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On symmetric n-derivations in C^*-algebras

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Abstract

This paper addresses the outcomes that elucidate the characteristics of symmetric linear n-derivations within the realm of C^*-algebras. Basically, we show that in a primitive C^*-algebra $\mathscr{A}$, if $\mathfrak{D}, \mathcal{G} : \mathscr{A}^n \to \mathscr{A}$ are two nonzero symmetric linear n-derivations such that $f(a)a + ag(a) = 0$ holds $\forall\ a \in \mathscr{W}$, a nonzero left ideal of $\mathscr{A}$ where f and g are traces of $\mathfrak{D}$ and $\mathcal{G}$ respectively, then either $\mathscr{A}$ is commutative or $\mathcal{G}$ acts as a left n-multiplier. Ultimately, we provide a comprehensive characterization of symmetric n-derivations by means of left n-multipliers.

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1 Introduction

Derivations on Banach algebras have long been the focus of intensive research due to their intrinsic connections with both algebraic structures and functional analysis. The interplay between algebraic properties and topological structures in Banach algebras creates a fertile ground for investigating various linear and nonlinear operations that preserve algebraic and analytical properties simultaneously. Biderivations, which provide a parallel perspective on derivations unveiling complex patterns that have a big impact on areas like operator theory, differential equations, and functional analysis. Throughout the discussion, unless otherwise mentioned, $\mathscr{A}$ will denote C^*-algebra with $\mathscr{Z}(\mathscr{A})$ as its centre. However, $\mathscr{A}$ may or may not have unity. The symbols $[x, y]$ and $x \circ y$ denote the commutator $xy - yx$ and the anti-commutator $xy + yx$, respectively, for any $x, y \in \mathscr{A}$. An algebra $\mathscr{A}$ is said to be prime if $x\mathscr{A}y = \{0\}$ implies that either $x = 0$ or $y = 0$, and semiprime if $x\mathscr{A}x = \{0\}$ implies that $x = 0$, where $x, y \in \mathscr{A}$.

A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $||.||$ satisfying the multiplicative inequality; $||xy|| \leq ||x||||y||$ for all x and y in $\mathscr{A}$. An involution on an algebra $\mathscr{A}$ is a linear map $x \mapsto x^*$ of $\mathscr{A}$ into itself such that the following conditions are hold: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(x + \lambda y)^* = x^* + \bar{\lambda}y^*$ for all $x, y \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, the field of complex numbers, where $\bar{\lambda}$ is the conjugate of λ. An algebra equipped with an involution is called a $*$-algebra or algebra with involution. A Banach $*$-algebra is a Banach algebra $\mathscr{A}$ together with an isometric involution $||x^*|| = ||x||$ for all $x \in \mathscr{A}$. A C^*-algebra $\mathscr{A}$ is a Banach $*$-algebra with the additional norm condition $||x^*x|| = ||x||^2$ for all $x \in \mathscr{A}$. A C^*-algebra is prime if the intersection of any two nonzero (closed, two-sided) ideals is nonzero. A C^*-algebra $\mathscr{A}$ is primitive if its zero ideal is primitive, that is, if $\mathscr{A}$ has a faithful nonzero irreducible representation. A linear operator $\mathscr{D}$ on a C^*-algebra $\mathscr{A}$ is called a derivation if $\mathscr{D}(\vartheta\ell) = \mathscr{D}(\vartheta)\ell + \vartheta\mathscr{D}(\ell)$ holds $\forall\ \vartheta, \ell \in \mathscr{A}$. Consider the inner derivation δ_a implemented by an element a in $\mathscr{A}$, which is defined as $\delta_a(x) = xa - ax$ for every x in $\mathscr{A}$, as a typical example of a nonzero derivation in a noncommutative algebra.

