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The total graph of a module with respect to multiplicative-semiprime subsets

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Abstract

Let M be a module over a commutative ring R. A proper subset S of M is said to be multiplicative-semiprime if $rn \in S$, for all $r \in R, n \in S$ and if $r^2m \in S$ for some $r \in R$ and $s \in M$ then $rm \in S$. Total graph of a module with respect to multiplicative-semiprime subset is a graph with all elements of M as vertices, and for two distinct elements $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in S$. The main purpose of this paper is to extend the results given in [6] to a more general case, especially when S is a submodule of M. We prove some properties about connectedness and completeness of this graph, type of graph, and the impact of girth and diameter of this graph to the cardinality of the factor module of M.

1 Introduction

Throughout this paper, R is a commutative ring with nonzero identity and M is a unitary R-module. The notion of total graph has been introduced by some previous authors, for examples in [2], [5], and [3]. Total graph of a module is a simple graph with the vertex set M and two distinct vertices x and y are adjacent if $x + y \in T(M)$, where $T(M)$ is the set of all torsion elements in M.

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Moreover, some authors define the total graph of module with respect to another proper submodule. We recall first the set of $\{r \in R | rM \subseteq N\}$ which is denoted by $(N : M)$. Total graph of a module M with respect to a proper submodule N is a simple graph with vertex set M and two distinct vertices x and y are adjacent if $x + y \in M(N)$, where $M(N) = \{m \in M | rm \in N, \text{ for some } r \in R \backslash (N : M)\}$. This adjacency is the generalization of adjacency in the previous total graph of a module we mentioned before. For more detailed results, the readers are suggested to read [1].

A nonempty proper subset U of M is called a multiplicative-prime subset of M if it satisfies the following two properties: (i) $ru \in U$ for all $r \in R$ and $u \in U$; (2) if $ru \in U$ for some $r \in R$ and $u \in M$, then $r \in (U : M)$ or $u \in U$. For any multiplicative-prime subset U of M, we obtain a generalized total graph $GT_U(M)$ with vertices in M and any two vertices $x, y \in M$ are adjacent if $x+y \in U$. Saraei et.al [6] study some properties of this graph related to its diameter and girth.

Motivated by the notion of semiprime submodule in [4], we define a multiplicative -semiprime subset. In this paper, we introduce and investigate total graph of M with respect to a multiplicative-semiprime subset S and denoted by $GT_S(M)$. This graph is an undirected graph with all elements of M as vertices, and for two distinct elements $m, n \in M$, they are adjacent if $m + n \in S$.

Further observation of $GT_S(M)$ can be divided into two cases: S is a submodule of M or not. In this paper we only discuss when S is a submodule of M. We obtain some results of connectedness and completeness of this total graph, type of the graphs and the relationship between girth and diameter of this graph to the cardinality of factor module of M. We also conclude that the set $A = \{m \in M \backslash S | 2m \in S\}$ is not always an empty set if S is a multiplicative-semiprime subset. However, A is always empty if S is a multiplicative-prime subset of M. This fact influences the form of related total graph of a module M.

For any simple graph Γ, we denote the set of all edges and vertices of Γ, by $E(\Gamma)$ and $V(\Gamma)$, respectively. We write $a \sim b$ if the vertices a and b are adjacent. We recall that a graph is connected if there exists a path connecting any two of its distinct vertices. We say that a graph is a totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by $d(a, b)$, is the length of a shortest path connecting them (if such path does not exist, then $d(a, b) = \infty$. We define $d(a, a) = 0$. The diameter of a graph Γ, denoted by $diam(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted by $gr(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise; $gr(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n

and the complete bipartite graph of on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). For a graph Γ, the degree of a vertex $v \in \Gamma$, denoted by $deg(v)$, is the number of edges incident with v. We say that two induced subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertices of Γ_1 is adjacent in Γ to some vertex Γ_2. For other basic definitions in graph theory, we refer to [7].

