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Rings with annihilator conditions on power values of generalized derivations

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Abstract

Let R be a prime ring with its Utumi ring of quotient U, $C = Z(U)$, the extended centroid of R, G a generalized derivation of R and λ a nonzero ideal of R. Suppose that there exists $0 \neq b \in R$ such that $b([x,y]^t[G([x,y]),[x,y]][x,y]^s)^m = 0$ or $b((x\circ y)^t[G(x\circ y),(x\circ y)](x\circ y)^s)^m$ for all $x, y \in \lambda$, where $t \geq 1$, $s \geq 0$, $m \geq 1$ are fixed integers. Then either R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$ in four variables x_1, x_2, x_3, x_4 and $G(x) = qx + xq + \alpha x$ for some $q \in U$ and $\alpha \in C$ or $G(x) = \alpha x$ for all $x \in R$ with $\alpha \in C$.

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1 Introduction

Throughout this paper, R will represent a prime ring with centre $Z(R)$, extended centroid C and U its utumi quotient ring. We shall write for any pair of elements $x, y \in R$, the commutator $[x, y] = xy - yx$ and skew commutator $x \circ y = xy + yx$. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1, x_2, x_3, x_4) = \Sigma_{\sigma \in s_4}(-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$, where $(-1)^{\sigma}$ is $+1$ or -1 according to σ being an even or odd permutation in the permutation group S_4. An additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ is a derivation. Starting from this definition Bresar [10] introduced the concept of a generalized derivation. An additive mapping $F : R \to R$ associated with a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ is called a generalized derivation. One may observe that concept of generalized derivation includes the concept of derivation, also of the left multiplier when $d = 0$. Let $a, b \in R$, an additive mapping $F : R \to R$ defined by $F(x) = ax + xb$ for all $x \in R$ is an example of a generalized derivation. Generalized derivations have been primarily studied in operator algebras. Therefore, any investigation from algebraic point of view might be interesting [6]. In [18], it is proved that if R is a prime ring and d is a derivation of R such that $ad(R) = 0$, then either $a = 0$ or $d = 0$. Bresar [16] proved that if R is a $(n-1)$!-torsion free semiprime ring with $ad(x)^n = 0$ for all $x \in R$ and $a \in R$, $n \geq 1$ a fixed integer, then $ad(R) = 0$. When R is a prime ring, it is obvious that either $a = 0$ or $d = 0$. In [11], Lee and Lin extended Bresar's result for Lie ideal case by deleting the restriction on R to be $(n-1)$!-torsion free. For one-sided ideal, Chang and Lin [15] considered the case when $d(x)x^n = 0$ for all $x \in I$, a nonzero ideal right ideal of R. They showed that if R is a prime ring and d is a nonzero derivation of R and n is a fixed positive integer, then $d(I)I = 0$ and if $x^n d(x) = 0$ for all $x \in I$, then $R \cong M_2(F)$, the 2×2 matrices over a field F of two elements. Later, for noncommuting Lie ideal L of R, Dhara and Sharma [14] proved that if $(u^s[d(u), u]u^t)^n \in Z(R)$ for all $u \in L$, where $s \geq 0$, $t \geq 0$, $n \geq 1$ are fixed integers, then R satisfies s_4.
Following this line of investigation, we prove the following theorem:

Theorem 1.1. *Let R be a prime ring with characteristic different from 2, U its Utumi quotient ring, C its extended centroid, λ a nonzero ideal of R and G a nonzero generalized derivation with associated derivation d of R, $s \geq 0$, $t \geq 1$, $m \geq 1$ fixed integers and $0 \neq b \in R$. Assume that $b([x, y]^t [G([x, y]), [x, y]][x, y]^s)^m = 0$ for all $x, y \in \lambda$. Then one of the following holds:*

(i) R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$ in four variables and $G(x) =$

$qx + xq + \alpha x$ for some $q \in U$ and $\alpha \in C$;

(ii) $G(x) = \alpha x$ for all $x \in R$ and $\alpha \in C$.

2 Preliminaries

In all that follows, R always denotes a prime ring, U its Utumi quotient ring. The definition and axiomatic formulation of Utumi quotient ring U can be found in [4] and [5], respectively. We have the following properties which we need:

1. $R \subseteq U$;
2. U is prime ring with identity;
3. The centre of U denoted by C and is called the extended centroid of R, C is a field.