In order to broaden the scope of derivation, Maksa [14] introduced the concept of symmetric bi-derivations. A bi-linear map $\mathfrak{D} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is said to be a bi-derivation if

$$\mathfrak{D}(\vartheta\vartheta', \ell) = \mathfrak{D}(\vartheta, \ell)\vartheta' + \vartheta\mathfrak{D}(\vartheta', \ell)$$

$$\mathfrak{D}(\vartheta, \ell\ell') = \mathfrak{D}(\vartheta, \ell)\ell' + \ell\mathfrak{D}(\vartheta, \ell')$$

holds for any $\vartheta, \vartheta', \ell, \ell' \in \mathcal{A}$. The foregoing conditions are identical if $\mathfrak{D}$ is also a symmetric map, that is, if $\mathfrak{D}(\vartheta, \ell) = \mathfrak{D}(\ell, \vartheta)$ for every $\vartheta, \ell \in \mathcal{A}$. In this case, $\mathfrak{D}$ is referred to as a symmetric bi-derivation of $\mathcal{A}$.

In this paper we briefly discuss the various extensions of the notion of derivations on C^*-algebras. The most general and important one among them is the notion of a symmetric n-derivations on C^*-algebras. The idea of symmetric n-derivations is given by Park [18].

Definition 1.1. *Let $n \geq 2$ be a fixed integer and $\mathcal{A}^n = \underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{n\text{-times}}$. A map $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is said to be symmetric (permuting) if*

$$\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \mathfrak{D}(\vartheta_{\pi(1)}, \vartheta_{\pi(2)}, \ldots, \vartheta_{\pi(n)})$$

for all permutations $\pi(t) \in S_n$ and $\vartheta_t \in \mathcal{A}$, where $t = 1, 2, \ldots, n$. A n-linear map $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is said to be a permuting(symmetric) linear n-derivation on $\mathcal{D}$ if $\mathcal{D}$ is permuting and $\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_i\vartheta_i', \ldots, \vartheta_n) = \mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_i, \ldots, \vartheta_n)\vartheta_i' + \vartheta_i\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_i', \ldots, \vartheta_n)$ hold for all $\vartheta_i, \vartheta_i' \in \mathcal{A},\ i = 1, 2, \ldots, n$.

A map $\mathfrak{d} : \mathcal{A} \to \mathcal{A}$ defined by $\mathfrak{d}(\vartheta) = \mathfrak{D}(\vartheta, \vartheta, \ldots, \vartheta)$ is called the trace of $\mathfrak{D}$. If $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is permuting and n-linear, then the trace $\mathfrak{d}$ of $\mathfrak{D}$ satisfies the relation

$$\mathfrak{d}(\vartheta + \ell) = \mathfrak{d}(\vartheta) + \mathfrak{d}(\ell) + \sum_{k=1}^{n-1} {}^nC_k\ h_k(\vartheta; \ell)$$

$\forall\ \vartheta, \ell \in \mathcal{A}$, where ${}^nC_k = \binom{n}{k}$ and

$$h_k(\vartheta; \ell) = \mathfrak{D}(\underbrace{\vartheta, \ldots, \vartheta}_{(n-k)\text{-times}}, \underbrace{\ell, \ldots, \ell}_{k\text{-times}}).$$

A 1-derivation is obviously a derivation, and a 2-derivation is a symmetric bi-derivation on C^*-algebras. The idea of a permuting n-multiplier on rings was initially suggested by Ashraf et al. in [3] where they proved some interesting results.

Definition 1.2. *A permuting n-linear map $\Lambda : \mathcal{A}^n \to \mathcal{A}$ is called a permuting left n-multiplier (resp. permuting right n-multiplier) if*

$$\Lambda(i_1, i_2, \ldots, i_t i_t',\ \ldots, i_n) = \Lambda(i_1, i_2, \ldots, i_t, \ldots, i_n)i_t'$$

$$(\textit{resp. } \Lambda(i_1, i_2, \ldots, i_t i'_t, \ldots, i_n) = i_t \Lambda(i_1, i_2, \ldots, i'_t, \ldots, i_n)$$

holds $\forall\, i_t, i'_t \in \mathcal{A},\ t = 1, 2, \ldots, n$*. If* Λ *is both a permuting left* n*-multiplier and a permuting right* n*-multiplier, it is referred to as a permuting* n*-multiplier.*