2 Main Results

In this section, we begin with the definition of multiplicative-semiprime subset of a module and total graph of a module with respect to multiplicative-semiprime subset.

Definition 2.1. *Let M be a module over a commutative ring R. A nonempty proper subset S of M is called a multiplicative-semiprime subset of M if it satisfies the following conditions :*

1. *$rn \in S$, for all $r \in R$ and $n \in S$.*
2. *For all $r \in R$ and $n \in M$, if $r^2n \in S$ then $rn \in S$.*

The notion of total graph of a module is given below:

Definition 2.2. *Let M be a module over a commutative ring. The total graph of a module with respect to a multiplicative-semiprime subset, denoted by $GT_S(M)$, is a simple undirected graph with all elements of M as vertices, and for two distinct elements $m, n \in M$, the vertices m and n are adjacent if $m + n \in S$.*

Example 2.1. *Let $\mathbb{Z}_{12}$ be a module over $\mathbb{Z}_{12}$. The set $S = \{0, 6\}$ is multiplicative-semiprime in $\mathbb{Z}_{12}$. In Figure 1 we give the illustration of total graph of $\mathbb{Z}_{12}$-module $\mathbb{Z}_{12}$ with respect to $\{0, 6\}$.*

Next we study the case when S is a semiprime submodule of M. If $S = M$, then it is clear that $GT_S(M)$ is a complete graph. In Proposisition 2.1 we will see the relation between $GT_S(S)$ and $GT_S(M\backslash S)$.

Proposition 2.1. *Let M be a module over a commutative ring R and S a semiprime submodule of M. Then $GT_S(S)$ is a complete subgraph of $GT_S(M)$ and is disjoint from $GT_S(M\backslash S)$. In particular, $GT_S(S)$ is connected.*

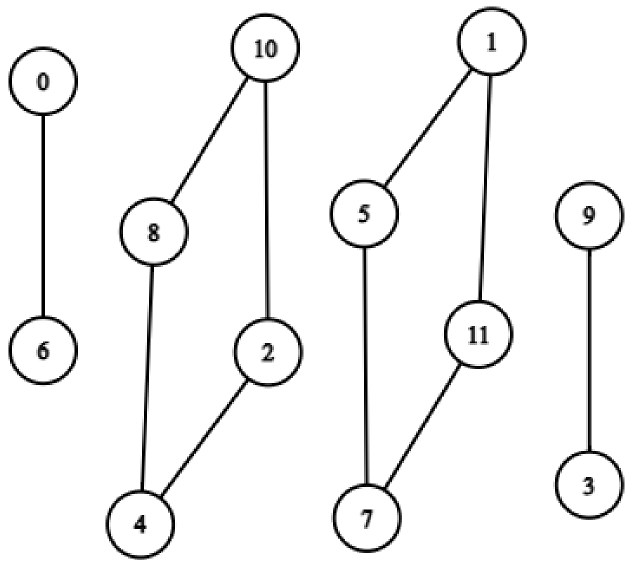

Figure 1: The total graph of $\mathbb{Z}_{12}$-module $\mathbb{Z}_{12}$ with respect to $\{0,6\}$.

Proof. Let $x, y \in S$. Since S is a submodule of M, $x + y \in S$. It means induced subgraph $GT_S(S)$ is a complete graph. Suppose that $GT_S(S)$ and $GT_S(M\backslash S)$ are not disjoint. Then, there exist $a \in S$ and $b \in M\backslash S$ such that $a + b \in S$. Since S is a submodule of M, $b = a + b - a \in S$. It is a contradiction to our assumption that $b \in M\backslash S$. Therefore, $GT_S(S)$ and $GT_S(M\backslash S)$ are disjoint. Since $GT_S(S)$ is a complete graph, $GT_S(S)$ is a connected graph. $\square$

We give more results of $GT_S(M\backslash S)$ in the following proposition.

