The Aligarh Bulletin of Mathematics
Volume 43, Number 1 (2024), 39 - 48
ISSN: 0304-9787

On proximity pairs, best proximity point and best approximation

Sangeeta[1] **and T. D. Narang**[2]

[1]Department of Mathematics
Amardeep Singh Shergill Memorial College
Mukandpur-144507 Punjab (India)
[2]Department of Mathematics
Guru Nanak Dev University
Amritsar -143005 (India)
Emails: seetzz_20@yahoo.co.in, tdnarang1948@yahoo.co.in

(Received: March 19, 2024 Accepted: August 1, 2024)

Abstract

The proximity pair associated with the pair (A, B) of non-empty subsets A, B of a metric space (X, d) is (A_o, B_o), where $A_o = \{x \in A : d(x, y) = dist(A, B),$ for some $y \in B\}$ and $B_o = \{y \in B : d(x, y) = dist(A, B),$ for some $x \in A\}$. The pair (A, B) is said to have (d)-property (respectively, weak (d)-property) if $d(x_1, y_1) = dist(A, B)$, $d(x_2, y_2) = dist(A, B)$ imply $d(x_1, x_2) = d(y_1, y_2)$ (respectively, $d(x_1, x_2) \leq d(y_1, y_2)$), where $x_1, x_2 \in A$ and $y_1, y_2 \in B$. For a mapping $T : A \to B$, a point $x_o \in A$ is called a best proximity point of T if $d(x_o, Tx_o) = dist(A, B)$. In this paper, we discuss proximity pairs and apply (d)-property and weak (d)-property to discuss the uniqueness of best approximation and the existence and uniqueness of best proximity points when the underlying spaces are metric spaces and linear metric spaces. It is also shown that if a linear metric space (X, d) is strictly convex, then each pair (A, B) of non-empty closed and convex subsets of X has (d)-property.

Keywords and phrases: Best proximity point, proximity pair, convex metric space, strict convexity, metric projection.
2020 AMS Subject Classification: 41A50, 41A65, 46B20.

1 Introduction

Let A and B be non-empty subsets of a metric space (X, d) and $dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. The proximity pair associated with the pair (A, B), denoted by (A_o, B_o) is defined as $A_o = \{x \in A : d(x, y) = dist(A, B)$, for some $y \in B\}$ and $B_o = \{y \in B : d(x, y) = dist(A, B)$, for some $x \in A\}$. The pair (A, B) is said to have (d)-property (respectively, weak (d)-property) if $d(x_1, y_1) = dist(A, B)$, $d(x_2, y_2) = dist(A, B)$ imply $d(x_1, x_2) = d(y_1, y_2)$ (respectively, $d(x_1, x_2) \leq d(y_1, y_2)$), where $x_1, x_2 \in A$ and $y_1, y_2 \in B$. Clearly, (d)-property implies weak (d)-property but converse is not true (Example 2.2). A metric space (X, d) is said to have (d)-property (weak (d)-property) if for every two non-empty subsets A, B of X, the pair (A, B) has the (d)-property(weak (d)-property). Basha and Veeramani [3] showed that in a normed linear space X, $A_o \subseteq Bd(A)$, $B_o \subseteq Bd(B)$; A_o and B_o are closed and convex subsets of A and B respectively if A and B are closed and convex subsets of X. Kirk et al. [5] discussed sufficient conditions which ensure the non-emptyness of A_o and B_o when the underlying spaces are normed linear spaces. Raj [7] introdued (d)-property in normed linear spaces and proved that any pair (A, B) of non-empty closed and convex subsets of a real Hilbert space H has (d)-property. Caballero et al. [4] used (d)-property to prove a best proximity point theorem for Geraghty-contraction map in complete metric spaces. Zhang et al. [11] introduced weak (d)-property and used it to generalize the best proximity point theorem of Caballero et al. [4]. Raj and Eldred [8] characterized strictly convex normed linear spaces in terms of (d)-property, and used (d)-property to prove the uniqueness of best approximation and also to prove best proximity point theorems in normed linear spaces. Bajracharya and Damai [2] used (d)-property and weak (d)-property to discuss best approximation problems.

In this paper we shall discuss some of the results of [2], [3], [7] and [8] in spaces more general than normed linear spaces.

