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# Some results on $g$-noncommuting graphs

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**Abstract**

Let $G$ be a finite group, $g$ be a fixed non-identity element of $G$, and $Z(G)$ be the center of $G$. We denote by $\Delta_G^g$, the $g$-noncommuting graph, where the vertex set is $G \setminus Z(G)$. Two distinct vertices, $x$ and $y$, are adjacent if $[x, y]$ is not equal to $g$ and $g^{-1}$. Recall that $[x, y]$ is the commutator of two elements $x$ and $y$ of $G$ and is equal to $x^{-1}y^{-1}xy$. In this paper, we prove that the $g$-noncommuting graphs associated with the generalized quaternion and dihedral groups are isomorphic and survey the locating chromatic number of $g$-noncommuting graph of this family of groups.

## 1 Introduction

Let $G$ be a finite group and $g$ be a non-identity element of $G$. Then, the $g$-noncommuting graph of $G$, $\Delta_G^g$, is defined with vertex set $G \setminus Z(G)$ such that two distinct vertices $x$ and $y$ join by an edge whenever $[x, y] \neq g$ and $[x, y] \neq g^{-1}$,

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where $[x, y] = x^{-1}y^{-1}xy$ is the commutator of $x$ and $y$ [10]. Recall that for any group $G$, $K(G) = \{[x, y] : x, y \in G\}$ is the set of all commutators of $G$ and $G' = \langle K(G) \rangle$, where $G'$ is the commutator subgroup of $G$. Therefore, if $g \notin K(G)$, then the $g$-noncommuting graph $\Delta_G^g$ is a complete graph. Hence, we assume that $g$ is a non-identity element in $K(G)$. If two finite groups $G$ and $H$ are isomorphic, then $\Delta_G^g \cong \Delta_H^h$ for some $g \in G$ and $h \in H$, but the converse is not generally true. In [7], authors proved that two groups with isomorphic $g$-noncommuting graphs have the same order. Also, the $g$-noncommuting graphs associated with symmetric, alternating, and dihedral groups have been studied in [6]. Now, we investigate the $g$-noncommuting graph associated with quaternion groups and prove that the $g$-noncommuting graphs for quaternion and dihedral groups are generally isomorphic.

One of the essential concepts in graph theory is graph coloring; there are many kinds of graph coloring, such as vertex coloring, edge coloring, locating coloring, etc. We remind locating coloring as follows:

Let $c$ be a proper $k$-coloring of a graph and $\pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of the vertex set into the resulting color classes. The *color code of vertex* $v$ with respect to $\pi$, denoted by $c_\pi(v)$, is defined as the ordered $k$-tuple $(d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$ such that $d(v, V_i)$ is the minimum distance from $v$ to each other vertex in $V_i$ for $1 \leqslant i \leqslant k$. If distinct vertices of the graph have distinct color codes, then $c$ is called a *locating coloring*. The locating chromatic number, denoted by $\chi_L$, is the minimum number of colors needed for locating graph coloring. Obviously, $\chi(\Gamma) \leq \chi_L(\Gamma)$, for any graph $\Gamma$. See [1, 2, 4, 5] for more results in the subject and related subjects.

In [6], the proper coloring of $g$-noncommuting graph of dihedral groups was investigated. Hence, in this article, we will survey the locating coloring of $g$-noncommuting graph on dihedral groups.

Here, our notations and terminologies are standard, and one can refer to graph terminologies to [3].

## 2 The results

We know that the generalized quaternion group of order $4n$ is denoted by $Q_{4n}$ and has the following presentation

$$Q_{4n} = \langle x, y \ : \ x^{2n} = e, \ x^n = y^2, \ x^y = x^{-1} \rangle.$$

Indeed, $Z(Q_{4n}) = \{e, x^n\}$ and $(Q_{4n})' = \langle x^2 \rangle$. Also, the following properties occur.

$$\begin{array}{ll} [x^k, x^l] = e & k, l = 1, 2, \ldots, 2n, \\ [x^k y, x^l] = x^{2l} & k, l = 1, 2, \ldots, 2n, \\ [x^k y, x^l y] = x^{2(k-l)} & k, l = 1, 2, \ldots, 2n. \end{array} \tag{2.1}$$

Therefore, if $g$ is a non-identity element and $g = x^{2i}$ for some $1 \leq i \leq n-1$, then by definition of $g$-noncommuting graph and equalities (2.1), we have the following properties in $g$-noncommuting graph associated to $Q_{4n}$.