Proposition 2.2. *Let M be a module over a commutative ring R and S a semiprime submodule of M. Then the following assertions hold:*

1. *Suppose that G is an induced subgraph of $GT_S(M\backslash S)$ and let m and m' be distinct vertices of G that are connected by a path in G. Then there exists a path in G of length 2 between m and m'. In particular, if $GT_S(M\backslash S)$ is connected, then $diam(GT_S(M\backslash S)) \leq 2$.*
2. *Let m and m' be distinct elements of $GT_S(M\backslash S)$ that are connected by a path. If $m + m' \notin S$ then $m \sim (-m) \sim m'$ and $m \sim (-m') \sim m'$ are paths of length 2 between m and m' in $GT_S(M\backslash S)$.*

Proof.
1. Let m_1, m_2, m_3 and m_4 are distinct vertices of G. It suffices to show that there is a path $m_1 \sim m_2 \sim m_3 \sim m_4$ from m_1 to m_4, then m_1 and m_4 are adjacent. Now, $m_1 + m_2, m_2 + m_3, m_3 + m_4 \in S$. Then, we get

$$m_1 + m_4 = (m_1 + m_2) - (m_2 + m_3) - (m_3 + m_4) \in S.$$

Thus, m_1 and m_4 are adjacent. If $GT_S(M\backslash S)$ is connected, then $diam(GT_S(M\backslash S)) \leq 2$.

2. Since $m, m' \notin S$ and $m + m' \notin S$, there exists $w \in GT_S(M\backslash S)$ such that $m \sim w \sim m'$ is a path of length 2 by part (1). Thus, $w + m, w + m' \in S$ and $m - m' = (m + w) - (w + m') \in S$. Since $m, m' \notin S$, $m \neq -m'$ and $m' \neq -m'$. Thus, $m \sim (-m') \sim m'$ is a path from m to m' in $GT_S(M\backslash S)$. $\square$

As we can see in Figure 1, an induced subgraph with vertex set $M\backslash S$ of total graph of module with respect to multiplicative-semiprime subset is not always a connected graph. In the following proposition, we give the necessary and sufficient condition such that $GT_S(M\backslash S)$ is a connected graph.

Proposition 2.3. *Let M be a module over a commutative ring R and S a semiprime submodule of M. Then the following statements are equivalent:*

1. *$GT_S(M\backslash S)$ is a connected graph.*
2. *Either $m + m' \in S$ or $m - m' \in S$(but not both) for all $m, m' \in M\backslash S$.*
3. *Either $m + m' \in S$ or $m + 2m' \in S$ for all $m, m' \in M\backslash S$. In particular, if (3) is satisfied, then either $2m \in S$ or $3m \in S$ for all $m \in M\backslash S$.*

Proof. $(1) \Rightarrow (2)$ Take any $m, m' \in M\backslash S$ such that $m + m' \notin S$. We prove that $m - m' \in S$. If $m = m'$, then $m - m' = 0 \in S$. If $m \neq m'$, then by Proposition 2.2 (2) we get that $m \sim (-m') \sim m'$ is a path from m to m'. Then $m - m' \in S$.

$(2) \Rightarrow (3)$ Take any distinct elements $m, m' \in M\backslash S$ such that $m + m' \notin S$. Then $(m+m')+m' \in S$ or $(m+m')-m' \in S$ by assumption. If $(m+m')+m' \in S$, then $m \in S$, that is a contradiction. Therefore, $(m+m')+m' = m+2m' \in S$. In particular, $m + m = 2m \in S$ or $m + 2m = 3m \in S$ for all $m \in M\backslash S$. Both $2m$ and $3m$ can not be in S, since $m = 3m - 2m \in S$. It contradicts with our assumption that $m \notin S$.