2 Preliminaries

In this section, we recall few definitions and examples related to results proved in the paper. A subset K of a metric space (X, d) is said to be

1. proximinal if for each $x \in X$ there exists a point $k_o \in K$, called a best approximation to x in K, such that $d(x, k_o) = dist(x, K) \equiv \inf\{d(x, k) : k \in K\}$,

2. semi-Chebyshev if each $x \in X$ has at most one best approximation in K,

3. Chebyshev if each $x \in X$ has exactly one best approximation in K.

Since

$$P_K(x) = \begin{cases} x, & \text{if } x \in K \\ \phi, & \text{if } x \in \overline{K} \setminus K, \end{cases}$$

every proximinal set is closed. The mapping $P_K : X \to 2^K \equiv$ collection of all subsets of K which takes each element x of X to the set K is called a metric projection.

A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure [10] on X if for all $x, y \in X$ and $\lambda \in [0,1]$, $d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$ holds for all $u \in X$. A metric space (X,d) together with a convex structure is called a convex metric space and is denoted by (X,d,W). A nonempty subset K of a convex metric space (X,d,W) is said to be convex [10] if $W(x,y,\lambda) \in K$ for every $x, y \in K$ and $\lambda \in [0,1]$. Every normed linear space is a convex metric space but there are many convex metric spaces which are not normed linear spaces (see [6], [10]). A linear space X equipped with a metric d is called a linear metric space (see [1]) if both addition and scalar multiplication in X are continuous, and the metric d is translation invariant, i.e., $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$.

A linear metric space (X,d) is said to be strictly convex [1] if for $x, y \in X$, $x \neq y$, $d(x,0) \leq r$, $d(y,0) \leq r$ imply $d(\frac{x+y}{2}, 0) < r$, $r > 0$. A subset K of a linear metric space (X,d) is convex if for each $x, y \in K$, the line segment $[x,y]$ joining x and y lies in K. If A, B are non-empty subsets of a metric space (X,d) and $T : A \to B$, a point $x_o \in A$ is called a best proximity point of T if $d(x_o, Tx_o) = dist(A,B)$.

Example 2.1. [8] *In the Euclidean space* $(\mathbb{R}^2, |\cdot|_2)$, *let*

$$A = \{(x,y) : -2 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$B = \{(x,y) : 1 \leq x \leq 2, 0 \leq y \leq 1\}.$$

Then A, B *are non-empty closed and convex subsets with* $dist(A,B) = 2$. *It is easy to verify that* $A_o = \{(-1,y) : 0 \leq y \leq 1\}$, $B_o = \{(1,y) : 0 \leq y \leq 1\}$.

Example 2.2. [11] *In the Euclidean space* $(\mathbb{R}^2, |\cdot|_2)$, *let* $A = \{(0,0)\}$, $B = \{y : y = 1+\sqrt{1-x^2}\}$. *Then* $dist(A,B) = \sqrt{2}$, $A_o = \{(0,0)\}$, $B_o = \{(-1,1),(1,1)\}$, $d((0,0),(-1,1)) = d((0,0),(1,1)) = \sqrt{2}$. *The pair* (A,B) *satisfies the weak d-property but not the d-property as* $0 = d((0,0),(0,0)) < d((-1,1),(1,1)) = 2$.

3 The results

It is easy to see that in any metric space (X, d), the sets A_o and B_o are closed subsets of A and B respectively if A and B are closed subsets of X. The following proposition deals with the convexity of these sets:

Proposition 3.1. *If A is a convex subset of a convex metric space (X, d, W), then A_o is convex.*

Proof. Let $x_1, x_2 \in A_o$. Then, $d(x_1, y) = d(x_2, y) = dist(A, B)$ for some $y \in B$. We claim that $W(x_1, x_2, \lambda) \in A_o$ for $0 \leq \lambda \leq 1$. Consider

$$\begin{aligned}
d(W(x_1, x_2, \lambda), y) &\leq \lambda d(x_1, y) + (1 - \lambda) d(x_2, y) \\
&= \lambda dist(A, B) + (1 - \lambda) dist(A, B) \\
&= dist(A, B) \\
&\leq d(W(x_1, x_2, \lambda), y)) \text{ as } W(x_1, x_2, \lambda) \in A.
\end{aligned}$$