Moreover, we will use frequently some important theory of generalized polynomial identities and differential identities. We recall some of the facts.

Fact 1. *If B is a basis of U over C, then any element of $T = U *_C C\{x_1, \dots, x_n\}$, the free product over C of U and the free C-algebra $C\{x_1, \dots, x_n\}$, can be written in the form of $g = \sum_i \alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials, that is $m_i = q_0 x_1 q_1, \dots, x_k q_k$, with $q_i \in B$ and $x_i \in \{x_1, \dots, x_n\}$. In [5] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if all α_i are zero. Let $a_1, a_2, \dots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, x_2, \dots, x_n) + a_2 g_2(x_1, x_2, \dots, x_n) + \cdots + a_k g_k(x_1, x_2, \dots, x_n) = 0 \in T$, for some $g_1, g_2, \dots, g_k \in T$. If for any i, $g_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} x_j hj(x_1, x_2, \dots, x_n)$ and $h_j(x_1, x_2, \dots, x_n) \in T$, then $g_1(x_1, x_2, \dots, x_n), g_2(x_1, x_2, \dots, x_n), \dots, g_k(x_1, x_2, \dots, x_n)$ are zero element of T. The same conclusion holds if $g_1(x_1, x_2, \dots, x_n)a_1 + g_2(x_1, x_2, \dots, x_n)a_2 + \cdots + g_k(x_1, x_2, \dots, x_n)a_k = 0 \in T$ and $g_1(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} h_j(x_1, x_2, \dots, x_n)x_j$ for some $h_j(x_1, x_2, \dots, x_n) \in T$.*

We refer the reader to [4] for more details of generalized polynomials identities.

Fact 2. *[7, Theorem 2] If I is a two-sided ideal of R, then I and U satisfy the same differential identities.*

Fact 3. *[6, Theorem 3] Let R be a semiprime ring. Then every generalized derivation F on a dense right ideal of R is uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Moreover, a and d are uniquely determined by the generalized derivation F.*

Fact 4. *[5, Theorem 2] If I is a two-sided ideal of R, I and U satisfy the same generalized polynomial identities with coefficients in U.*

Fact 5. *[12, Theorem 2] Let R be a prime ring and d be a nonzero derivation on R and I be a nonzero ideal of R. By Kharchenko's Theorem if I satisfies the differential polynomial identity $P(x_1, x_2, \dots, x_n, d(x_1), d(x_2), \dots, d(x_n)) = 0$, then either d is an inner derivation or I satisfies the generalized polynomial identity $P(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = 0$.*

Fact 6. *[17, Lemma 2.2] Let K be a field, R be a dense ring of K-linear transformations (over a vector space V) with $dim_K V \geq 3$, $b, q \in R$ and $q \notin K$. Assume $bv = 0$ for any $v \in V$ such that $\{v, qv\}$ is linearly K-independent, then $b = 0$.*

3 Main results

We begin with the following lemmas:

Lemma 3.1. *Let R be a prime ring, Utumi quotient ring U, extended centroid C and $p, q \in U$. If there exists $0 \neq b \in R$ such that $b([x,y]^t[p[x,y]+[x,y]q,[x,y]][x,y]^s)^m = 0$ for all $x, y \in R$, where $s \geq 0, t \geq 1, m \geq 1$ are fixed integers, then either R satisfies a nontrivial generalized polynomial identity (GPI) or $p, q \in C$.*

Proof. Let R does not satisfy any nontrivial GPI. Let $T = U *_C C\{x, y\}$, the free product over C of U and $C\{x, y\}$, the free C-algebra in noncommuting indeterminates x and y. Then $b([x,y]^t[p[x,y]+[x,y]q,[x,y]][x,y]^s)^m$ is zero element in $T = U *_C C\{x, y\}$. Thus $b([x,y]^t[p[x,y]+[x,y]q,[x,y]][x,y]^s)^m = 0 \in T$, that is,

$$b([x,y]^t(p[x,y]^2 + [x,y](q-p)[x,y] - [x,y]^2 q)[x,y]^s)^m = 0 \in T. \quad (3.1)$$