$(3) \Rightarrow (1)$ Let $m, m' \in M\backslash S$ be distinct elements of M such that $m+m' \notin S$. By hypothesis $m + 2m' \in S$ and we get $2m' \notin S$. Thus $3m' \in S$ by assumption. Moreover, since $m+m' \in S$ and $3m' \in S$, $m \neq 2m'$. Therefore $m \sim (2m') \sim m'$ is a path from m to m' in $GT_S(M\backslash S)$. Thus $GT_U(M\backslash S)$ is connected. □

Let M be a module over a commutative ring R, U is a multiplicative-prime subset of M, and S is a multiplicative-semiprime subset of M. If $2 \notin (U : M)$, then for all $m \in M\backslash U$ we get $2m \notin U$. This is true since U is a multiplicative prime-subset. It means, if U is a multiplicative-prime subset of M then the set $\{m \in M\backslash U | 2m \in U\}$ is always an empty set. However, this condition is not necessarily true for S as a multiplicative-semiprime subset. If $2 \notin (S : M)$, then the set $\{m \in M\backslash S | 2m \in S\}$ is not always empty. In the following theorem, we provide the result regarding the type of graph formed by this graph, depending on whether 2 is an element of $(S : M)$ or not and whether the set $m \in M\backslash S | 2m \in S$ is empty or not.

Theorem 2.1. *Let M be a module over a commutative ring R and S be a semiprime submodule of M. Let $A = \{m \in M\backslash S | 2m \in S\}$, $|S| = \alpha$, $|M/S| = \beta$, and $|\{m + S | m \in A\}| = \gamma$.*

1. *If $2 \in (S : M)$, then $GT_S(M\backslash S)$ is the union of $\beta - 1$ disjoint K^{α}'s.*
2. (a) *If $2 \notin (S : M)$ and $A = \emptyset$, then $GT_S(M\backslash S)$ is the union of $\frac{\beta-1}{2}$ disjoint $K^{\alpha,\alpha}$'s.*

   (b) *If $2 \notin (S : M)$ and $A \neq \emptyset$, then $GT_S(M\backslash S)$ is the union of $\frac{\beta-1-\gamma}{2}$ disjoint $K^{\alpha,\alpha}$'s and γ disjoint K^{α}'s.*

Proof. 1. Note that $m+S \subseteq M\backslash S$ for all $m \notin S$. Let $m+n_1, m+n_2 \in m+S$. Since $2 \in (S : M)$ and S a submodule of M, $2m + (n_1 + n_2) \in S$. So, each coset $m + S$ induces a complete subgraph of $GT_S(M\backslash S)$. Moreover, every distinct coset of S form disjoint induced subgraph of $GT_S(M\backslash S)$. Suppose that $m + S$ and $m' + S$ are two distinct coset but there is a vertex in $m + S$ that adjacent to a vertex in $m' + S$. Let $m+n$ and $m'+n'$ are adjacent where $n, n' \in S$. Then $m + n + m' + n' \in S$. We get that $m + m' = (m + n) + (m' + n') - (n + n') \in S$. Consequently, $m - m' = (m + m') - 2m \in S$. In other words $m + S = m' + S$. It contradicts with our assumption that $m + S$ and $m' + S$ are two distinct coset. It means, every coset of S form disjoint induced subgraph of $GT_S(M\backslash S)$. Since $|M/S| = \beta$, then

$$M/S = \{S, m_1 + S, \ldots, m_{\beta-1} + S\}.$$

Therefore, $GT_S(M\backslash S)$ is union of $\beta - 1'$s induced subgraph $m_i + S, i \in \{1, \ldots, \beta - 1\}$, where each subgraph is complete subgraph with cardinality of vertex set is α.

2. (a) Let $m \in M\backslash S$. Assume that $2 \notin (S : M)$ and $A = \emptyset$. We show that no two distinct elements in $m + S$ are adjacent. Suppose there are two distinct elements of $m + S$ are adjacent, i.e., $m + m_1$ and $m + m_2$ for some $m_1, m_2 \in S$. Then $(m + m_1) + (m + m_2) = 2m + m_1 + m_2 \in S$. Since $m_1, m_2 \in S$ and S is a submodule of M, $2m = 2m + m_1 + m_2 - (m_1 + m_2) \in S$. It contradicts with our assumption that $A = \emptyset$. So, $(m+S)\cup(-m+S)$ is a complete bipartite