This gives $d(W(x_1, x_2, \lambda), y) = dist(A, B)$ and so $W(x_1, x_2, \lambda) \in A_o$ for all $\lambda \in [0, 1]$. Hence A_o is convex. The proof for the convexity of B_o is similar. □

The following proposition shows that $A_o \subseteq Bd(A)$:

Proposition 3.2. *If A is a convex subset of a convex metric space (X, d, W), then $A_o \subseteq Bd(A)$.*

Proof. Let $x \in A_o$ be arbitrary. Then there exists some $y \in B$ such that $d(x, y) = dist(A, B)$. Suppose $x \in int(A)$, then some open ball $B_\epsilon(x) \subseteq A$, $\epsilon > 0$. Let $z_n = W(y, x, \frac{1}{n})$. Then, for all n,

$$\begin{aligned}
d(z_n, x) &= d(W(y, x, \frac{1}{n}), x) \\
&\leq \frac{1}{n} d(y, x) = \frac{1}{n} d(x, y) \\
&= \frac{1}{n} dist(A, B).
\end{aligned}$$

This implies $d(z_m, x) < \epsilon$ for large m, i.e., $z_m \in B_\epsilon(x) \subset A$ for large m. Consider

$$\begin{aligned} d(W(y, x, \frac{1}{m}), y) &\leq (1 - \frac{1}{m})d(x, y) \\ &< dist(A, B), \end{aligned} \tag{3.1}$$

i.e., $d(W(y, x, \frac{1}{m}), y) < dist(A, B))$, a contradiction. Hence $x \in Bd(A)$, i.e., $A_o \subseteq Bd(A)$, i.e., $A_o \subseteq Bd(A)$. □

Recall that a normed linear space is said to be strictly convex (see [8]) if for any two distinct points in the closed unit sphere, the mid-point of the line segment joining them lies strictly inside the closed unit sphere. Raj and Eldred [8] established the following characterization of strictly convex normed linear spaces:

Theorem 3.1. *A normed linear space X is strictly convex if and only if every pair (A, B) of non-empty closed and convex subsets of X has the (d)-property.*

Using this characterization, the following results were obtained in [8]:

Corollary 3.1. *[8] Let M be non-empty, closed and convex subset of a strictly convex normed linear space X. Then, M has at most one point of minimum norm.*

Corollary 3.2. *[8] Let A and B be non-empty closed and convex subsets of a strictly convex normed linear space X such that A_o is non-empty. Then, the restriction of the metric projection mapping P_{A_o} to B_o is an isometry, i.e., P_{A_o} : $B_o \to A_o$ is an isometery.*

As an application of the above characterization theorem, the following best proximity point theorem was proved in [8]:

Theorem 3.2. *Let A, B be non-empty closed and convex subsets of a strictly convex Banach space X and $T : A \to B$ be a contraction mapping (i.e., $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in A$ and $k < 1$) such that $A_o \neq \phi$ and $T(A_o) \subseteq B_o$ where $A_o = \{x \in A : d(x, y) = dist(A, B)$, for some $y \in B\}$ and $B_o = \{y \in B : d(x, y) = dist(A, B)$, for some $x \in A\}$. Then there exists a unique $x^* \in A$ such that $\|x^* - Tx^*\| = dist(A, B)$. Further, for each fixed x_o in A_o, there is a sequence $< x_n >$ such that for each $n \in \mathbb{N}$, $\|x_{n+1} - Tx_n\| = dist(A, B)$ and $< x_n >$ converges to best proximity point x^* of the map T.*

We have the following generalizations of the corresponding results of V. Shankar Raj [7] proved for Hilbert spaces and of Raj and Eldrd [8] proved for normed linear spaces.

Theorem 3.3. *If a linear metric space (X, d) is strictly convex, then every pair (A, B) of non-empty closed and convex subsets of X has the (d)-property.*