There has been considerable interest in the structure of linear derivation and linear bi-derivation on C^*-algebras and more generic Banach algebras. Derivations on C^*-algebras were described in various ways by different authors. For example, in 1966, Kadison [11] proved that each linear derivation of a C^*-algebra annihilates its centre. In 1989, Mathieu [16] extended the Posner's first result [19] on C^*-algebras. Basically, he proved that "if the product of two linear derivations d and d' on a C^*-algebra is a linear derivation then $dd' = 0$". The question "under which conditions all linear derivations are zero on a given $*$-algebra" have attracted much attention of authors (for instance, see [10] and [12]). Very recently, Ekrami and Mirzavaziri [7] showed that "if $\mathcal{A}$ is a C^*-algebra admitting two linear derivations d and d' on $\mathcal{A}$, then there exists a linear derivation D on $\mathcal{A}$ such that $dd' + d'd = D^2$ if and only if d and d' are linearly dependent".

In [2], Ali and Khan proved that if $\mathcal{A}$ is a C^*-algebra admitting a symmetric bilinear generalized $*$-biderivation $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with an associated symmetric bilinear $*$-biderivation $\mathcal{B} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, then $\mathcal{H}$ maps $\mathcal{A} \times \mathcal{A}$ into $Z(\mathcal{A})$. In [6], Dhara and Ali characterized the n-centralizing generalized derivations on C^*-algebras. Basically, they proved that "if n a fixed positive integer and $\mathcal{A}$ be a C^*-algebra and $\mathcal{F}$ a linear generalized derivation with an associated linear derivation $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{F}$ is n-centralizing on $\mathcal{A}$, then either $\mathcal{A}$ is commutative or $\mathcal{F}(x) = qx + \mathcal{D}(x)$ for all $x \in \mathcal{A}$, where $q \in \mathcal{C}$, the extended centroid of $\mathcal{A}$ and $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$. In particular, $\mathcal{D}$ is commuting on $\mathcal{A}$" (see also [1, 8, 13, 20] for recent results).

In the prospect of above motivation, we prove some results based on linear n-derivations of C^*-algebras. Our main intention is to investigate the structure of primitive C^*-algebra and the forms of maps (traces of n-derivations) satisfying the functional identity $f(\vartheta)\vartheta + \vartheta g(\vartheta) = 0\ \forall\, \vartheta \in \mathcal{W}$, a nonzero left ideal of $\mathcal{A}$, where f and g are traces of linear n-derivations $\mathfrak{D}$ and $\mathcal{G}$, respectively. Precisely, we prove that if $\mathcal{A}$ is a C^*-algebra admitting two symmetric n-derivations $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ and $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with traces f and g, respectively satisfying $f(\vartheta)\vartheta + \vartheta g(\vartheta) = 0$ $\forall\, \vartheta \in \mathcal{W}$, a left ideal of $\mathcal{A}$, then either $\mathcal{A}$ is commutative or $\mathcal{G}$ acts as a left n-multiplier on $\mathcal{W}$. Moreover, we also characterize the traces of q-iterations of linear n-derivations in primitive C^*-algebra and prove that for a fixed integer $n \geq 2$, if $\mathcal{A}$

is a primitive C^*-algebra and $q \geq 1$, a fixed integer admitting q-iterations of linear n-derivations $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q : \mathcal{A}^n \to \mathcal{A}$ such that the product of the traces of $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q$ respectively, is zero on a nonzero ideal of $\mathcal{A}$, then either $\mathfrak{D}_1 = 0$ or the rest of $\mathfrak{D}_i's$ act as n-multipliers on $\mathcal{A}$.

2 The results

In order to establish the proofs of our main theorems, we first state a result which we use frequently in the proof of our main results.