We suppose that $p \notin C$, then p and 1 are linearly independent over C. Thus,

$$b([x,y]^t(p[x,y]^2+[x,y](q-p)[x,y]-[x,y]^2q)[x,y]^s)^{m-1}([x,y]^t p[x,y]^{2+s}) = 0 \in T. \tag{3.2}$$

Again since p and 1 are linearly independent over C, we get

$$b([x,y]^t(p[x,y]^2+[x,y](q-p)[x,y]-[x,y]^2q)[x,y]^s)^{m-2}([x,y]^t p[x,y]^{2+s})^2 = 0 \in T.$$

Arguing in the similar manner as above, we have

$$b([x,y]^t p[x,y]^{2+s})^m = 0 \in T. \tag{3.3}$$

Which implies that $p = 0$, a contradiction. Therefore, we conclude that $p \in C$ and R satisfies

$$b([x,y]^{t+1}[[x,y],q][x,y]^s)^m = 0 \in T. \tag{3.4}$$

Which yields that $q \in C$. □

Lemma 3.2. *Let $R = M_2(F)$ be a ring of 2×2 matrices over the field F of characteristic not 2. Suppose there exists $0 \neq b \in R$ such that $b([x,y]^t[p[x,y]^2+[x,y](q-p)[x,y]-[x,y]^2q][x,y]^s)^m = 0$ for all $x,y \in R$, where $s \geq 0, t \geq 1, m \geq 1$ are fixed integers. Then, $q-p \in Z(R)$.*

Proof. By assumption, we have

$$b([x,y]^t[p[x,y]^2+[x,y](q-p)[x,y]-[x,y]^2q][x,y]^s)^m = 0 \text{ for all } x,y \in R. \tag{3.5}$$

Let $x = e_{21}$, $y = e_{12} \in R$, so $[x,y] = e_{22} - e_{11}$. Also let $q - p = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Our hypothesis becomes

$$b((e_{22}-e_{11})^t \begin{pmatrix} 0 & -2a_{12} \\ -2a_{21} & 0 \end{pmatrix} (e_{22}-e_{11})^s)^m = 0.$$

That is,

$$b\begin{pmatrix} 0 & \pm 2a_{12} \\ \mp 2a_{21} & 0 \end{pmatrix}^m = 0.$$

Since characteristic of $R \neq 2$, we have

$$b\begin{pmatrix} 0 & \pm a_{12} \\ \mp a_{21} & 0 \end{pmatrix}^m = 0. \tag{3.6}$$

If m is odd, then we have

$$b\begin{pmatrix} 0 & \pm a_{12}^{(m+1)/2} a_{21}^{(m-1)/2} \\ \mp a_{21}^{(m+1)/2} a_{12}^{(m-1)/2} & 0 \end{pmatrix} = 0.$$

If m is even then, we get

$$b\begin{pmatrix} a_{12}^{(m+1)/2} a_{21}^{(m-1)/2} & 0 \\ 0 & a_{21}^{(m+1)/2} a_{12}^{(m-1)/2} \end{pmatrix} = 0.$$

In both the cases, we get $b_{kl}a_{12}a_{21} = 0$ for all $k, l = 1, 2$. As $b \neq 0$, we have some nonzero b_{kl}. In this case $a_{12}a_{21} = 0$. For any automorphism θ of R, $\theta(b), \theta(p)$ and $\theta(q)$ enjoy the same property as b, p, q have, namely, for all $x, y \in R$

$$\theta(b)([x,y]^t[\theta(p)[x,y]^2 + [x,y](\theta(q) - \theta(p))[x,y] - [x,y]^2\theta(q)][x,y]^s)^m = 0. \tag{3.7}$$

Hence $\theta(b)_{kl}\theta(a)_{12}\theta(a)_{21} = 0$, where $\theta(a)_{ij}$ is the (i,j)-entry of $\theta(p-q)$. Let $a_{12} = a_{21} = 0$ and $\theta_1(x) = (1 - e_{21})x(1 + e_{21})$ be an inner automorphism of R. Then $\theta_1(p-q)_{12} = 0$, i.e., $a_{11} = a_{22}$. That is, $p - q$ is a scalar matrix and hence $p-q \in Z(R)$. Therefore, we assume that $a_{12} \neq 0$ and $\theta_2(x) = (1+e_{21})x(1-e_{21})$ is an inner automorphism of R. Then

$$\theta_2(p-q)_{12}\theta(p-q)_{21} = a_{12}(a_{21} - a_{22} + a_{11} - a_{12}) = 0$$

$$\theta_1(p-q)_{12}\theta(p-q)_{21} = a_{12}(a_{21} + a_{22} - a_{11} - a_{12}) = 0.$$