Proof. Let (X, d) be a strictly convex linear metric space and A, B be non-empty closed and convex subsets of X. Suppose $x_1, x_2 \in A$ and $y_1, y_2 \in B$ be such that $d(x_1, y_1) = d(x_2, y_2) = dist(A, B) > 0$ as A and B are closed subsets of X. Put $u = x_1 - y_1$, $v = x_2 - y_2$. If $u = v$, then $x_1 - y_1 = x_2 - y_2$, i.e., $x_1 - x_2 = y_1 - y_2$ and so $d(x_1, x_2) = d(y_1, y_2)$ Thus (A, B) has the (d)-property. Suppose $u \neq v$, then

$$\begin{aligned} dist(A,B) &\leq d\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \text{ as } \frac{x_1+x_2}{2} \in A \text{ and } \frac{y_1+y_2}{2} \in B \\ &= d\left(\frac{x_1-y_1}{2}, \frac{y_2-x_2}{2}\right) \\ &= d\left(\frac{u}{2}, \frac{-v}{2}\right) \\ &= d\left(\frac{u+v}{2}, 0\right) \\ &< dist(A,B). \end{aligned}$$

As $d(u, 0) = d(x_1 - y_1, 0) = d(x_1, y_1) = dist(A, B)$, $d(v, 0) = d(x_2 - y_2, 0) = d(x_2, y_2) = dist(A, B)$ and $u \neq v$. Therefore, $u \neq v$ is not possible. Hence, every pair (A, B) of closed convex subsets of X has the (d)-property. □

Theorem 3.4. *If M is a non-empty closed convex subset of a linear metric space (X, d) in which every pair of non-empty closed convex subsets has (d)-property, then each element $x \in X$ has atmost one best approximation to x in M.*

Proof. Take $A = \{x\}$ and $B = M$. Suppose some $x \in X$ has two distinct best approximations in M, i.e.,

$$d(x, y_1) = d(x, y_2) = d(x, M) = dist(\{x\}, M).$$

Since the pair ({x}, M) has (d)-property,

$$0 = d(x, x) = d(y_1, y_2).$$

Hence, $y_1 = y_2$. □

Corollary 3.3. *If M is a non-empty closed and convex subset of a strictly convex linear metric space (X, d), then M is semi-Chebyshev.*

Corollary 3.4. *Let A, B be non-empty closed and convex subsets of a linear metric space (X, d) in which (A, B) has (d)-property (or (X, d) is a strictly convex linear metric space) and $A_o \neq \phi$. Then the restriction of the metric projection P_{A_o} to B_o i.e., $P_{A_o} : B_o \to A_o$. is an isometery.*

Proof. Let $y_1, y_2 \in B_o$. Then by using Theorem 3.4, there exists unique pair $(x_1, x_2) \in A_o \times A_o$ such that $d(x_1, y_1) = d(x_2, y_2) = dist(A, B)$, i.e., $P_{A_o}(y_i) = x_i$, $i = 1, 2$. By the (d)-property,

$$d(P_{A_o}(y_1), P_{A_o}(y_2)) = d(x_1, x_2) = d(y_1, y_2).$$

Hence, $P_{A_o} : B_o \to A_o$. is an isometery. □

It can be easily seen that the proof of Theorem 3.2 for normed linear spaces given in [8] can easily be extended to linear metric spaces and so, we have:

Theorem 3.5. *Let A, B be non-empty closed and convex subsets of a complete linear metric space (X, d) in which (A, B) has (d)-property (or (X, d) is a complete strictly convex linear metric space) and $T : A \to B$ be a contraction mapping such that $A_o \neq \phi$. and $T(A_o) \subseteq B_o$. Then, there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$. Further, for each x_o in A_o, there is a sequence $< x_n >$ such that for each $n \in \mathbb{N}$, $d(x_{n+1}, Tx_n) = dist(A, B)$ and $< x_n >$ converges to the best proximity point x^* of the mapping T.*

Remark 3.1. *Whereas it was shown in [8] that the converse of Theorem 3.3 also holds in normed linear spaces, it is not known whether the converse holds in linear metric spaces too.*

Remark 3.2. *It can be easily seen that Theorem 3.4, its two corollaries and Theorem 3.5 hold in convex metric spaces. It will be interesting to prove Theorem 3.3* *and its converse for convex metric spaces. We may recall (see [6]) that a convex metric space (X, d, W) is said to be strictly convex if for any $r > 0$ and $x, y, p \in X$ with $d(x, p) \leq r$, $d(y, p) \leq r$ and $x \neq y$ we have $d(W(x, y, \frac{1}{2}), p) < r$.*

Remark 3.3. *The condition of convexity of the sets A and B in Theorem 3.5 can not be relaxed even in case of normed linear spaces (see [8], Example 4.1)*