Lemma 2.1. *[5] Let n be a fixed positive integer and R a $n!$-torsion free ring. Suppose that $y_1, y_2, \ldots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ for $\lambda = 1, 2, \ldots, n$. Then $y_i = 0$ for $i = 1, 2, \ldots, n$.*

This section deals with the study of permuting n-multipliers. In the present section, we look about the action of symmetric linear n-derivations satisfying the functional identity $f(i)i + ig(i) = 0 \ \forall\ i \in \mathcal{W}$, a nonzero left ideal of $\mathcal{A}$ where f and g are the traces of symmetric n-derivations $\mathfrak{D}$ and $\mathcal{G}$ respectively. According to Brešar's proof in [4, Theorem 4.1], if $\mathcal{R}$ is a prime ring, $\mathcal{W}$ a nonzero left ideal of $\mathcal{R}$, and d and g are nonzero derivations of $\mathcal{R}$ satisfying $d(\vartheta)\vartheta - \vartheta g(\vartheta) \in \mathcal{Z}(\mathcal{R}) \ \forall\ \vartheta \in \mathcal{W}$, then $\mathcal{R}$ is commutative. We expand the previous result by demonstrating the following theorem for the traces of linear n-derivations of $\mathcal{A}$.

Theorem 2.1. *Let $\mathcal{A}$ be a primitive C^*-algebra and $\mathcal{W}$ a nonzero left ideal of $\mathcal{A}$. Suppose that $\mathcal{A}$ admit two symmetric linear n-derivations $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ and $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with f and g as traces of $\mathfrak{D}$ and $\mathcal{G}$, respectively. If $f(\vartheta)\vartheta + \vartheta g(\vartheta) = 0$ $\forall\ \vartheta \in \mathcal{W}$, then either $\mathcal{A}$ is commutative or $\mathcal{G}$ acts as a left n-multiplier on $\mathcal{W}$. Furthermore, in the last case, either $\mathfrak{D} = 0$ or $\mathcal{W}[\mathcal{W}, \mathcal{W}] = \{0\}$.*

Proof. We have given that $\mathcal{D}$, $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ be two symmetric linear n-derivations of a primitive C^*-algebra $\mathcal{A}$ such that $f(\vartheta)\vartheta + \vartheta g(\vartheta) = 0 \ \forall\ \vartheta \in \mathcal{W}$, a nonzero left ideal of $\mathcal{A}$. Therefore, $\mathcal{A}$ is prime by [17, Theorem 5.4.5] because $\mathcal{A}$ is a primitive C^*-algebra. Now replacing ϑ by $\vartheta + m\ell$ in the hypothesis for $\ell \in \mathcal{W}$ and $1 \leq m \leq n-1$, we get

$$f(\vartheta + m\ell)(\vartheta + m\ell) + (\vartheta + m\ell)g(\vartheta + m\ell) = 0 \ \forall\ \vartheta, \ell \in \mathcal{W}.$$

By the definitions of f and g and using the given condition, we get

$$f(\vartheta)m\ell + f(m\ell)\vartheta + \vartheta g(m\ell) + m\ell g(\vartheta) + \Big(\sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\vartheta,\dots,\vartheta}_{(n-t)\text{-times}}, \underbrace{m\ell,\dots,m\ell}_{t\text{-times}})\Big)(\vartheta + m\ell) + (\vartheta + m\ell)\Big(\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\vartheta,\dots,\vartheta}_{(n-t)\text{-times}}, \underbrace{m\ell,\dots,m\ell}_{t\text{-times}})\Big) = 0$$

$\forall\ \vartheta, \ell \in \mathcal{W}$. On using Lemma 2.1, we get

$$f(\ell)\vartheta + n\mathfrak{D}(\vartheta,\ell,\dots,\ell)\ell + \vartheta g(\ell) + n\ell\mathcal{G}(\vartheta,\ell,\dots,\ell) = 0 \tag{2.1}$$

$\forall\ \vartheta, \ell \in \mathcal{W}$. Replace ϑ by ϑk to obtain

$$f(\ell)\vartheta k + n\vartheta\mathfrak{D}(k,\ell,\dots,\ell)\ell + n\mathfrak{D}(\vartheta,\ell,\dots,\ell)k\ell + \vartheta k g(\ell) + n\ell\vartheta\mathcal{G}(k,\ell,\dots,\ell) + n\ell\mathcal{G}(\vartheta,\ell,\dots,\ell)k = 0\ \forall\ \vartheta,\ell,k \in \mathcal{W}. \tag{2.2}$$