subgraph of $GT_S(M\backslash S)$. Moreover, if $m+x_1$ is adjacent to $m'+x_2$ for some $m, m' \in M\backslash S$ and $x_1, x_2 \in S$, then $m+x_1+m'+x_2 \in S$. Hence, $m+m' = m+x_1+m'+x_2-(x_2+x_2) \in S$. Therefore, $m+S = -m+S$. Thus, $GT_S(M\backslash S)$ is the union of $\frac{\beta-1}{2}$ disjoint subgraph $(m+S)\cup(-m+S)$, each subgraph is $K^{\alpha,\alpha}$, i.e., a complete bipartite subgraph with cardinality of vertex set is α where $\alpha = |S| = |m+S|$.

(b) Let $m \in M\backslash S$. Assume that $2 \notin (S : M)$ and $A \neq \emptyset$. We show that no two distinct elements of $m+S$ where $m \notin A$ are adjacent. Suppose there are two distinct adjacent elements in $m+S$, i.e. $m+m_1$ and $m+m_2$ for some $m_1, m_2 \in S$. Hence, $(m+m_1)+(m+m_2) = 2m+m_1+m_2 \in S$. Since $m_1, m_2 \in S$ and S is a submodule of M, $2m = 2m+m_1+m_2-(m_1+m_2) \in S$. Moreover, since S is a submodule of M, $2m \in S$ for all $m \in S$. Therefore, for all $m \in M$ we get $2m \in S$ or equivalently $2 \in (S : M)$. It contradicts with our hypothesis that $2 \notin (S : M)$. Thus $(m+S)\cup(-m+S)$ with $m, -m \notin A$ is a complete bipartite induced subgraph of $GT_S(M\backslash S)$. Let $m \in A$. We show that each elements of $m+S$ are adjacent. Let $m+n_1, m+n_2 \in m+S$. Since $m \in A$ and S is a submodule of M, $2m+(n_1+n_2) \in S$. So, each coset $m+S$ induces a complete subgraph of $GT_S(M\backslash S)$. Moreover, every distinct coset $m+S$ where $m \in (M\backslash S)\backslash A$ form a disjoint induced subgraph of $GT_S(M\backslash S)$. Suppose that $m+S$ and $m'+S$ are two distinct coset but there is a vertex in $m+S$ that adjacent to a vertex in $m'+S$. Let $m+n$ and $m'+n'$ are adjacent where $n, n' \in S$. Then $m+n+m'+n' \in S$. We get that $m+m' = (m+n)+(m'+n')-(n+n') \in S$. Consequently, $m-m' = (m+m')-2m \in S$. In other words $m+S = m'+S$. It contradicts with our assumption that $m+S$ and $m'+S$ are two distinct coset. It means, every distinct coset $m+S$ where $m \in (M\backslash S)\backslash A$ form disjoint induced subgraph of $GT_S(M\backslash S)$. Since $|\{m+S | m \in A\}| = \gamma$, then $GT_S(M\backslash S)$ is union of disjoint $\frac{\beta-1-\gamma}{2}{}'$s induced complete bipartite subgraph $K^{\alpha,\alpha}$ and γ's induced complete subgraph K^{α}.