1) All vertices $x^k$ and $x^l$, $1 \leq k, l \leq 2n$, are adjacent together. Hence, the subgraph induced by these vertices is complete.

2) The vertices $x^k y$ and $x^l$ are adjacent if and only if $l \not\equiv i \pmod{n}$ and $l \not\equiv n - i \pmod{n}$.

3) The vertices $x^k y$ and $x^l y$ are adjacent if and only if $k - l \not\equiv i \pmod{n}$ and $k - l \not\equiv n - i \pmod{n}$.

The $g$-noncommuting graph of $Q_{4n}$ for specific element $g = x^2$, $n = 2$ and $n = 3$ are as follows:

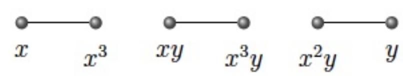


Figure 1: $\Delta_{Q_8}^{x^2}$

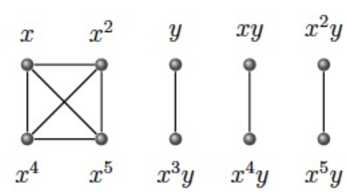


Figure 2: $\Delta_{Q_{12}}^{x^2}$

The following discusses the connectivity of $\Delta_{Q_{4n}}^{g}$.

**Theorem 2.1.** *$\Delta^g_{Q_{4n}}$ is a connected graph if and only if $n \geq 4$. Moreover, diam $(\Delta^g_{Q_{4n}}) = 2$.*

*Proof.* As we know, the induced subgraph on $\langle x \rangle \setminus Z(Q_{4n})$ is complete, and the vertices $x^k y$ and $x^l$ are adjacent if and only if $l \not\equiv i \pmod n$ and $l \not\equiv n - i \pmod n$. Since $n \geq 4$, a vertex $x^l$ exists such that $x^l$ joins to all vertices like $x^k y$. Therefore, the $g$-noncommuting graph is connected. Also, the diameter is two because at least two vertices are not adjacent. □

It is clear that the $g$-noncommuting graphs associated with isomorphic groups are isomorphic, but the converse is not true.

Now, consider the generalized quaternion group of order $2^n$ as follows:

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = e,\ x^{2^{n-2}} = y^2,\ x^y = x^{-1} \rangle.$$

Hence, $Z(Q_{2^n}) = \{e,\ x^{2^{n-2}}\}$ and $(Q_{2^n})' = \langle x^2 \rangle$.

One can see that $Q_{2^n} \cong Q_{4(2^{n-2})}$. So, the $g$-noncommuting graphs associated with this family of groups are isomorphic.

Recall that the dihedral group $D_{2n}$ of order $2n$, $D_{2n} = \langle a, b \mid a^n = b^2 = e, a^b = a^{-1} \rangle$ has trivial center and $D'_{2n} = \langle a \rangle$ if $n$ is odd and $Z(D_{2n}) = \{e, a^{n/2}\}$ and $D'_{2n} = \langle a^2 \rangle$ when $n$ is even. Nasiri [6] investigated the $g$-noncommuting graph associated with dihedral groups. Moreover, Supu et al. [9] investigated the topological indices of relative $g$-noncommuting graph of dihedral groups, based on [8].

The dihedral group and generalized quaternion group are not isomorphic. Still, the following theorem proves that the $g$-noncommuting graphs for dihedral and generalized quaternion groups are isomorphic.

**Theorem 2.2.** *For any integer $n$, $\Delta^{x^{2i}}_{Q_{4n}} \simeq \Delta^{a^{2i}}_{D_{2(2n)}}$.*

*Proof.* Consider the following map.

$$\begin{aligned} f : V(\Delta^{x^{2i}}_{Q_{4n}}) &\longrightarrow V(\Delta^{a^{2i}}_{D_{2(2n)}}) \\ x^k &\longmapsto a^k \\ y &\longmapsto b \\ x^k y &\longmapsto a^k b \end{aligned}$$

