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Symmetric products of spheres and homology of their covering spaces

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Abstract

In this paper, first we compute covering space of $SP^n(S^n)$ and then we compute homology of the covering spaces $\widetilde{SP^n(S^n)}$ and $\widetilde{SP^\infty(S^n)}$ of $SP^n(S^n)$ and $SP^\infty(S^n)$ respectively.
Also, if m and n are distinct non-negative integers, then using Homology of covering space of infinite symmetric products of spheres, we deduce the following results:
i) $H_p(\widetilde{SP^\infty(S^m)}) \ncong H_p(\widetilde{SP^\infty(S^n)})$;
ii) $H_p(\widetilde{SP^\infty(S^m)}) \nsim H_p(\widetilde{SP^\infty(S^n)})$.

1 Introduction and preliminaries

The unit n-sphere of radius 1 is defined as : $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$. The n-sphere is a Riemannian manifold of positive curvature and is orientable. The n-sphere admits, for every point $x_0 \in S^n$, a CW-structure with one 0-cell x_0 and one n-cell $S^n - x_0$. Hence S^n is an n-dimensional CW-complex(cell complex).
Now, we recall the following definitions and statements:-

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Definition 1.1. ***(Symmetric product)*** *Let X be a topological space with base point $x_0 \in X$. For $n \geq 0$, we define the n fold symmetric product of X , denoted by $SP^n(X)$ by $SP^0(X) = x_0, SP^n(X) = X^n/S_n$ for $n \geq 1$,where X^n denotes the n fold cartesian product of X with itself and S_n denotes the symmetric group on n objects regarding as acting on X^n by permuting the coordinates.*
Hence for $n \geq 1$, $SP^n(X) = \{(x_1, \ldots, x_n) : x_i \in X)\}$.
There is an embedding $SP^n(X) \hookrightarrow SP^{n+1}(X)$ given by
$j(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, x_0)$. Thus $SP^n(X)$ can naturally considered as a subset of $SP^{n+1}(X)$ and is given by a sequence
$SP^1(X) \subset SP^2(X) \subset \cdots \subset SP^n(X) \subset SP^{n+1}(X) \subset \cdots$.
We define the infinite symmetric product of X is the colimit $SP^\infty(X) \simeq Colim$ $SP^n(X)$ according to the above sequence.
Rana [6] showed that SP^n and SP^∞ are covariant functor from the category of pointed topological spaces and base point preserving continuous maps to the category of pointed topological spaces and base point preserving continuous maps.

Definition 1.2. ***(CW-complex)*** *A pair (X,ε) consisting of a Hausdorff space X and a cell decomposition ε of X is called a CW-complex if the following axioms are satisfied:*
(i) ***(Characteristic map)****: For each n-cell $e^n_\alpha \in \varepsilon$ there is a map $\phi_\alpha : (D^n, S^{n-1}) \to (X, X^{n-1})$ restricting to a homeomorphism $\phi_{\alpha|D^n - S^{n-1}} : D^n - S^{n-1} \to e^n_\alpha$ and taking S^{n-1} into X^{n-1}.*
(ii) ***(Closure finite)****: For any cell $e_\alpha \in \varepsilon$ the closure $\overline{e_\alpha}$ intersects only a finite number of other cells in ε.*
(iii) ***(Weak topology)****: A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e_\alpha}$ is closed in X for each $e_\alpha \in \varepsilon$.*

Definition 1.3. ***(Eilenberg-Maclane space)*** *A pointed CW-complex is called an Eilenberg MacLane space if it has only one nontrivial homotopy group. If G is a group and n is a positive integer, the Eilenberg-MacLane space of type (G,n) is a pointed CW-complex X whose homotopy groups vanish in all dimensions except n, where G = $\pi_n(X)$ and G is to be abelian for n > 1,*
we can write the notation K(G, n) for a CW-complex which represents an Eilenberg-MacLane space of type (G, n).
The unit circle S^1 with $G = \mathbb{Z} : K(\mathbb{Z}, 1) \simeq S^1$.

The infinite dimensional real projective space $\mathbb{R}P^{\infty}$ *with* $G = \mathbb{Z}_2 : K(\mathbb{Z}_2, 1) \simeq \mathbb{R}P^{\infty}$.
The infinite dimensional complex projective space $\mathbb{C}P^{\infty}$ *with* $G = \mathbb{Z} : K(\mathbb{Z}, 2) \simeq \mathbb{C}P^{\infty}$.