On comparing (2.1) and (2.2), we get

$$\vartheta[k, g(\ell)] + n\mathfrak{D}(\vartheta,\ell,\dots,\ell)[k,\ell] + n\vartheta\mathfrak{D}(k,\ell,\dots,\ell)\ell + n\ell\vartheta\mathcal{G}(k,\ell,\dots,\ell) = 0 \tag{2.3}$$

$\forall\ \vartheta, \ell, k \in \mathcal{W}$. Substitute $r\vartheta$ for ϑ in (2.3) to get

$$r\vartheta[k, g(\ell)] + nr\mathfrak{D}(\vartheta,\ell,\dots,\ell)[k,\ell] + n\mathfrak{D}(r,\ell,\dots,\ell)\vartheta[k,\ell] + nr\vartheta\mathfrak{D}(k,\ell,\dots,\ell)\ell + n\ell r\vartheta\mathcal{G}(k,\ell,\dots,\ell) = 0 \tag{2.4}$$

$\forall\ \vartheta, \ell, k \in \mathcal{W}, r \in \mathcal{A}$. Comparing (2.3) and (2.4), we find that

$$n\mathfrak{D}(r,\ell,\dots,\ell)\vartheta[k,\ell] + n\ell r\vartheta\mathcal{G}(k,\ell,\dots,\ell) - nr\ell\vartheta\mathcal{G}(k,\ell,\dots,\ell) = 0 \quad \forall\ \vartheta,\ell,k, \in \mathcal{W}, r \in \mathcal{A}.$$

On simplifying, we obtain

$$\mathfrak{D}(r,\ell,\ldots,\ell)\vartheta[k,\ell]+[\ell,r]\vartheta\mathcal{G}(k,\ell,\ldots,\ell)=0 \tag{2.5}$$

$\forall\,\vartheta,\ell,k\in\mathcal{W}, r\in\mathcal{A}$. Replacing ℓ by k in (2.5), we see that

$$[k,r]\vartheta g(k)=0\;\forall\,\vartheta,k\in\mathcal{W}, r\in\mathcal{A}.$$

Substituting $r\vartheta$ for ϑ, we get

$$[k,r]\mathcal{A}\vartheta g(k)=0\;\forall\,\vartheta,k\in\mathcal{W}.$$

Since every primitive C^*-algebra is a prime, so the last relation yields that either $[k,r]=0$ or $\vartheta g(k)=0$. If $[k,r]=0\;\forall\,k\in\mathcal{W}$ and $r\in\mathcal{A}$, then replacing k by sk, we get $[s,r]k=0\;\forall\,k\in\mathcal{W}, r,s\in\mathcal{A}$. Again, replace k by rk such that $[s,r]\mathcal{A}k=0\;\forall\,k\in\mathcal{W}, r,s\in\mathcal{A}$. Henceforward, we conclude that $\mathcal{A}$ is commutative. Next, if $\vartheta g(k)=0\;\forall\,\vartheta,k\in\mathcal{W}$, then replacing k by $k+m\ell$ for $1\leq m\leq n-1$, we get

$$\vartheta g(k+m\ell)=0\;\forall\,\vartheta,\ell,k\in\mathcal{W}.$$

That is,

$$\vartheta g(k)+\vartheta g(m\ell)+\vartheta\sum_{t=1}^{n-1}{}^{n}C_t\mathcal{G}(\underbrace{k,\ldots,k}_{(n-t)\text{-times}},\underbrace{m\ell,\ldots,m\ell}_{t\text{-times}})=0\;\forall\,\vartheta,\ell,k\in\mathcal{W}.$$

By using Lemma 2.1, we get

$$\vartheta\mathcal{G}(k,\ell,\ldots,\ell)=0\;\forall\,\vartheta,\ell,k\in\mathcal{W}.$$

This implies that

$$\mathcal{G}(\vartheta k,\ell,\ldots,\ell)=\mathcal{G}(\vartheta,\ell,\ldots,\ell)k.$$

Hence, $\mathcal{G}$ acts as a left n-multiplier. Since $\vartheta\mathcal{G}(k,\ell\ldots,\ell)=0\;\forall\,\vartheta,\ell,k\in\mathcal{W}$, using (2.5), we arrive at

$$\mathfrak{D}(r,\ell,\ldots,\ell)\vartheta[k,\ell]=0\;\forall\,\vartheta,\ell,k,\in\mathcal{W}, r\in\mathcal{A}.$$