□

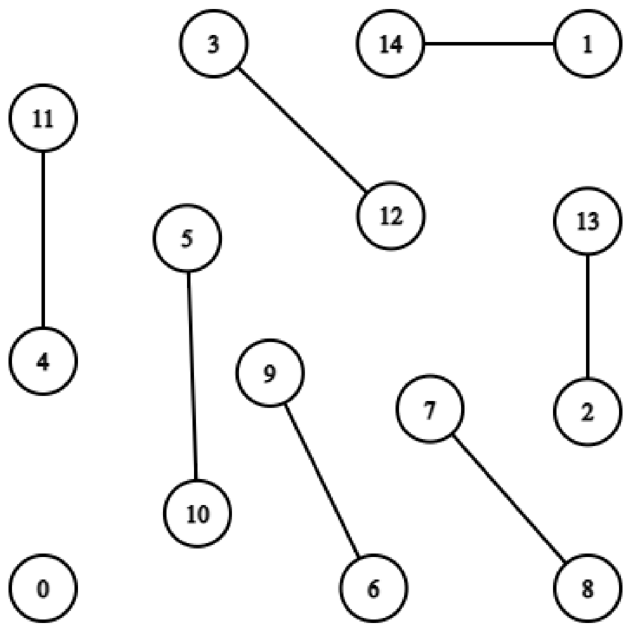

Figure 2: $GT_{\{0\}}(\mathbb{Z}_{15})$.

Example 2.2. *1. Let $\mathbb{Z}_{15}$ be a module over $\mathbb{Z}_{15}$ and $S = \{0\}$ a multiplicative-semiprime subset of $\mathbb{Z}_{15}$. Since there is no $m \in \mathbb{Z}_{15} \backslash \{0\}$ such that $2m \in \{0\}$, then $A = \emptyset$ and $2 \notin (S : M)$. In Figure 2, $GT_S(M \backslash S)$ is the union of a complete bipartite graphs as stated in Theorem 2.1 (2)(a).*

2. Let $\mathbb{Z}_{18}$ be a module over $\mathbb{Z}_{18}$ and $S = \{0, 6, 12\}$ a multiplicative-semiprime subset of $\mathbb{Z}_{18}$. It is easy to see that $2 \notin (S : M)$. We get that $A = \{3, 9, 15\}$. In Figure 3, we show that induced subgraph with vertex set A forms complete subgraph and the rest form complete bipartite subgraphs as stated in Theorem 2.1 (2)(b).

As we can see induced subgraph $GT_S(M \backslash S)$ in Figure 2 and Figure 3 is neither a complete graph nor a connected graph. In the following theorem, we give some necessary and sufficient conditions of induced subgraphs $GT_S(M \backslash S)$ to be a connected graph or a complete graph.

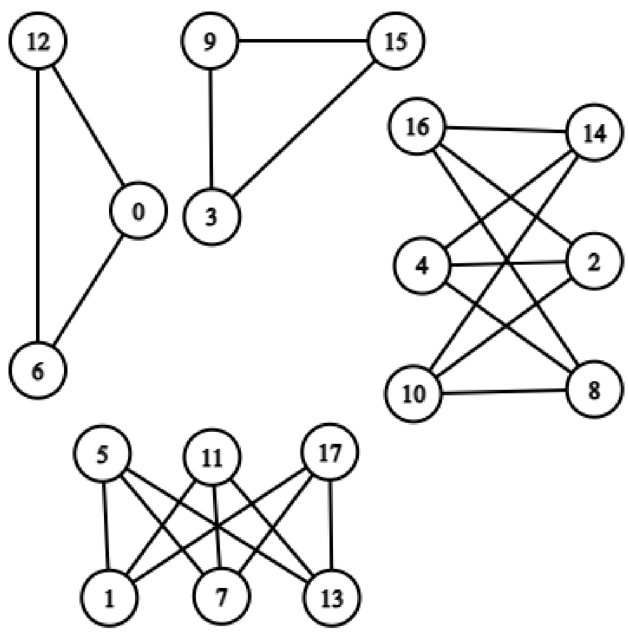

Figure 3: $GT_{\{0,6,12\}}(\mathbb{Z}_{18})$.