Definition 1.4. ***(Homotopy)*** *Let X be a space and* x_0 *a base point of X . For a given positive integer n, consider the set* $F_n(X, x_0)$ *of all continuous maps* α *from the unit n-cube* I^n *into X for which* $\alpha(\partial I^n) = x_0$.
Define an equivalence relation $\sim x_0$ *on* $F_n(X, x_0)$ *as follows:*
For α *and* β *in* $F_n(X, x_0)$, α *is equivalent module* x_0 *to* β *written as* $\alpha \sim x_0\beta$*, if there is a homotopy* $H : I^n \times I \to X$ *such that*
$H(t_1, t_2, \ldots, t_n, 0) = \alpha(t_1, t_2, \ldots, t_n)$
$H(t_1, t_2, \ldots, t_n, 1) = \beta(t_1, t_2, \ldots, t_n), (t_1, t_2, \ldots, t_n) \in I^n$ *and* $H(t_1, t_2, \ldots, t_n, s) = x_0, (t_1, t_2, \ldots, t_n) \in (\partial I^n), s \in I$,
Under this equivalence relation on $F_n(X, x_0)$*, the equivalence class determined by* α *is denoted by* $[\alpha]$ *and is called the homotopy class of* α *module* x_0 *or simply the homotopy class of* α.
Define $(\alpha \circ \beta)(t) = \begin{cases} \alpha(2t_1, t_2, \ldots, t_n) & \textit{if} \quad 0 \leq t_1 \leq \frac{1}{2} \\ \beta(2t_1 - 1, t_2, \ldots, t_n) & \textit{if} \quad \frac{1}{2} \leq t_1 \leq 1 \end{cases}$
with this operation, the set of equivalence classes of $F_n(X, x_0)$ *is a group called the n-th homotopy group of X at* x_0 *denoted by* $\pi_n(X, x_0)$.

Definition 1.5. ***(Homology)*** *A sequence* $C_* = \{C_n, \partial_n\}, n \in \mathbb{Z}$ *of additive abelian groups* C_n *together with a sequence of group homomorphisms* $\partial_n : C_n \to C_{n-1}$ *such that* $\partial_n \circ \partial_{n+1} = 0$ *is called a chain complex and* ∂_n *is called a boundary homomorphism. More precisely*
$C_* \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} 0$ *is called a chain complex if* $\partial_n \circ \partial_{n+1} = 0, \forall n \in \mathbb{Z}$.
The elements of $Z_n = ker(\partial_n)$ *are called n-cycles and the elements of* $B_n = Im(\partial_{n+1})$ *are called n -boundaries of the chain complex* C_*.
As $\partial_n \circ \partial_{n+1} = 0,\ Im(\partial_{n+1}) \subset ker(\partial_n)$*, so* B_n *is a normal subgroup of* $Z_n, \forall n$.
Define $H_n = \frac{Z_n}{B_n} = \frac{ker(\partial_n)}{Im(\partial_{n+1})}$ *, is called n-th homology group.*

Definition 1.6. ***(Cohomology)*** *A sequence* $C^* = \{C^n, \partial^n\}, n \in Z$ *of additive abelian groups* C^n *together with a sequence of group homomorphisms* $\partial^n : C^{n-1} \to C^n$ *such that* $\partial^{n+1} \circ \partial^n = 0$ *is called a cochain complex and* ∂^n *is called a*

coboundary homomorphism. More precisely,
$C^* \cdots \to C^{n-1} \xrightarrow{\partial^n} C^n \xrightarrow{\partial^{n+1}} C^{n+1} \to \cdots$ *is called a cochain complex if* $\partial^{n+1} \circ \partial^n = 0$, $\forall n \in \mathbb{Z}$.
The elements of $Z^n = ker(\partial^{n+1})$ *are called n-cocycles and the elements of* $B^n = Im(\partial^n)$ *are called n -coboundaries of the cochain complex* c^*.
As $\partial^{n+1} \circ \partial^n = 0$ *,* $Im(\partial^n) \subset ker(\partial^{n+1})$*, so* B^n *is a normal subgroup of* $Z^n, \forall n \in \mathbb{Z}$.
Define $H^n = \frac{Z^n}{B^n} = \frac{ker(\partial^{n+1})}{Im(\partial^n)}$ *, is called n-th cohomology group.*