One can check that $f$ is a one-to-one corresponding. If $x^k$ joins to $x^l$, then $[x^k, x^l] = e$. So by map $f$, $[a^k, a^l] = e$, hence $a^k$ joins to $a^l$ and conversely. Now, assume that the vertices $x^ky$ and $x^l$ are adjacent. Therefore, $[x^ky, x^l] \neq x^{2i}$ and $[x^ky, x^l] \neq x^{2n-2i}$ if and only if $l \neq i, n-i, n+i$ and $2n-i$. Hence, $[a^kb, a^l] = a^{2l} \neq a^{2i}$ and $a^{2n-2i}$ and it follows $a^kb$ and $a^l$ are adjacent. Similarly, $x^ky$ and $x^ly$ are adjacent if and only if $a^kb$ and $a^lb$ are adjacent. So, those two graphs are isomorphic. □

In [6], Nasiri et al. computed the chromatic number of dihedral groups. Now, we find the locating chromatic number of these families of groups.

**Theorem 2.3.** *Let $D_{2n}$ be dihedral group of order $2n$, where $n$ is an odd integer, $g = a^i$, $1 \leq i \leq n-1$, $i$ is an even integer and $m = \frac{3i}{2}$. If $n = \frac{3i}{2}$, then $\chi_L(\Delta^g_{D_{2n}}) = 2n-3$. Otherwise*

$$\chi_L(\Delta^g_{D_{2n}}) = n - 3 + [\frac{n}{m}]i + \beta$$

*when*

$$\beta = \begin{cases} n - [\frac{n}{m}]m & ; \quad 0 \leqslant n - [\frac{n}{m}]m < \frac{i}{2} \\ \frac{i}{2} & ; \quad \frac{i}{2} \leqslant n - [\frac{n}{m}]m < i \\ n - [\frac{n}{m}]m - \frac{i}{2} & ; \quad i \leqslant n - [\frac{n}{m}]m < m \end{cases} .$$

*Proof.* We know that two vertices $a^rb$ and $a^sb$ are adjacent if and only if $r - s$ is not equal to $\frac{i}{2}$ and $n - \frac{i}{2}$ where $0 \leq r, s \leq n-1$. Similarly $a^r$ is adjacent to $a^sb$ if and only if $r$ is not equal to $\frac{i}{2}$ and $n - \frac{i}{2}$ where $0 \leq s \leq n-1$ and $1 \leq r \leq n-1$. Suppose that all vertices are partitioned into the sets of cardinality $\frac{i}{2}$. We know that all $\frac{i}{2}$ vertices in the first set are adjacent together, so we need $\frac{i}{2}$ distinct colors for coloring them. We can use the previous colors to color the second set because the vertices in the second set are not adjacent to some of the vertices in the first set. In the third set, we need new colors since the vertices in the third set are not adjacent to some vertices in the second set but are adjacent to the first set. If we color the vertices in the fourth set the same as in the third set, then the color codes of vertices in the fourth and third sets are the same, and it is impossible by the definition of the locating chromatic number. Hence, we need $\frac{i}{2}$ new colors and can use them for coloring the vertices in the fifth set. This process continues. Therefore, for any set of cardinality $\frac{3i}{2}$ we need $i$ distinct colors. Finally we have $n = \frac{3i}{2}q + r$ where $0 \leq r < \frac{3i}{2}$ and for coloring the vertices to form $a^rb$ we need $qi + \beta$ distinct colors such that

$$\beta = \begin{cases} n - [\frac{n}{m}]m & ; \quad 0 \leqslant n - [\frac{n}{m}]m < \frac{i}{2} \\ \frac{i}{2} & ; \quad \frac{i}{2} \leqslant n - [\frac{n}{m}]m < i \\ n - [\frac{n}{m}]m - \frac{i}{2} & ; \quad i \leqslant n - [\frac{n}{m}]m < m \end{cases}.$$

Also, all vertices $a^r$ where $r$ is not equal to $\frac{i}{2}$ and $n - \frac{i}{2}$ are adjacent to all vertices $a^s b$, so we need $n-3$ distinct new colors and we can color $a^{\frac{i}{2}}$ and $a^{n-\frac{i}{2}}$ with the previous colors for coloring $a^j b$.
That way, this coloring is a locating coloring because the color codes for every vertex are different, and by coloring process, this is minimum coloring. Hence $\chi_L(\Delta^g_{D_{2n}}) = n - 3 + [\frac{n}{m}]i + \beta$ and the proof is completed. □

The following is an example for $D_{10}$ ($n = 5$) where $g = a^2$.