Replace r by sr to get

$$\mathfrak{D}(s,\ell,\ldots,\ell)\mathcal{A}\vartheta[\hbar,\ell]=\{0\}\ \forall\ \vartheta,\ell,\hbar\in\mathcal{W}.$$

Since $\mathcal{A}$ is a primitve C^{*}-algebra, we get that either $\mathfrak{D}(s,\ell,\ldots,\ell)=0$ or $\vartheta[\hbar,\ell]=0\ \forall\ \vartheta,\ell,\hbar\in\mathcal{W}, s\in\mathcal{A}$. If $\mathfrak{D}\neq 0$, the later result is $\mathcal{W}[\mathcal{W},\mathcal{W}]=\{0\}$. $\square$

Following the same vein, we can also prove the next result:

Theorem 2.2. *Let $\mathcal{A}$ be a primitive C^{*}-algebra and $\mathcal{W}$ a nonzero right ideal of $\mathcal{A}$. Assume that $\mathfrak{D}$ and $\mathcal{G}$ be two symmetric linear n-derivations of $\mathcal{A}$ with trace f and g, respectively. If $f(\vartheta)\vartheta+\vartheta g(\vartheta)=0\ \forall\ \vartheta\in\mathcal{W}$, then either $\mathcal{A}$ is commutative or $\mathfrak{D}$ acts as a left n-multiplier on $\mathcal{W}$. Furthermore, in the last case either, $\mathcal{G}=0$ or $\mathcal{W}[\mathcal{W},\mathcal{W}]=\{0\}$.*

In view of the above result, we obtain the following corollary for bilinear symmetric biderivations:

Corollary 2.1. *Let $\mathcal{A}$ be a primitive C^{*}-algebra, $\mathcal{W}$ a nonzero left ideal of $\mathcal{A}$ and Δ_1, Δ_2 be bilinear symmetric bi-derivations of $\mathcal{A}$ with trace δ_1 and δ_2, respectively. If $\Delta_1(\vartheta,\vartheta)\vartheta+\vartheta\Delta_2(\vartheta,\vartheta)=0\ \forall\ \vartheta\in\mathcal{W}$, then either $\mathcal{A}$ is commutative or Δ_2 acts as a left bi-multiplier on $\mathcal{W}$. Moreover, in the last case either $\Delta_1=0$ or $\mathcal{W}[\mathcal{W},\mathcal{W}]=\{0\}$.*

Corollary 2.2. *Let $\mathcal{A}$ be a primitive C^{*}-algebra, $\mathcal{W}$ a nonzero right ideal of $\mathcal{A}$ and Δ_1, Δ_2 be symmetric bilinear bi-derivations of $\mathcal{A}$ with trace δ_1 and δ_2, respectively. If $\Delta_1(\vartheta,\vartheta)\vartheta+\vartheta\Delta_2(\vartheta,\vartheta)=0\ \forall\ \vartheta\in\mathcal{W}$, then either $\mathcal{A}$ is commutative or Δ_2 acts as a left bi-multiplier on $\mathcal{W}$. Moreover, in the last case either $\Delta_1=0$ or $\mathcal{W}[\mathcal{W},\mathcal{W}]=\{0\}$.*

The next result is a generalization of Vukman's result [21]. Indeed, Vukman showed that if $\mathcal{R}$ is a prime ring of characteristic different from two and three and there exist symmetric bi-derivations $\mathfrak{D}_1:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ and $\mathfrak{D}_2:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$, such that $f_1(a)f_2(a)=0,\ \forall\ a\in\mathcal{R}$ holds, where f_1 and f_2 are the traces of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively, then either $\mathfrak{D}_1=0$ or $\mathfrak{D}_2=0$. We extend this theorem for q-iterations of linear n-derivations of C^{*}-algebras.