Definition 1.7. ***(Covering space)*** *A covering space of a space* (X, x) *is a triple* $(\tilde{X}, \tilde{x}, p)$ *consisting of a pointed space* (X, x) *and a continuous surjective map* $p : (\tilde{X}, \tilde{x}) \to (X, x)$ *such that each point* $x \in X$ *has a path connected open neighborhood* U *such that each path component of* $p^{-1}(U)$ *is mapped homeomorphically onto* U *by* p*, that is, each point* $x \in X$ *has a path connected open neighborhood* U *such that* $p^{-1}(U)$ *is a disjoint union of open sets , each of which is mapped homeomorphically onto* U *by* p.

Definition 1.8. ***(Fiber bundle)*** *Let* Y*,* X *and* F *be topological spaces, called total space, base space and Fiber respectively. A fiber bundle is a structure* (Y, X, q, F) *with continuous surjection* $q : Y \to X$ *satisfying the following conditions:*
(i) For any $x \in X$ *the pre-image* $q^{-1}(x)$ *is homeomorphic to* F *and is called the fiber over* x.
(ii) For any $x \in X$ *there is an open neighbourhood* $U \subseteq X$ *of* x *such that there is a homeomorphism* $\phi : q^{-1}(U) \to U \times F$ *with subspace topology and the following diagram commutes:*

$$\begin{array}{ccc} q^{-1}(U) & \xrightarrow{\phi} & U \times F \\ {\scriptstyle q}\downarrow & \swarrow {\scriptstyle p_1} & \\ U & & \end{array}$$

where p_1 *is the natural projection onto the first coordinate. The set of all* $\{U_i, \phi_i\}$ *is called a local trivialization of the bundle.*
Note that when the fiber is a vector space, the bundle is called a vector bundle.

Definition 1.9. ***(Pullback bundle)*** *Let $q : Y \to X$ be a fiber bundle with fiber F and let $f : X' \to X$ be a continuous map. Define the pullback bundle by $f^*Y = \{(x', y) \in X' \times Y | f(x') = q(y)\} \subseteq X' \times Y$ and the projection map $q' : f^*Y \to X'$, given by the projection onto the first coordinate and $g : f^*Y \to Y$, given by the projection onto the second coordinate, such that the following diagram commutes:*

$$\begin{array}{ccc} f^*Y & \xrightarrow{g} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{f} & X \end{array}$$

If (U, ϕ) is a local trivialization of Y, then $(f^{-1}(U), \psi)$ is a local trivialization of f^*Y, where $\psi(x', y) = (x', p_2(\phi(y)))$. It then follows that f^*Y is a fiber bundle over X' with fiber F and the bundle is called the pullback bundle of Y by f.

Definition 1.10. *A sequence of abelian groups and homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is called exact at B if $ker(g) = Im(f)$.*

2 Some useful results

Lemma 2.1. ([5]) *Let $h : C \to D$ be a morphism of chain complexes such that $p_i : C_i \to D_i$ is an isomorphism for $i \leq n$.*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C_{i+1} & \longrightarrow & C_i & \xrightarrow{\sigma_i} & C_{i-1} & \longrightarrow & \cdots & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_{i-1}} & & & & \\ \cdots & \longrightarrow & D_{i+1} & \longrightarrow & D_i & \xrightarrow{\delta_i} & D_{i-1} & \longrightarrow & \cdots & \longrightarrow & 0 \end{array}$$

Then, $H_i(C) \cong H_i(D)$ for $i \leq n - 1$.

Theorem 2.1. ***(Dold-Thom)*** ([5]) *Let X be a connected cell complex. Then, there is a homotopy equivalence $SP^\infty(X) \simeq \prod_{n=1}^{\infty} K(H_n(X, \mathbb{Z}), n)$.*

Theorem 2.2. *Let Σ_m be a compact Riemann surface of genus m. Then there is a homotopy equivalence such that $SP^{\infty}(\Sigma_m) \cong CP^{\infty} \times (S^1)^{2m}$.*

Proof. We know that $H_n(\Sigma_m;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for} \quad n = 0, 2 \\ \mathbb{Z}^{2m} & \text{for} \quad n = 1 \\ 0 & \text{otherwise} \end{cases}$.