Theorem 2.2. *Let M be a module over a commutative ring R and S a semiprime submodule of M. Then*

1. *$GT_S(M\backslash S)$ is a complete graph if and only if $|M/S| = 2$ or $|M/S| = |M| = 3$.*

2. *$GT_S(M\backslash S)$ is a connected graph if and only if $|M/S| = 2$ or $|M/S| = 3$.*

3. *$GT_S(M\backslash S)$ (and hence $GT_S(S)$ and $GT_S(M)$) is totally disconnected if and only if $S = \{0\}$ and $2 \in (S : M)$.*

Proof. 1. ($\Rightarrow$) Let $GT_S(M\backslash S)$ be a complete subgraph of $GT_S(M)$. Then by Theorem 2.1, $GT_S(M\backslash S)$ is single K^α or $K^{1,1}$. If $GT_S(M\backslash S)$ is K^α, then $\beta - 1 = 1$. Hence, $\beta = 2$ and $|M/S| = 2$. If $GT_S(M\backslash S)$ is $K^{1,1}$, then $\frac{\beta-1}{2} = 1$ and $\alpha = 1$. Thus, $\beta = 3$ and $\alpha = 1$. Therefore, $|M/S| = 3$ and $S = \{0\}$. Hence, $|M/S| = |M| = 3$.

($\Leftarrow$) Let $|M/S| = 2$ and $M/S = \{S, x + S\}$ where $x \notin S$. Then $x + S = -x + S$ and we get $2x \in S$. Thus, $2 \in (S : M)$. Next we show that $GT_S(M\backslash S)$ is a complete graph. Let $m, m' \in M\backslash S$. Then $m + m' = (m + x) + (m' + x) - 2x \in S$. Thus $GT_S(M\backslash S)$ is a complete graph. Let $|M/S| = |M| = 3$. In this case, we show that $2 \notin (S : M)$. Suppose $2 \in (S : M)$. Then $2m \in S$ for all $m \in M$. Thus, we get $2(m+S) = 0_{M/S}$ for all $m \in M$. It is a contradiction since we assume M/S is a cyclic group with order 3. So, $2 \notin (S : M)$. By Theorem 2.1, $GT_S(M\backslash S)$ is a union of $\frac{3-1}{2} = 1$ disjoint $K^{1,1}$'s. Then every case leads to $GT_S(M\backslash S)$ is a complete graph.

2. ($\Rightarrow$) Let $GT_S(M\backslash S)$ be a connected graph. By Theorem 2.1, $GT_S(M\backslash S)$ is single K^α or $K^{\alpha,\alpha}$. If $2 \in (S : M)$, by Theorem 2.1 $\beta - 1 = 1$. Then $|M/S| = 2$. If $2 \notin (S : M)$, then $\frac{\beta-1}{2} = 1$. Thus $|M/S| = 3$.

   ($\Leftarrow$) By point 1 above, we assume that $|M/S| = 3$. We will show that $2 \notin (S : M)$. Suppose that $2M \subseteq S$ and $M/S = \{S, x + S, y + S\}$ where $x, y \notin S$. Since M/S is a cyclic group with order 3, then $x + y \in S$. Thus, x is adjacent to y. It contradicts with the statement that $GT_S(M\backslash S)$ is a union of two disjoint subgraph, $x + S$ and $y + S$. Hence, $2 \notin (S : M)$.

By Theorem 2.1, we get $GT_S(M\backslash S)$ is union of $\frac{3-1}{2} = 1$ disjoint $K^{\alpha,\alpha}$'s. Therefore, $GT_S(M\backslash S)$ is a connected graph.

3. $GT_S(M\backslash S)$ is totally disconnected if and only if it is a disjoint union of K^1's. So, by Theorem 2.1, $GT_S(M\backslash S)$ is totally disconnected if and only if $2 \in (S : M)$, $|S| = 1$, and $|M/S| = 1$.

□

Now, we give some results on the diameter of $GT_S(M\backslash S)$.