Above two equations give that $2a_{12}^2 = 0$. Since, characteristic of $R \neq 2$, we get $a_{12} = 0$, a contradiction. Similarly if $a_{21} \neq 0$, we get the contradiction $a_{21} = 0$. □

Lemma 3.3. *Let $R = M_3(F)$ be a ring of 3×3 matrices over the field F of characteristic not 2. Suppose there exists $0 \neq b \in R$ such that $b([x,y]^t[p[x,y]^2 +$*

$[x,y](q-p)[x,y]-[x,y]^2q][x,y]^s)^m = 0$ *for all* $x,y \in R$, *where* $s \geq 0,\ t \geq 1,\ m \geq 1$ *are fixed integers. Then,* $q-p \in Z(R)$.

Proof. By assumption, we have

$$b([x,y]^t[p[x,y]^2+[x,y](q-p)[x,y]-[x,y]^2q][x,y]^s)^m = 0 \text{ for all } x,y \in R. \tag{3.8}$$

Let $p-q=(a_{kl}), p=(p_{kl}), q=(q_{kl})$ for $a_{kl}, p_{kl}, q_{kl} \in F$, $k,l=1,2,3$. Also let $x=e_{21}, y=e_{12} \in R$. Thus

$$b\left((e_{22}-e_{11})^l \begin{pmatrix} p_{11}-q_{11}-a_{11} & p_{12}-q_{12}+a_{12} & 0 \\ p_{21}-q_{21}+a_{21} & p_{22}-q_{22}-a_{22} & -q_{23} \\ p_{31} & p_{32} & 0 \end{pmatrix} (e_{22}-e_{11})^s\right)^m = 0. \tag{3.9}$$

As $p_{11}-q_{11}=a_{11}$, $p_{12}-q_{12}=a_{12}$, $p_{21}-q_{21}=a_{21}$, $p_{22}-q_{22}=a_{22}$, above equation becomes

$$b\left((e_{22}-e_{11})^l \begin{pmatrix} 0 & 2a_{12} & 0 \\ 2a_{21} & 0 & -q_{23} \\ p_{31} & p_{32} & 0 \end{pmatrix} (e_{22}-e_{11})^s\right)^m = 0. \tag{3.10}$$

Let $a_{12}a_{21} \neq 0$. We show that this leads a contradiction. The proof is divided into a number of steps:

Step-1 Let $s \neq 0$. In this case equation (3.10) becomes

$$b\begin{pmatrix} 0 & 2a_{12} & 0 \\ 2a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^m = 0.$$

Since characteristic of $R \neq 2$, we have

$$b\begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^m = 0. \tag{3.11}$$

If m is even, then

$$b\begin{pmatrix} a_{12}^{m/2}a_{21}^{m/2} & 0 & 0 \\ 0 & a_{12}^{m/2}a_{21}^{m/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

If m is odd, we get

$$b\begin{pmatrix} 0 & a_{12}^{m+1/2}a_{21}^{m-1/2} & 0 \\ a_{12}^{m-1/2}a_{21}^{m+1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

In either case, we have $b_{kl} = 0$ for all $k = 1, 2, 3$ and $l = 1, 2$. Now consider the two following inner automorphism of R, $f_1(x) = (1 + e_{31})x(1 - e_{31})$ and $f_2(x) = (1-e_{31})x(1+e_{31})$. If $f_1(p-q)_{12}f_1(p-q)_{21} = f_2(p-q)_{12}f_2(p-q)_{21} = 0$, then $a_{12}(a_{21} - a_{22}) = 0$ and $a_{12}(a_{21} + a_{22}) = 0$, i.e., $a_{12}a_{21} = 0$, a contradiction. Hence one of them is zero. Assume $f_1(p - q)_{12}f_1(p - q)_{21} \neq 0$. This gives that $f_1(b)_{ij} = 0$ for all $i = 1, 2, 3$ and $j = 1, 2$. By calculation, we have $f_1(b)_{i1} = b_{i1} - b_{i3} = -b_{i3} = 0$ for $i \neq 3$ and $f_1(b)_{31} = b_{31} + b_{11} - b_{33} - b_{13} = -b_{33} = 0$. This gives that $b = 0$, a contradiction.