Now, as Σ_m is a connected cell complex , the Dold-Thom theorem (Theorem 2.1) implies:

$SP^{\infty}(\Sigma_m) \simeq \prod_{n=1}^{\infty} K(H_n(\Sigma_m, \mathbb{Z}), n) = K(H_1(\Sigma_m, \mathbb{Z}, 1) \times K(H_2(\Sigma_m, \mathbb{Z}), 2)$

$= K(\mathbb{Z}^{2m}, 1) \times K(\mathbb{Z}, 2) = (S^1)^{2m} \times \mathbb{C}P^{\infty}$.

Hence, $SP^{\infty}(\Sigma_m) \cong CP^{\infty} \times (S^1)^{2m}$. □

As $SP^{\infty}(\Sigma_m)$ is the colimit of $SP^n(\Sigma_m)$, $SP^{\infty}(\Sigma_m)$ has a cell complex structure for which the $SP^n(\Sigma_m)$ are subcomplexex such that the natural inclusion $i : SP^n(\Sigma_m) \hookrightarrow SP^{\infty}(\Sigma_m)$ is an isomorphism upto the n-skeletons: $(SP^n(\Sigma_m))_n \cong (SP^{\infty}(\Sigma_m)_n$.

Theorem 2.3. *Let Σ_n be a compact Riemann surface of genus m with an nth symmetric product space $SP^n(\Sigma_m)$ and infinite symmetric product space $SP^{\infty}(\Sigma_m)$. Then for a filed $\mathbb{F}$, $H_k(SP^n(\Sigma_m), \mathbb{F}) \cong H_k(SP^{\infty}(\Sigma_m), \mathbb{F})$ for $k = 0, 1, 2, \ldots, n-1$.*

Proof. The proof follows directly by Lemma 2.1. □

Theorem 2.4. *If X is a cell complex with the n-skeleton X_n and $\tilde{X}$ is a covering space with the covering map p, then $\tilde{X}$ is a cell complex with n-skeleton $p^{-1}(X_n) = \tilde{X}_n$.*

Proof. The proof can be found in Hatcher [1]. □

Corollary 2.1. *Let $p : \tilde{X} \to X$ be a covering map and $f : Y \to X$ be a continuous map. Then, Pullback f_p^* of the covering map p along f is a covering map.*

$$\begin{array}{ccc} f^*(\tilde{X}) & \xrightarrow{\tilde{f}} & \tilde{X} \\ \Big\downarrow f_p^* & & \Big\downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

Proof. Let $f(y) = x$ in X for any $y \in Y$. Since p is a covering map, there exists an open neighbourhood $U_x \subset X$ such that $p^{-1}(U_x) = \bigcup\limits_{i \in I} V_i$, where each V_i is open in $\tilde{X}$ for $i \in I$ and maps homeomorphically onto U_x by p. Now, since f is continuous , $f^{-1}(U_x)$ is an open set. Let $U_y = f^{-1}(U_x)$ be the open neighbourhood of y.
Claim: U_y is evenly covered by f_p^*.
That is $(f_p^*)^{-1}(U_y) = (f_p^*)^{-1}(f^{-1}(U_x)) = \tilde{f}^{-1}(p^{-1}(U_x)) = \tilde{f}^{-1}(\bigcup\limits_{i \in I} V_i) = \bigcup\limits_{i \in I} \tilde{f}^{-1}(V_i)$. So we need to check that each $\tilde{f}^{-1}(V_i)$ is mapped homeomorphically onto U_y by f_p^*. By Corollary 2.2 we have $\tilde{f}$ is a homeomorphism and hence we have the result. □

Corollary 2.2. *Let $p : \tilde{X} \to X$ be a covering map and $f : Y \to X$ be a homeomorphism. If the pullback of p along f is $\tilde{Y}$, and the covering map $f_p^* : \tilde{Y} \to Y$, then the function $\tilde{f} : \tilde{Y} \to \tilde{X}$ is a homeomorphism.*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \Big\downarrow {\scriptstyle f_p^*} & & \Big\downarrow {\scriptstyle p} \\ Y & \xrightarrow[\cong]{f} & X \end{array}$$

Theorem 2.5. *The long sequence of homomorphisms*

$$\cdots \longrightarrow H_{n+1}(X,A) \xrightarrow{\sigma} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\sigma} H_{n-1}(A) \longrightarrow \cdots \longrightarrow 0$$

is exact and is called the long exact homology sequence associated to the pair (X, A).

