n-1},$$

$$y'' = c\delta^{''}(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n} + 2c\delta^{'}(t)\sum_{n=0}^{\infty}\frac{(-3n)(3n)!}{n!3^n}t^{-3n-1}$$

$$+c\delta(t)\sum_{n=0}^{\infty}\frac{(-3n)(-3n-1)(3n)!}{n!3^n}t^{-3n-2}.$$

After substituting the above equations into Equation (1.1), and using property (2.7), we obtain

$$y'' - ty = 2c\delta(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n-2} - 2c\delta(t)\sum_{n=0}^{\infty}\frac{(-3n)(3n)!}{n!3^n}t^{-3n-2}$$

$$+c\delta(t)\sum_{n=0}^{\infty}\frac{(-3n)(-3n-1)(3n)!}{n!3^n}t^{-3n-2} - c\delta(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n+1}$$

$$= c\delta(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n-2}(9n^2+9n+2) - c\delta(t)\sum_{n=0}^{\infty}\frac{(3n)!}{n!3^n}t^{-3n+1}$$

$$= c\delta(t)\left(\frac{2}{t^2}+\frac{3!}{3t^5}(20)+\frac{6!}{2!3^2t^8}(56)+\cdots\right) - c\delta(t)\left(t+\frac{3!}{3t^2}+\frac{6!}{2!3^2t^5}+\frac{9!}{3!3^3t^8}+\cdots\right)$$

$$= -ct\delta(t) = 0.$$

4 Conclusion

In this paper, we have applied Fourier transform method to solve Airy equation. However, the results show that the proposed method is an accurate method to solve this equation. Also, we noted the solution can be written in terms of Dirac delta distribution.

Acknowledgment

We sincerely wish to thank the reviewers for their helpful comments.

References

[1] O. Vallée, M. Soares, Airy functions and applications to physics. World Scientific, 2010.

[2] M. Khalid, M. Sultana, F. Zaidi, *Numerical solution of Airy differential equation by using Haar wavelet*, Math. Th. Model, **4(10)** (2014), 142–148.

[3] M. Turkyilmazoglu, *The Airy equation and its alternative analytic solution*, Phys. Scr., **86(5)** (2012), 055004.

[4] M. Ehrhardt and R. E. Mickens, *Solutions to the discrete Airy equation: Application to parabolic equation calculations*, J. Comput. Appl. Math., **172(1)** (2004), 183–206.

[5] F. J. Narcowich, and A. Boggess, *A first course in wavelets with Fourier analysis*, Hoboken, 2009.

[6] R. Bracewell and P. B. Kahn, *The Fourier transform and its applications*, Amer. J. Phys., **34(8)** (1966), 712–712.

[7] L. Debnath and D.Bhatta, *Integral transforms and their applications*, Chapman and Hall/CRC, 2016.

[8] N. Wheeler, *Simplified production of Dirac delta function identities*, Reed College, 1997.

[9] B. Osgood, *The Fourier transform and its applications*, Lecture notes for EE, 2009.

[10] M. Düz, A. Issa, S. Avezov, *A new computational technique for Fourier transforms by using the Differential transformation method*, Bull. Int. Math. Virtual Inst., **12(2)** (2022), 287–295.

[11] S. Avezov, A. Issa, M. Düz, *Solving difference equations using fourier transform method*, Sigma J. Eng. Nat. Sci., **42(4)** (2024), 1239–1244.

[12] M. Düz, S. Avezov, A. Issa, *Solutions to Differential-Differential Difference Equations with Variable Coefficients by Using Fourier Transform Method*, Süleyman Demirel Univ. Fac. Arts J. Science Sci., **18(3)** (2023), 259–267.