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POLOID AND MONOID

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Abstract

In this study, group(monoid) and structure of polyloid are investigated. The conditions in the construction of the group and the poloid structure are compared. Even the structure of a non-commutative group is not poloid, the poloid is also a group. The condition (P4) added to a group structure further enriched a group structure. New research is obtained with some properties, lemmas and theorems this condition by group structure. The condition (P4) is given that a monoid retains the only unit "e" element. It is shown that all elements different from the unit can be written by processing with a common element.

1 Introduction

Definition 1.1. *A group is a set* P *equipped with a binary operation* $\bigstar : P \times P \to P$ *that associates an element* $a\bigstar b \in P$ *to every pair of elements* $a, b \in P$*, and having the following properties:* $\bigstar$ *is associative, has an identity element* $e \in P$ *, and every element in* P *is invertible (w.r.t.*$\bigstar$*). More explicitly, this means that the following equations hold for all*$a, b, c, d \in P$*:*
(P1) $a\bigstar(b\bigstar c) = (a\bigstar b)\bigstar c$*, (associativity);*
(P2) $a\bigstar e = e\bigstar a = a$*, (identity);*
(P3) For every $a \in P$*, there is some* $a^{-1} \in P$ *such that* $a\bigstar a^{-1} = a^{-1}\bigstar a = e$*.*

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(inverse);
(P4) For every $a \in P$, there are some $d, f \in P$ such that $b \bigstar f = f \bigstar d = a$ with $b \neq d$. (escort).

A set P together with an operation $\bigstar : P \times P \to P$ and an element e satisfying only conditions (P1), (P2), (P3) and (P4) is called a poloid. It is denoted by $(P, *)$ in [2, 5, 6].

The set P that satisfies the (P1), (P2) and (P3) conditions is called a monoid considered in [1, 7, 8, 9]. An element e satisfying (P1) is called a neutral unit element in [3, 4]. If $(P, \bigstar)$ is a poloid and $a \in P$ then

$$a \bigstar a = a \Longleftrightarrow a = e.$$

Norwegian mathematician Niels Henrik Abel, (1802–1829) abelian, a group took the name abelian or commutative group by satisfying the following property: For all $x, y \in P$

$$x \bigstar y = y \bigstar x \text{ [5]}.$$

A poloid is an algebraic structure between the abelian group and the monoid or non-commutative group. All elements of a poloid can be commutative and some of its elements can be commutative according to the given operation. Briefly, the monoid, i.e., the group structure, expanded with the condition (P4).

Lemma 1.1. *(See [2, 3, 4, 10, 11]). Let $(P, \bigstar)$ be a poloid and for all $a, b, c, d \in P$ the followings hold.*

(i) If e' is a second such unit element, then

$$e' = e.$$

(ii) If $b \bigstar a = e$ and $a \bigstar c = e$, then

$$b = c.$$

(iii) If $a \bigstar b = a \bigstar c$ and $b \bigstar a = c \bigstar a$, then $b = c$.

(iv) If $e \notin \{a, b, c\}$, $c \neq a$, $b \bigstar a = c$ and $a \bigstar d = c$, then

$$d \neq b.$$

Proof. Let $(P, \bigstar)$ be a poloid. For any $a, b, c, d \in P$.

(i) If e' is a second such unit element, then,

$$e' = e' \bigstar e = e.$$

(ii) If $b \bigstar a = e$ and $a \bigstar c = e$, then

$$b = b \bigstar e = b \bigstar (a \bigstar c) = (b \bigstar a) \bigstar c = e \bigstar c = c.$$

(iii) If $a \bigstar b = a \bigstar c$, then,

$$a \bigstar b = a \bigstar c \Longrightarrow a^{-1} \bigstar (a \bigstar b) = a^{-1} \bigstar (a \bigstar c)$$

$$\Longrightarrow (a^{-1} \bigstar a) \bigstar b = (a^{-1} \bigstar a) \bigstar c \Longrightarrow b = c,$$

and

$$b \bigstar a = c \bigstar a \Longrightarrow (b \bigstar a) \bigstar a^{-1} = (c \bigstar a) \bigstar a^{-1}$$

$$\Longrightarrow b \bigstar (a \bigstar a^{-1}) = c \bigstar (a \bigstar a^{-1}) \Longrightarrow b = c.$$

(iv) If $e \notin \{a, b, c\}$, $c \neq a$, $b \bigstar a = c$ and $a \bigstar d = c$, then,

$$b \bigstar a = c \Longrightarrow b = c \bigstar a^{-1},$$

and

$$a \bigstar d = c \Longrightarrow d = a^{-1} \bigstar c$$

$$b \neq d.$$

□

The set of all matrices of order n over a field F is denoted by $\mathbb{M}_n(F)$.