Proof. Proof can be found in Hatcher [1]. □

Lemma 2.2. ***(The Five-Lemma)*** *In a commutative diagram of abelian groups,*

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \rho \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

if the two rows are exact and $\alpha, \beta, \delta, \rho$ are isomorphism, then γ is also an isomorphism.

Proof. Proof can be found in Hatcher [1]. □

Theorem 2.6. ***(Whitehead Theorem)***([5]) *Let $f : X \to Y$ be a continuous map between connected cell complexes. Then f is a homotopy equivalence if and only if $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all $k \geq 1$, where $\pi_k(X)$ and $\pi_k(Y)$ are k-th homotopy groups of X and Y respectively for $k \geq 1$.*

Theorem 2.7. ([5]) *Given a fiber bundle (Y, X, q, F) and choosing a base point $y_0 \in Y$; then, there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_2(F, y_0) \longrightarrow \pi_2(Y, y_0) \to \pi_2(X, q(y_0)) \to \pi_1(F, y_0) \longrightarrow \pi_1(Y, y_0) \to \pi_1(X, q(y_0)).$$

3 Homology of $SP^n(S^n)$ and $SP^\infty(S^n)$

Definition 3.1. ***(Betti numbers)*** *Let X be a topological space and $H_n(X)$ be the nth homology group of X. Then, for a non-negative integer p, the pth Betti number $b_p(X)$ of X is the dimension of $H_p(X)$, i.e, $H_p(X, \mathbb{F}) = \mathbb{F}^{b_p}$ for a field $\mathbb{F}$.*

Example 3.1. *For the n-dimensional sphere S^n, we have,*

$$H_p(S^n, \mathbb{F}) = \begin{cases} \mathbb{F} & \textit{for} \quad p = 0, n \\ 0 & \textit{otherwise} \end{cases}$$

Therefore, $b_0(S^n) = b_n(S^n) = 1$ and all other Betti numbers are 0.

Definition 3.2. ***(Poincaré polynomial)*** *For a fixed coefficient field $\mathbb{F}$ the Poincaré polynomial $P_X(t)$ of a topological space X is the generating power series of its Betti numbers, i.e, $P_X(t) = \sum_i b_i t_i$ where b_i is the dimension of $H_i(X, \mathbb{F})$ as a vector space of $\mathbb{F}$, i.e, the ith Betti number of X.*

Example 3.2. *For the n-dimensional sphere S^n, The Betti numbers are $b_0(S^n) = b_n(S^n) = 1$ and all other Betti numbers are 0 (by Example 3.1). Therefore, $P_{S^n}(t) = 1 + t^n$.*

Theorem 3.1. ([5]) *Let X and Y be two topological spaces. Then, the Poincaré polynomial of the tensor product $X \times Y$ can be written as $P_{X\times Y}(t) = P_X(t)P_Y(t)$.*

Theorem 3.2. ([10]) *Let X be a finite cell complex. If for any field $\mathbb{F}$ and all $k \geq 0$ the dimension of $H_k(X, \mathbb{F})$ is independent of $\mathbb{F}$, then $H_k(X, \mathbb{F})$ is a free abelian group of the same rank as the Betti number.*

By Theorem 2.1 and Theorem 3.1, we have the Poincaré polynomial for $SP^\infty(S^n)$ as $P_{SP^\infty(S^n)}(t) = P_{\mathbb{C}P^\infty \times (S^1)^{2n}}(t) = P_{\mathbb{C}P^\infty}(t)P_{(S^1)^{2n}}(t) = (1+t)^{2n}(\frac{1}{1-t^2})$
$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\begin{pmatrix} 2n \\ i \end{pmatrix} t^{i+2j}$.
Note that the pth Betti number b_p of a space is the coefficient of t^p of its own Poincaré polynomial.
Hence, $b_p(SP^\infty(S^n))$

$$= \begin{cases} \sum_{i=0}^{p/2}\begin{pmatrix} 2n \\ 2i \end{pmatrix} & \text{for} \quad p = 0, even \\ \sum_{i=0}^{(p-1)/2}\begin{pmatrix} 2n \\ 2i+1 \end{pmatrix} & \text{for} \quad p = odd \end{cases}$$