Step-2 Let $s = 0$. In this case equation (3.10) becomes

$$b\begin{pmatrix} 0 & -2a_{12} & 0 \\ 2a_{21} & 0 & -q_{23} \\ 0 & 0 & 0 \end{pmatrix}^m = 0. \tag{3.12}$$

Right multiplying by $e_{11} + e_{22}$, if m is even, then

$$b\begin{pmatrix} 2^m a_{12}^{m/2}a_{21}^{m/2} & 0 & 0 \\ 0 & 2^m a_{12}^{m/2}a_{21}^{m/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

and if m is odd, we get

$$b\begin{pmatrix} 0 & -2^m a_{12}^{m+1/2} a_{21}^{m-1/2} & 0 \\ 2^m a_{12}^{m-1/2} a_{21}^{m+1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

As above, we get a contradiction. Therefore, we have $a_{12}a_{21} = 0$. Let $a_{21} = 0$ and $a_{12} \neq 0$. For any automorphism $\theta_1(x) = (1 - e_{21})x(1 + e_{21})$ and $\theta_2(x) = (1 + e_{12})x(1 - e_{12})$ of R, we have

$$\theta_1(q-p)_{12}\theta_1(q-p)_{21} = \theta_2(q-p)_{12}\theta_2(q-p)_{21} = 0.$$

Then,

$$a_{12}(a_{21} + a_{11} - a_{12} - a_{12}) = 0,\ a_{12}(a_{21} - a_{11} + a_{12} - a_{12}) = 0.$$

We have $a_{12} = 0$ which is a contradiction. Therefore, $a_{12} = 0 = a_{21} = 0$. Arguing in the similar manner, we can show that $a_{kl} = 0$ for $k \neq l$ that is $p-q$ is a diagonal matrix. Let $\theta(x) = (1 - e_{kl})x(1 + e_{kl})$, $k \neq l$ be an inner automorphism of R. Then, $\theta(p-q)_{kl} = a_{ll} - a_{kk} + a_{kl} - a_{lk} = a_{ll} - a_{kk} = 0$. That is $a_{ll} = a_{kk}$. Hence $q - p$ is a scalar matrix. □

Lemma 3.4. *Let R be a prime ring with characteristic different from 2, U its Utumi quotient ring, C extended centroid of R, λ an ideal of R and $p, q \in U$. If there exists $0 \neq b \in R$ such that $b([x,y]^t[p[x,y] + [x,y]_q, [x,y]][x,y]^s)^m = 0$ for all $x, y \in \lambda$ where $s \geq 0,\ t \geq 1,\ m \geq 1$ are fixed integers, then either R satisfies s_4 and $p + 2q \in C$ or $p, q \in C$.*

Proof. By hypothesis, we have

$$P(x,y) = b([x,y]^t[p[x,y] + [x,y]_q, [x,y]][x,y]^s)^m = 0 \text{ for all } x, y \in \lambda. \quad (3.13)$$

By Fact-4, I, R, U satisfy the same generalized polynomials identity with coefficients in U and we have

$$P(x,y) = b([x,y]^t[p[x,y] + [x,y]_q, [x,y]][x,y]^s)^m = 0 \text{ for all } x, y \in R. \quad (3.14)$$