Theorem 1.1. *(See [2, 6, 7]). Let $A, B, X \in M_n(F)$ be such that regular matrices and X unknowns matrix. Then, in the solution of the equation $AX = B$, there are regular matrices $A = B_2A_3$,$B = B_2B_3$, such as B_2, A_3 and B_3, and the rational matrix $\dfrac{B_3}{A_3}$ is the solution of the equation $AX = B$. This solution is equal to the rational matrix $\dfrac{B}{A}$.*

Matrices obtained as a result of these operations are also regular.

2 Poloid and Monoid

The fact that the matrices are poloid allowed the Theorem 1.1. given above to be extended over poloid.

Theorem 2.1. *Let* $(P, \bigstar)$ *be a poloid. For all* $a \in P$ *and* $i, j, k = 1, ..., n;\ n \in \mathbb{Z}^+, \exists\, p_i, p_j, p_k \in P$*, then*

$$a = p_i \bigstar p_j \text{ and } a = p_j \bigstar p_k, \text{ where } p_i \neq p_j \neq p_k \text{ for } i \neq j \neq k.$$

Proof. In a poloid, each element is written in terms of another element. □

Corollary 2.1. *Let* $a_i, a_k \in P$*. Then,*

(i) $e = a_i \bigstar a_k$.

(ii) $a_i^{-1} = a_k$.

Proof. Let $a_i, a_k \in P$. Then,

(i)

$$p_j = a \bigstar a_i, p_k = a \bigstar a_k$$

$$a = p_j \bigstar p_k = a \bigstar (a_i \bigstar a_k)$$

$$e = a_i \bigstar a_k.$$

As the result of (i), the following (ii) is written.

(ii) $a_i^{-1} = a_k$.

□

Theorem 2.2. *Let* $(P, \bigstar)$ *be a poloid, let* $p \in P \diagdown \{e\}$*, be a constant and* $n \in \mathbb{Z}^+$ *for* $i = 1, ..., n,$ *and* $\forall\, p_i \in P \diagdown \{p, e\}$*, then the followings hold.*

(i)

$$p_i = p \bigstar q_i, \text{ where } q_i \in P \diagdown \{e\}.$$

(ii)

$$p_i = r_i \bigstar p, \text{ where } r_i \in P \diagdown \{e\}.$$

Proof. The proof of the theorem is done explicitly by the Lemma 1.1. (iii),(iv) and (P4). □

Theorem 1.1. extends clause (P4) even more. The generator definition of a poloid is given below for the first time.

Definition 2.1. *Let* $(P, \bigstar)$ *be a poloid and let* $c \in P$ *be a constant element. If* $p = c\bigstar p_1, \exists\, p_1 \in P$ *for all* $p \in P$ *then, the* $(P, \bigstar)$ *poloid is called to be generate by element c. It is denoted by* $\langle c \rangle$, *and*

$$\langle c \rangle = \{x | x = c\bigstar c_i \in P, i \in \mathbb{Z}^+\}.$$

The existence of c_i for generator c is obvious from condition P4.

Theorem 2.3. *Let* $(P, \bigstar)$ *be a poloid. Then the followings hold.*

(i) For all $p \in P$,

$$\langle p \rangle = P.$$

(ii) For all $p_n \in P, n \in \mathbb{Z}^+$,

$$P = \cap_{n=1}^{\infty} \langle p_n \rangle.$$

Proof. Let $(P, \bigstar)$ be a poloid.

(i) For any $p \in P$,

$$\langle p \rangle = \{x | x = p\bigstar p_n \in P, \exists\, n \in \mathbb{Z}^+\}.$$

$$\langle p \rangle = P.$$

(ii) For all $p_n \neq p$, $p_n \in P$,

$$p_n \in \langle p \rangle.$$

It is clear that $\langle p \rangle = \langle p_n \rangle$. By Theorem 2.3. (i),

$$P = \cap_{n=1}^{\infty} \langle p_n \rangle.$$

□

3 Conclusions and Discussions

Discussing the intricacies between poloid and group structures led to the following results.

(i) Each non-unit element of the poloid is written in terms of other elements.

(ii) Some elements of the poloid are commutative.

(iii) Each element of the poloid is represented by the same element.

(iv) The results of two or more binary operations to be defined to the poloid structure are left as the subject of research.

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