Now the homology of a space X, $H_p(X, \mathbb{F}) = \mathbb{F}^{b_p}$ for a field $\mathbb{F}$.
Hence, by Theorem 3.2 we have, $H_p(SP^\infty(S^n), \mathbb{Z})$

$$= \begin{cases} \mathbb{Z}^{\sum_{i=0}^{p/2}\begin{pmatrix} 2n \\ 2i \end{pmatrix}} & \text{for} \quad p = 0, even \\ \mathbb{Z}^{\sum_{i=0}^{(p-1)/2}\begin{pmatrix} 2n \\ 2i+1 \end{pmatrix}} & \text{for} \quad p = odd \end{cases}.$$

Now by the above result and by Theorem 2.3 we have,

$$H_p(SP^n(S^n), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\sum_{i=0}^{p/2}\begin{pmatrix} 2n \\ 2i \end{pmatrix}} & \text{for} \quad p = 0, even \\ \mathbb{Z}^{\sum_{i=0}^{(p-1)/2}\begin{pmatrix} 2n \\ 2i+1 \end{pmatrix}} & \text{for} \quad p = odd \end{cases}$$

for $p = 0, 1, 2, \ldots, n-1$.

4 Covering space of $SP^n(S^n)$

First we will define the Abel-Jacobi map.
Let Σ_m be a compact Riemann surface of genus m. Let $\gamma_1, \gamma_2, \ldots, \gamma_{2m}$ are smooth closed loops representing a basis $[\gamma_1], [\gamma_2], \ldots, [\gamma_{2m}]$ for $H_1(\Sigma_m; \mathbb{Z}) \cong \mathbb{Z}^{2m}$. Let $H^0(\Sigma_m; \Omega^{1,0})$ be the vector space of holomorphic 1-forms on Σ_m. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a basis for $H^0(\Sigma_m; \Omega^{1,0}) \cong \mathbb{C}^m$.
Let L be a $2m$ dimensional lattice defined by $L = \{\sum_{i=1}^{2m} n_i w_i | n_i \in \mathbb{Z}\} \leq \mathbb{C}^m$, generated by the basis of $w_1, w_2, \ldots, w_{2m} \in L$ such that $w_i = (\int_{\gamma_i} \alpha_1, \int_{\gamma_i} \alpha_2, \ldots, \int_{\gamma_i} \alpha_m)$.
The Jacobian of the Σ_m, denoted by $J(\Sigma_m)$ is the compact quotient space, $J(\Sigma_m) = \mathbb{C}^m/L \cong \mathbb{R}^{2m}/L$.
Note that $\mathbb{C}^m/L$ is a $2m$ dimensional torus which is homeomorphic to $(S^1)^{2m}$ as a topological space as L is a discrete subgroup of $\mathbb{C}^m$ of maximal rank.
Now, fix a point $x_0 \in \Sigma_m$. The Abel-Jacobi map is a map $AJ : \Sigma_m \to J(\Sigma_m)$. For every point $x \in \Sigma_m$, choose a curve γ from x_0 to x and define the map AJ as $AJ(x) = (\int_{x_0}^{x} \alpha_1, \int_{x_0}^{x} \alpha_2, \ldots, \int_{x_0}^{x} \alpha_m) + L$.
As $J(\Sigma_m)$ is an abelian group, the Abel-Jacobi map AJ can be extended to a symmetric product, $AJ_n : SP^n(\Sigma_m) \to J(\Sigma_m)$ defined by $AJ_n(p) = AJ_1(y_1) + AJ_2(y_2) + \ldots + AJ_n(y_n)$, where $p = (y_1, y_2, \ldots, y_n) \in SP^n(\Sigma_m)$.
Now, we will construct a homomorphism $J(S^n) \to J(S^n)$ which is also a covering map.The Jacobian is the compact quotient space $J(S^n) = \mathbb{R}^{2n}/L$. Let $\{u_1, u_2, \ldots, u_{2n}\} \in \mathbb{R}^{2n}$ be a basis for L and $C = [u_1, u_2, \ldots, u_{2n}]$ be the column matrix for the basis. Let A be a $2n \times 2n$ matrix with integer entries such that $det(A) \neq 0$. Then $B := CAC^{-1}$ is a surjective linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that sends L to L. This is determined to be a surjective homomorphism $p : J(S^n) \to J(S^n)$ by $p([x]) = [Bx]$ for $x \in \mathbb{R}^{2n}$.
Then p determines a covering map $J(S^n) \to J(S^n)$, where $|det(A)|$ represents the number of sheets. Now we shall consider the pullback diagram of p along the Abel-Jacobi map:

$$
\begin{array}{ccc}
\widetilde{SP^n(S^n)} & \longrightarrow & J(S^n) \\
\Big\downarrow f_p^* & & \Big\downarrow p \\
SP^n(S^n) & \xrightarrow{AJ} & J(S^n)
\end{array}
$$

By Corollary 2.1, the pullback f_p^* of the covering map p is also a covering map with $|det(A)|$ number of sheets. Hence, $\widetilde{SP^n(S^n)}$ is a covering space of $SP^n(S^n)$.

5 Homology of the covering spaces $\widetilde{SP^\infty(S^n)}$ and $\widetilde{SP^n(S^n)}$

In this section, we will find the homology groups of the covering spaces $\widetilde{SP^\infty(S^n)}$ and $\widetilde{SP^n(S^n)}$. Also we will show that a covering space of $SP^\infty(S^n)$ has the same homology as $SP^\infty(S^n)$.

Consider the following diagram of topological spaces and continuous maps which is commutative:

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xleftarrow{g'} & Z' \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \phi_y} & & \downarrow{\scriptstyle \phi_z} \\ X & \xrightarrow{f} & Y & \xleftarrow{g} & Z \end{array}$$

Then we have two respective pullback spaces $X' \times_{Y'} Z'$ and $X \times_Y Z$ to the diagrams $X' \to Y' \leftarrow Z'$ and $X \to Y \leftarrow Z$ such that
$X' \times_{Y'} Z' = \{(x', z') \in X' \times Z' | f'(x') = g'(z')\}$ and $X \times_Y Z = \{(x, z) \in X \times Z | f(x) = g(z)\}$.

Hence, we can define a function $\psi : X' \times_{Y'} Z' \to X \times_Y Z$ such that $\psi(x', z') = (\phi_x(x'), \phi_z(z'))$.

So, we can define the pullback space for the diagram

$$\mathbb{C}P^\infty \times J(S^n) \xrightarrow{p_2} J(S^n) \xleftarrow{\quad p \quad} J(S^n)$$

as a covering space $\widetilde{\mathbb{C}P^\infty \times J}(S^n)$ of $\mathbb{C}P^\infty \times J(S^n)$.

Similarly, we can define the pullback space for the diagram

$$SP^\infty(S^n) \xrightarrow{AJ_\infty} J(S^n) \xleftarrow{\quad p \quad} J(S^n)$$

as a covering space $\widetilde{SP^\infty(S^n)}$ of $SP^\infty(S^n)$.

As, the pullback of a trivial fiber bundle is a trivial fiber bundle with the same fiber, $\widetilde{\mathbb{C}P^\infty \times J}(S^n) :\to J(S^n)$ is a trivial bundle.

Moreover, since we have the pullback as the covering space, $\widetilde{\mathbb{C}P^\infty \times J}(S^n) \cong \mathbb{C}P^\infty \times J(S^n)$.

Lemma 5.1. *$\psi : \widetilde{SP^\infty(S^n)} \to \widetilde{\mathbb{C}P^\infty \times J}(S^n)$ is a homotopy equivalence.*

Proof. As we have pullback diagrams of a trivial fiber bundle and fiber bundle, we can say that $\mathbb{C}P^\infty \to \widetilde{\mathbb{C}P^\infty \times J}(S^n) \to J(S^n)$ is a trivial bundle and $\mathbb{C}P^\infty \to \widetilde{SP^\infty(S^n)} \to J(S^n)$ is a fiber bundle.
Now, we have a long exact sequence of homotopy groups (by Theorem 2.7), which make the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_{n+1}(J(S^n)) & \longrightarrow & \pi_n(\mathbb{C}P^\infty) & \to & \pi_n(\widetilde{\mathbb{C}P^\infty \times J}(S^n)) & \to & \pi_n(J(S^n)) & \longrightarrow & \pi_{n-1}(\mathbb{C}P^\infty) & \longrightarrow & \cdots \\
 & & \downarrow Id & & \downarrow f_* & & \downarrow \phi_* & & \downarrow Id & & \downarrow f_* & & \\
\cdots & \longrightarrow & \pi_{n+1}(J(S^n)) & \longrightarrow & \pi_n(AJ_\infty^{-1}(*)) & \longrightarrow & \pi_n(\widetilde{SP^\infty(S^n)}) & \longrightarrow & \pi_n(J(S^n)) & \longrightarrow & \pi_{n-1}(AJ_\infty^{-1}(*)) & \longrightarrow & \cdots
\end{array}$$