If R does not satisfy any nontrivial generalized polynomials identity, then by Lemma 3.1 we are done. Let R satisfies a nontrivial generalized polynomial identity. In the light of Fact-4, U satisfies $P(x, y)$. In case C is infinite, we have $P(x, y) = 0$ for all $x, y \in U \bigotimes_c \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_c \bar{C}$ are prime and centrally closed [8], we may replace R by U or $U \bigotimes_c \bar{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $P(x, y) = 0$ for all $x, y \in R$. By Martindale's Theorem [22], R is a primitive ring having nonzero $soc(R)$ with C as associative division ring. Hence by Jacobson' Theorem [13], R is isomorphic to dense ring of linear transformations of vector space V over C. If V is finite dimensional over C, then $R \cong M_n(C)$. If $n = 2$, then we are done by Lemma 3.2. If $n = 3$, then by Lemma 3.3, we get $p - q \in Z(R)$. Therefore, our hypothesis becomes

$$b([x, y]^t([x, y]^2 q - q[x, y]^2)[x, y]^s)^m = 0 \text{ for all } x, y \in R. \tag{3.15}$$

For some $v \in V$, if $\{v, qv\}$ is linearly independent over C, then there exists $w \in V$ such that $\{v, qv, w\}$ is linearly independent over C. By Jacobson's Theorem there exist $x_1, x_2 \in R$ such that

$$x_2 v = w,\ x_2 qv = w,\ x_1 v = 0,\ x_1 qv = 0,\ x_1 w = v.$$

Multiplying equation (3.15) by v from right, we get $bv = 0$, hence $b = 0$ by Fact-6 which is a contradiction to $b \neq 0$. Hence $\{v, qv\}$ is linearly dependent over C, i.e., $q \in C$. If $n > 3$ and for some $v \in V$, $\{v, pv\}$ is linearly independent over C, then there exist $w, r \in V$ such that $\{v, pv, w, r\}$ is linearly independent over C. In light of Jacobson's Theorem there exist $x_1, x_2 \in R$ such that

$$x_2 v = w,\ x_2 pv = -w,\ x_2 r = 0,\ x_2 qv = 0$$

$$x_1 v = 0,\ x_1 pv = r,\ x_1 w = v,\ x_1 qv = 0.$$

Multiplying equation (3.14) by v from right, to have $bv = 0$ and hence $b = 0$ by Fact-6 which is a contradiction to $b \neq 0$. Hence $\{v, pv\}$ is linearly dependent over

C, i.e., $p \in C$ and by hypothesis, we have

$$b([x,y]^{t+1}[q,[x,y]][x,y]^s)^m = 0 \text{ for all } x, y \in R. \tag{3.16}$$

Again let for some $v \in V$, $\{v, qv\}$ be linearly independent over C. Then $\{v, qv, w\}$ is linearly independent over C for some $w \in V$. Again by Jacobson's Theorem there exist $x_1, x_2 \in R$ such that

$$x_2 v = qv,\ x_2 qv = w,\ x_2 w = -v$$

$$x_1 v = w,\ x_1 qv = 0,\ x_1 w = qv - v.$$

Multiplying equation (3.16) by v from right, to have $bv = 0$ and hence $b = 0$ by Fact-6 which is a contradiction to $b \neq 0$. Hence $\{v, qv\}$ is linearly dependent over C, i.e., $q \in C$. Finally assume that V is infinite dimensional over C. Then as in Lemma 2 in [19], R satisfies

$$b(u^t(pu^2 + u(p-q)u) - u^2q)u^t)^m = 0. \tag{3.17}$$

For some $v \in V$ let $\{v, qv\}$ be linearly independent over C. Then $\{v, qv, w\}$ for some $w \in V$ is linearly independent over C. By Jacobson's Theorem there exists $x \in R$ such that

$$xv = v,\ uqv = -pv + w,\ xw = w - v.$$

Multiplying equation (3.17) by v from right, to have $bv = 0$ and hence $b = 0$ by Fact-6 which is a contradiction to $b \neq 0$. Hence $\{v, qv\}$ is linearly dependent over C that is $q \in C$. Therefore equation (3.17) becomes

$$b(u^t[p,u])u^{t+1})^m = 0. \tag{3.18}$$

Again let for some $v \in V$, $\{v, pv\}$ be linearly independent over C. By Jacobson's Theorem there exists $x \in R$ such that

$$xv = v,\ xpv = pv - v.$$

Multiplying equation (3.18)by v from right, to have $bv = 0$ and hence $b = 0$ by

Fact-6 which is a contradiction to $b \neq 0$. Hence $\{v, pv\}$ is linearly dependent over C, i.e., $p \in C$. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 By assumption, we have

$$b([x,y]^t[G([x,y]),[x,y]][x,y]^s)^m = 0 \text{ for all } x,y \in \lambda. \tag{3.19}$$