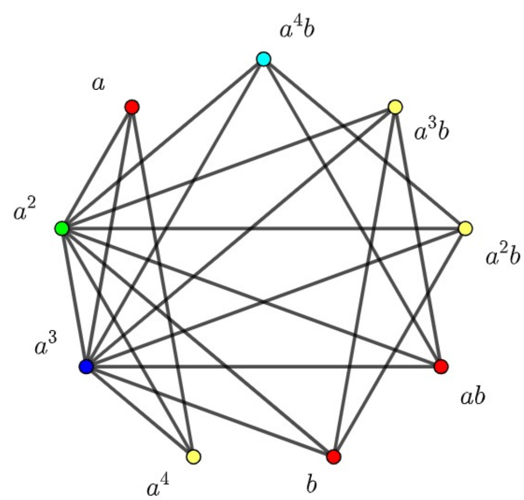


Figure 3: $\Delta^{a^2}_{D_10}$

If $n$ and $i$ are odd numbers and $g = a^i$, then we have a similar process, but it is enough to replace $\frac{i}{2}$ to $\frac{n-i}{2}$, and we divided vertices $a^r b$ into sets with $\frac{n-i}{2}$ members. So, we omitted the proof of the following theorem.

**Theorem 2.4.** *Let $D_{2n}$ be dihedral group of order $2n$, where $n$ is an odd integer, $g = a^i$, $1 \leq i \leq n-1$, $i$ is an odd integer and $m = \frac{3(n-i)}{2}$. If $n = \frac{3(n-i)}{2}$, then*

$\chi_L(\Delta^g_{D_{2n}}) = 2n-3$. *Otherwise*

$$\chi_L(\Delta^g_{D_{2n}}) = n-3+[\frac{n}{m}](n-i)+\beta,$$

*when*

$$\beta = \begin{cases} n-[\frac{n}{m}]m & ; \quad 0 \leqslant n-[\frac{n}{m}]m < \frac{n-i}{2} \\ \frac{n-i}{2} & ; \quad \frac{n-i}{2} \leqslant n-[\frac{n}{m}]m < (n-i) \\ n-[\frac{n}{m}]m-\frac{n-i}{2} & ; \quad (n-i) \leqslant n-[\frac{n}{m}]m < m \end{cases}.$$

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## References

[1] E. T. Baskoro and Asmiati, *Characterizing all trees with locating-chromatic number 3*, Electron. J. Graph Theorey Appl., **1(2)** (2013), 109–117.

[2] A. Behtoei and B. Omoomi, *On the locating chromatic number of the cartesian product of graphs*, Ars. Combin., **126**(2016), 221–235.

[3] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate texts in Mathematics, 244, Springer-Verlag, New York 2008.

[4] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater and P. Zhang, *The locating-chromatic number of a graph*, Bull. ICA., **36**(2002), 89–101.

[5] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater and P. Zhang, *Graphs of order $n$ with locating-chromatic number $n-1$*, Discrete Math., **269**(2003), 65–79.

[6] M. Nasiri, A. Erfanian, M. Ganjali, A. Jafarzadeh, *g-noncommuting graph of some finite groups*, J. Prime Res. Math., **12**(2016), 16–23.

[7] M. Nasiri, A. Erfanian, M. Ganjali, A. Jafarzadeh, *Isomorphic g-noncommuting graphs of finite groups*, Publ. Math. Debrecen., **91(1-2)** (2017), 33–42.

[8] M. Nasiri, A. Erfanian, Y. A. Farawi, I. Muchtadi-Alamsyah, *A kind of graph associated to a fixed element and a subgroup of group*, Southeast Asian Bull. of Maths., **44**(2020), 813-818.

[9] N. A. Supu, I. Muchtadi-Alamsyah, E. Suwastika, *Topological Indices of relative g-noncommuting graph of dihedral groups*, J. Indonesian Math. Soc., **3(29)** (2023), 271-288.

[10] B. Tolue, A. Erfanian and A. Jafarzadeh, *A kind of non-commuting graph of finite groups*, Journal of Sciences, Islamic republic of Iran, **25(4)** (2014), 379–384.