Bajracharya and Damai [2] used (d)-property and weak d-property to discuss existence and uniqueness of best approximation in metric spaces and Banach spaces. We also use these two properties to discuss existence and uniqueness of best approximation in metric spaces thereby extend and generalize some of the results proved in [2], We start with proving:

Theorem 3.6. *Let M be a proximinal convex subset of a convex metric space (X, d, W) in which each pair of non-empty closed and convex subsets has (d)-property, then M is Chebyshev.*

Proof. Since M is proximinal, for each $x \in X$ there is $m_o \in M$ such that $d(x, m_o) = dist(x, M)$. Suppose M is not Chebyshev, then there exists some $x \in X$ which has two distinct best approximations say $m_1, m_2 \in M$, i.e., $d(x, m_1) = dist(x, M)$, $d(x, m_2) = dist(x, M)$. Take $A = \{x\}$, $B = M$. Then A and B are non-empty closed and convex subsets of X. So by the (d)-property of the pair $(\{x\}, M)$, $0 = d(x, x) = d(m_1, m_2)$ and hence $m_1 = m_2$, which is a contradiction. Therefore M is Chebyshev. □

Remark 3.4. *Theorem 3.6 holds if the space X is such that in it each pair of non-empty closed and convex subsets has weak (d)-property. In this case $d(m_1, m_2) \leq d(x, x) = 0$ and so $m_1 = m_2$.*

Since compact, approximatively compact and boundedly compact subsets of a metric space are all proximinal (see [9]), we have

Corollary 3.5. *If M is a compact or approximatively compact or boundedly compact closed and convex subsets of a convex metric space (X, d, W) in which each pair of non-empty closed and convex subsets has (d)-property or weak (d)-property, then M is Chebyshev.*

Remark 3.5. *The following example shows that the convexity condition of the set M in Theorem 3.6 can not be dropped even in Banach spaces.*

Example 3.1. [2] *Consider the Euclidean space $\mathbb{R}^2$ with usual norm and $M = \mathbb{R}^2 \backslash B(0,1)$. Then M is proximinal but not Chebyshev as*

$$P_M(x) = \begin{cases} x, & \text{if } x \in M \\ \frac{x}{\|x\|}, & \text{if } x \in B(0,1) \setminus \{0\}, \\ \{y \in \mathbb{R}^2 : \|y\| = 1\}, & \text{if } x = 0. \end{cases}$$

Remark 3.6. *Theorem 3.6 was proved in [2] under stronger conditions on the set M as well as on the space X (the set M was assumed to be compact and the space X was taken to be one in which each pair (A, B) of non-empty closed subsets satisfy (d)-property or weak (d)-property).*

Acknowledgement: The authors are thankful to the learned referee for valuable suggestions, leading to an improvement of the paper.

REFERENCES

[1] G. C. Ahuja, T. D. Narang, and S. Trehan, *Best approximation on convex sets in metric linear spaces*, Math. Nachr., **78**(1977), 125-130.

[2] P. M. Bajracharya and G. R. Damai, *A note on existence and uniqueness of nearest points in a set*, International J. Research and Review, **8**(2018), 50-54.

[3] S. S. Basha and P. Veeraman, *Best proximity pair theorems for multifunctions with open fibres*, J. Approx. Theory, **103**(2000), 119-129.

[4] J. Caballero, J. Harjani and K. Sadarangani, *A best proximity point theorem for Garaghty-contractions*, J. Fixed Point Theory Appl., **231** (2012), 1-4.

[5] W. A. Kirk, S. Reichand and P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., **24**(2003), 119-129.

[6] M. Moosali, *The structure of fixed point set of a type of generalized non-expansive mappings in convex metric spaces*, Thai. J. Math., **19**(2021), 315-324.

[7] V. S. Raj, Ba*nach Contractions Principle for non-self mappings*, Preprint.

[8] V. S. Raj and A. A. Eldred, *A characterization of strictly convex spaces and applications*, J. Optim. Theory Appl., **160**(2014), 703-710.

[9] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, New York (1970).

[10] W. Takahashi, *A convexity in metric space and non-expansive mappings*, I. Kodai Math. Sem. Rep. **22**(1970), 142-149.

[11] J. Zhang, Y. Su and Q. Cheng, *A note on best proximity point theorem for Garaghty-contractions*, J. Fixed Point Theory and Appl., **99**(2013), 1-4.