where $*$ is the identity element of $J(S^n)$.
Since we have all the homotopy groups are abelian, by the Five-Lemma (Lemma 2.2) the map ψ_* is an isomorphism. Hence by Whitehead Theorem (Theorem 2.6) ψ is a homotopy equivalence. □

Corollary 5.1. $H_p(\widetilde{SP^\infty(S^n)}, \mathbb{Z}) = H_p(SP^\infty(S^n), \mathbb{Z})$.

Proof. by Lemma 5.1, we have the homotopy equivalence, $\widetilde{SP^\infty(S^n)} \cong \widetilde{\mathbb{C}P^\infty \times J}(S^n) \cong \mathbb{C}P^\infty \times J(S^n) \cong SP^\infty(S^n)$. Hence, $H_p(\widetilde{SP^\infty(S^n)}, \mathbb{Z}) = H_p(SP^\infty(S^n), \mathbb{Z})$. □

Hence, we have $H_p(\widetilde{SP^\infty(S^n)}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\sum_{i=0}^{p/2} \binom{2n}{2i}} & \text{for} \quad p = 0, even \\ \mathbb{Z}^{\sum_{i=0}^{(p-1)/2} \binom{2n}{2i+1}} & \text{for} \quad p = odd \end{cases}$.

Now, let us consider the relationship between the covering spaces.
Let $p : \tilde{X} \to X$ be a covering map and let X be a cell complex with n-skeleton X_n. Then $\tilde{X}$ is a cell complex with $\tilde{X}_n = p^{-1}(X_n)$ representing the n-skeleton of $\tilde{X}$.

Proposition 5.1. $H_p(\widetilde{SP^n(S^n)}, \mathbb{Z}) \cong H_p(\widetilde{SP^\infty(S^n)}, \mathbb{Z})$ *for* $p = 0, 1, 2, \ldots, n-1$.

Proof. Consider the pullback diagram:

$$\begin{array}{ccc}
\widetilde{SP^n(S^n)}_n & \xrightarrow{\tilde{f}} & \widetilde{SP^\infty(S^n)}_n \\
\Big\downarrow f_p^* & & \Big\downarrow p \\
SP^n(S^n)_n & \xrightarrow{f} & SP^\infty(S^n)_n
\end{array}$$

As we have a homeomorphism of n-skeletons $SP^n(S^n)_n \cong SP^\infty(S^n)_n$ for upto the nth skeletons, so, we have $\widetilde{SP^n(S^n)}_n \cong \widetilde{SP^\infty(S^n)}_n$ (By Corollary 2.2) for upto the nth skeletons. Now Lemma 2.1 gives the proof. □

Thus, as a result we have, $H_p(\widetilde{SP^n(S^n)}, \mathbb{Z})$

$$=\begin{cases} \mathbb{Z}^{\sum_{i=0}^{p/2}\binom{2n}{2i}} & \text{for} \quad p = 0, even \\ \mathbb{Z}^{\sum_{i=0}^{(p-1)/2}\binom{2n}{2i+1}} & \text{for} \quad p = odd \end{cases}$$

for $p = 0, 1, 2, \ldots, n-1$.

Proposition 5.2. *$H_p(\widetilde{SP^\infty(S^n)}) \ncong H_p(\widetilde{SP^\infty(S^m)}$ for distinct non-negative integers n and m.*

Proof. As we know $H_p(SP^\infty(S^n)) \ncong H_p(SP^\infty(S^m)$ for distinct non-negative integers n and m, the proof follows by Corollary 5.1 □

Proposition 5.3. *$H_p(\widetilde{SP^\infty(S^n)}) \nsim H_p(\widetilde{SP^\infty(S^m)}$ for distinct non-negative integers n and m.*

Proof. The proof follows by Proposition 5.2. □

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