By Fact-4 I, R, U satisfy the same generalized polynomial identity, we have

$$b([x,y]^t[G([x,y]),[x,y]][x,y]^s)^m = 0 \text{ for all } x,y \in U. \tag{3.20}$$

In the light of Fact-3, $G(x)$ can be written as $G(x) = px + d(x)$ for some $p \in U$ and a derivation d of U. Now equation (3.20) becomes

$$b([x,y]^t[p[x,y]+d([x,y]),[x,y]][x,y]^s)^m = 0 \text{ for all } x,y \in U. \tag{3.21}$$

Assume first that d is an inner derivation of U that is there exists $q \in U$ such that $d(x) = [q,x]$. Therefore, we have

$$b([x,y]^t[p[x,y]+[q,[x,y]],[x,y]][x,y]^s)^m = 0 \text{ for all } x,y \in U. \tag{3.22}$$

That is,

$$b([x,y]^t[(p+q)[x,y]-q[x,y],[x,y])[x,y]^s)^m = 0 \text{ for all } x,y \in U. \tag{3.23}$$

This can be written as

$$b([x,y]^t((p+q)[x,y]^2-[x,y]p[x,y]-q[x,y]^2)[x,y]^s)^m = 0 \text{ for all } x,y \in U. \tag{3.24}$$

By Lemma 3.4 either R satisfies s_4 and $p+2q \in C$ or $p+q,\ -q \in C$, that is, $p,q \in C$. In the first case R satisfies s_4, then we assume that $p+q = -q+\alpha$ for some $\alpha \in C$. Thus we have $G(x) = px+[q,x] = (p+q)x - xq = (-q+\alpha)x - xq = -qx - xq + \alpha x$ for all $x \in R$. If d is not an inner derivation of U, then by Kharchenko's Theorem [12], U satisfies the generalized polynomial identity

$$b([x,y]^t[p[x,y]+[z,y]+[x,w],[x,y]][x,y]^s)^m = 0 \text{ for all } x,y,w,z \in U. \tag{3.25}$$

In particular choosing $z = w = 0$, we obtain

$$b([x, y]^t [p[x, y], [x, y]][x, y]^s)^m = 0 \text{ for all } x, y \in U. \tag{3.26}$$

By [9, Lemma 5], we get $p \in C$. For $z = 0$, equation (3.25) becomes

$$b([x, y]^t [[x, w], [x, y]][x, y]^s)^m = 0 \text{ for all } x, y, w \in U. \tag{3.27}$$

By [21], we get

$$([x, y]^t [[x, w], [x, y]][x, y]^s)^m = 0 \text{ for all } x, y, w \in U. \tag{3.28}$$

It is a polynomial identity for U, so U is a P.I. ring. Since U is P.I. ring, it is well known that there exists a field K such that $U \subseteq M_t(K)$, the ring of $t \times t$ matrices over K. Moreover, U and $M_t(K)$ satisfy the same polynomial identity [20, Lemma 2]. If $t = 1$, then U is commutative and hence R is commutative, a contradiction. Suppose $t \geq 2$ and choose $w = e_{22}$ and $x = e_{12} - e_{21}, y = -e_{21}$. Since characteristic of $R \neq 2$, we obtain the following contradiction:

$$2^m (e_{12} + e_{21})^m = 0.$$

This completes the proof.

Similarly, we can prove the following theorem:

Theorem 3.1. *Let R be a prime ring with characterstic different from 2, U its Utumi quotient ring, C its extended centroid, λ a nonzero ideal of R and G a nonzero generalized derivation with associated derivation d of R, $s \geq 0, t \geq 1, m \geq 1$ fixed integers and $0 \neq b \in R$. Assume that $b((x \circ y)^t [G(x \circ y), (x \circ y)](x \circ y)^s)^m = 0$ for all $x, y \in \lambda$. Then one of the following holds:*

(i) R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$ in four variables and $G(x) = qx + xq + \alpha x$ for some $q \in U$ and $\alpha \in C$;

(ii) $G(x) = \alpha x$ for all $x \in R$ with $\alpha \in C$.

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