Theorem 2.3. *Let $\mathcal{A}$ be a primitive C^{*}-algebra, $\mathcal{W}$ a nonzero ideal of $\mathcal{A}$ and $q\geq 1$, a fixed integer. Consider $\mathfrak{D}_1,\mathfrak{D}_2,\ldots,\mathfrak{D}_q:\mathcal{A}^n\to\mathcal{A}$ to be linear n-*

derivations on $\mathscr{A}$ such that $d_1(i_1)d_2(i_2)\cdots d_q(i_q) = 0\ \forall\ i_1, i_2, \ldots, i_q \in \mathscr{W}$ where d_i's are traces of $\mathfrak{D}_i'$s respectively. Then one of the following holds:

1. *$d_1(i_1) = 0\ \forall\ i_1 \in \mathscr{W}$,*

2. *All $\mathfrak{D}_p$ act as left n-multipliers on $\mathscr{A}$ for $p = 2, 3, \ldots, q$.*

Proof. It is given that $\mathscr{A}$ is a primitive C^*-algebra. Therefore, $\mathscr{A}$ is prime C^*-algebra by [17, Theorem 5.4.5]. Now we will use induction to prove the theorem. If we put $q = 1$ in our hypothesis, then it is obvious that $d_1(i_1) = 0\ \forall\ i_1 \in \mathscr{W}$. Now consider $q = 2$, then by the hypothesis, we have

$$d_1(i_1)d_2(i_2) = 0\ \forall\ i_1, i_2 \in \mathscr{W}. \tag{2.6}$$

Replacing i_2 by $i_2 + m\ell_2$ for $\ell_2 \in \mathscr{W}$ and $1 \leq m \leq n-1$, we get

$$d_1(i_1)d_2(i_2 + m\ell_2) = 0\ \forall\ i_1, i_2, \ell_2 \in \mathscr{W}.$$

On simplifying, we get

$$d_1(i_1)d_2(i_2) + d_1(i_1)d_2(m\ell_2) + d_1(i_1)\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}_2(\underbrace{i_2, \ldots, i_2}_{(n-t)\text{-times}}, \underbrace{m\ell_2, \ldots, m\ell_2}_{t\text{-times}}) = 0 \tag{2.7}$$

$\forall\ i_1, i_2, \ell_2 \in \mathscr{W}$. Compare (2.6) and (2.7) and use Lemma 2.1 to get

$$nd_1(i_1)\mathfrak{D}_2(i_2, \ldots, i_2, \ell_2) = 0\ \forall\ i_1, i_2, \ell_2 \in \mathscr{W}.$$

This implies that

$$d_1(i_1)\mathfrak{D}_2(i_2, \ldots, i_2, \ell_2) = 0\ \forall\ i_1, i_2, \ell_2 \in \mathscr{W}. \tag{2.8}$$

Replacing ℓ_2 by $\ell_2 r$ in (2.8), we obtain

$$d_1(i_1)\ell_2\mathfrak{D}_2(i_2, \ldots, i_2, r) = 0\ \forall\ i_1, i_1, \ell_2 \in \mathscr{W}, r \in \mathscr{A}$$

i.e.,

$$\mathscr{d}_1(i_1)\ell_2\mathscr{A}\mathfrak{D}_2(i_2,\ldots,i_2,r)=\{0\}\ \forall\, i_1,i_2,\ell_2\in\mathscr{W}.$$

Since $\mathscr{A}$ is a primitive C^*-algebra, we can find either $\mathscr{d}_1(i_1)\ell_2=0$ or $\mathfrak{D}_2(i_2,\ldots,i_2,r)=0$. Consider the first case $\mathscr{d}_1(i_1)\ell_2=0$. This implies that $\mathscr{d}_1(i_1)=0$. Now consider the later case $\mathfrak{D}_2(i_2,\ldots,i_2,r)=0\ \forall\, i_2\in\mathscr{W},\ r\in\mathscr{A}$. A straightforward modification shows that $\mathfrak{D}_2(i_2,\ldots,i_2,w_1r)=\mathfrak{D}_2(i_2,\ldots,i_2,w_1)r\ \forall\, w_1\in\mathscr{W}, r\in\mathscr{A}$. Hence $\mathfrak{D}_2$ acts as a left n-multiplier as desired.
If $q=3$, then by the hypothesis, we can write