Proposition 2.4. *Let M be a module over a commutative ring R and S a semiprime submodule of M. Then $diam(GT_S(M\backslash S)) = 0, 1, 2, \infty$. In particular, if GT_S $(M\backslash S)$ is a connected graph, then*

$$diam(GT_S(M\backslash S)) \leq 2.$$

Proof. Let $GT_S(M\backslash S)$ be a connected graph. Then $GT_S(M\backslash S)$ is a complete graph or complete bipartite graph by Theorem 2.1. If $GT_S(M\backslash S)$ is a complete graph, then $diam(GT_S(M\backslash S)) = 1$ or $diam(GT_S(M\backslash S)) = 0$. If $GT_S(M\backslash S)$ is a complete bipartite graph, then $diam(GT_S(M\backslash S)) = 2$. Hence, if $GT_S(M\backslash S)$ is a connected graph, then $diam(GT_S(M\backslash S)) \leq 2$. If $GT_S(M\backslash S)$ is not a connected graph, then $diam(GT_S(M\backslash S)) = \infty$. □

Theorem 2.3. *Let M be a module over a commutative ring R and S a semiprime submodule of M.*

1. *$diam(GT_S(M\backslash S)) = 0$ if and only if $S = \{0\}$ and $|M| = 2$.*
2. *$diam(GT_S(M\backslash S)) = 1$ if and only if either*
   - *(a) $S \neq \{0\}$ and $|M/S| = 2$, or*
   - *(b) $|S| = \{0\}$ and $|M| = 3$.*
3. *$diam(GT_S(M\backslash S)) = 2$ if and only if $S \neq \{0\}$ and $|M/S| = 3$.*
4. *Otherwise, $diam(GT_S(M\backslash S)) = \infty$.*

Proof. The results follow from Theorem 2.1 and Theorem 2.2. □

We give some properties of the girth $GT_S(M\backslash S)$.

Proposition 2.5. *Let M be a module over a commutative ring R and S a semiprime submodule of M. Then $gr(GT_S(M\backslash S)) = 3, 4, \infty$. In particular, $gr(GT_S(M\backslash S)) \leq 4$ if $GT_S(M\backslash S)$ contains a cycle.*

Proof. If $GT_S(M\backslash S)$ contains a cycle, then by Theorem 2.1, $GT_S(M\backslash S)$ is a union of complete graphs or complete bipartite graphs with complete graphs. It means $GT_S(M\backslash S)$ contains either $3-$cycle or $4-$cycle. Thus, $gr(GT_S(M\backslash S) \leq 4$. If $GT_S(M\backslash S)$ contains no cycle, then $gr(GT_S(M\backslash S)) = \infty$. □

Theorem 2.4. *Let M be a module over a commutative ring R, S a semiprime submodule of M and $A = \{m \in M\backslash S | 2m \in S\}$.*

1. (a) *$gr(GT_S(M\backslash S)) = 3$ if and only if either*
      i. *$2 \in (S : M)$ and $|S| \geq 3$, or*
      ii. *$2 \notin (S : M)$, $A \neq \emptyset$, and $|S| \geq 3$.*

   (b) *$gr(GT_S(M\backslash S)) = 4$ if and only if $2 \notin (S : M)$, $|S| \geq 2$, and $A = \emptyset$.*

   (c) *Otherwise, $gr(GT_S(M\backslash S)) = \infty$.*

2. (a) *$gr(GT_S(M)) = 3$ if and only if $|S| \geq 3$.*

   (b) *$gr(GT_S(M)) = 4$ if and only if $2 \notin (S : M)$, $A = \emptyset$, and $|S| = 2$.*

   (c) *Otherwise, $gr(GT_S(M)) = \infty$.*

Proof. The results follow from Proposition 2.1, Theorem 2.1, and Proposition 2.5. □

3 CONCLUSION

The connectedness and completeness of total graph of module with respect to multiplicative-semiprime subset is related to the cardinality of its module or its factor module. Since the set $A = \{m \in M\backslash S | 2m \in S\}$ is not always an empty set, induced subgraph with vertex set A is not always a null graph. It means the type of graph $GT_S(M\backslash S)$ is not only a complete graph or bipartite graph but also can be both. We have shown in Theorem 2.3 that the diameter has an impact on the cardinality of the module or factor module. The girth of this graph not only has an impact on the cardinality of the module or factor module but also on the cardinality of A as shown in Theorem 2.4.

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