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\mathscr{d}_3(i_3)=0\ \forall\, i_1,i_2,i_3\in\mathscr{W}. \tag{2.9}$$

Replacing i_3 by $i_3+m\ell_3$ for $\ell_3\in\mathscr{W}$ and $1\leq m\leq n-1$ in (2.9) and taking account of Lemma 2.1, we get

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\mathfrak{D}_3(i_3,\ldots,i_3,\ell_3)=0\ \forall\, i_1,i_2,i_3,\ell_3\in\mathscr{W}. \tag{2.10}$$

Taking $\ell_3 s$ in place of ℓ_3 in (2.10) and using (2.10), we obtain

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\ell_3\mathfrak{D}_3(i_3,\ldots,i_3,s)=0\ \forall\, i_1,i_2,i_3,\ell_3\in\mathscr{W}, s\in\mathscr{A},$$

which gives

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\ell_3\mathscr{A}\mathfrak{D}_3(i_3,\ldots,i_3,s)=0\ \forall\, i_1,i_2,i_3,\ell_3\in\mathscr{W}.$$

Using primitiveness of $\mathscr{A}$, we have either $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\ell_3=0$ or $\mathfrak{D}_3(i_3,\ldots,i_3,s)=0$. Consider the first case, $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\ell_3=0$. Again using primitiveness of $\mathscr{A}$, we get $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)=0$. Then we are done by the previous case for $q=2$. Now consider $\mathfrak{D}_3(i_3,\ldots,i_3,s)=0\ \forall\, i_3\in\mathscr{W},\ s\in\mathscr{A}$, we can find $\mathfrak{D}_3(i_3,\ldots,i_3,w_2s)=\mathfrak{D}_3(i_3,\ldots,i_3,w_2)s=0\ \forall\, w_2\in\mathscr{W},\ s\in\mathscr{A}$, that is, $\mathfrak{D}_3$ acts as a left n-multiplier.
Next suppose that it is true for $n=q-1$ and we shall prove it for $n=q$. Let us assume the hypothesis,

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_q(i_q)=0\ \forall\, i_1,i_2,\ldots,i_q\in\mathscr{W}. \tag{2.11}$$

Replacing i_q by $i_q + m\ell_q$ for $\ell_q \in \mathscr{W}$ and $1 \leq m \leq n-1$ in (2.11) and taking account of Lemma 2.1, we get

$$n\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\mathfrak{D}_q(i_q,\ldots,i_q,\ell_q) = 0$$

$\forall\, i_1, i_2, \ldots, i_q, \ell_q \in \mathscr{W}$. The last relation gives

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\mathfrak{D}_q(i_q,\ldots,i_q,\ell_q) = 0. \tag{2.12}$$

Substituting $\ell_q u$ for ℓ_q in (2.12) and using (2.12), we arrive at

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\ell_q\mathfrak{D}_q(i_q,\ldots,i_q,u) = 0$$

i.e.,

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\ell_q\mathscr{A}\mathfrak{D}_q(i_q,\ldots,i_q,u) = \{0\}$$

$\forall\, i_1, i_2, \ldots, i_q, \ell_q \in \mathscr{W}, u \in \mathscr{A}$. Since every primitive C^*-algebra is a prime so the last relation gives that either $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1}) = 0$ or $\mathfrak{D}_q(i_q,\ldots,i_q,u) = 0\ \forall\ i_1, i_2, \ldots, i_q \in \mathscr{W},\ u \in \mathscr{A}$. If $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1}) = 0$, then we are done by the former case. If $\mathfrak{D}_q(i_q,\ldots,i_q,u) = 0\ \forall\ i_q \in \mathscr{W},\ u \in \mathscr{A}$, then we can easily compute that $\mathfrak{D}_q(i_q,\ldots,i_q,w_{q-1}u) = \mathfrak{D}_q(i_q,\ldots,i_q,w_{q-1})u\ \forall\ i_q, w_{q-1} \in \mathscr{W},\ u \in \mathscr{A}$. Hence $\mathfrak{D}_q$ acts as a left n-multiplier on $\mathscr{A}$ as desired. The theorem's proof is completed with this conclusion. □

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