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Zero inclusions and zero free regions for kakeya type polynomials

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Abstract

In this paper, we prove some interesting generalizations of results concerning the Eneström- Kakeya Theorem by giving more relaxation to only some of the coefficients of a polynomial.

1 Introduction and statement of results

In the present paper, our interest will be centered upon the study of bounds for the zeros of polynomials as functions of all their coefficients. Among the first contributors to this problem were Gauss and Cauchy [5]. Since then a lot of papers devoted to give new bounds for the zeros has been written contributing to the further growth of the subject [9]. However, restricted coefficients (real or complex numbers) often have appeared as part of close expressions of bounds ([1]-[3], [11]). In this paper we determine in the complex plane, the disks containing all the zeros of a polynomial involving restricted coefficients and the parameters which can be adapted appropriately to the intensity required.

A classical result of Cauchy which gives a bound for the moduli of all the zeros of a polynomial in terms of its coefficients is stated as follows.

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Theorem A

All the zeros of the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z| < 1 + M$, where $M = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|$.

The remarkable property of the bound in Theorem A is its simplicity of computations, which makes it rare from other such bounds. Cauchy type polynomials have been studied extensively in the past more than one-century. The research associated with this has sprawled into several directions and generates a plethora of publications for example see ([8], [9], [10], [14]). The research on mathematical objects associated with polynomials and relative position of their zeros has been active over a period; there are many research papers published in a variety of journals each year and different approaches have been taken for different purposes. The present article is concerned with zero containing and zero free regions for the polynomials with constraints over coefficients.

As a refinement of the Cauchy bound, but under the restricted coefficients the following result which is well known as Eneström Kakeya Theorem ([9], [10]) has made a great deal of progress towards the establishment for the moduli of zeros of a polynomial.

Theorem B

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n satisfying

$$a_n \geq a_{n-1} \geq ... \geq a_0 > 0,$$

then $P(z)$ has all its zeros in $|z| \leq 1$.

Several extensions and generalizations of Eneström - Kakeya Theorem are available in literature (viz.; [1]-[4], [6]). Joyal et al. [8] extended Eneström - Kakeya Theorem to the polynomials whose coefficients are monotonic but not necessarily non negative. In fact they proved the following result.

Theorem C

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0,$$

then $P(z)$ has all its zeros in the disk

$$|z| \leq \frac{|a_n| - a_0 + |a_0|}{|a_n|}.$$

Aziz and Zargar [4] relaxed the hypothesis of Theorem B in several ways and have proved some extensions and generalizations of this result and among the other things they proved.

Theorem D

If $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$ and $0 < \rho \leq 1$,

$$ka_n \geq a_{n-1} \geq ... \geq a_1 \geq \rho a_0 > 0, \tag{1.1}$$

then $P(z)$ has all its zeros in the disk

$$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho).$$

Remark

Since the circle $|z+k-1| \leq k+\frac{2a_0}{a_n}(1-\rho)$ is contained in the circle $|z| \leq (2k-1)+\frac{2a_0}{a_n}(1 - \rho)$, therefore it follows that all the zeros of polynomial $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ satisfying (1) lie in the circle $|z| \leq (2k - 1) + \frac{2a_0}{a_n}(1 - \rho)$.

Recently, Shah et al. [13] have proved the following generalization of Theorem C by relaxing the monotonicity of some coefficients. In fact, they proved the following result.

Theorem E

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_p z^p + a_{p-1} z^{p-1} + ... + a_q z^q + a_{q-1} z^{q-1} + ... + a_1 z + a_0$ be a polynomial of degree n satisfying

$$a_p \geq a_{p-1} \geq ... \geq a_q, \quad p \geq q,$$

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \;\; and \;\; N_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{M_p + N_q + a_p - a_q + |a_0|}{|a_n|}.$$

In the same paper, they also proved the following result concerning the zero free regions of a polynomial.

Theorem F

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_p z^p + a_{p-1} z^{p-1} + ... + a_q z^q + a_{q-1} z^{q-1} + ... + a_1 z + a_0$ be a polynomial of degree n satisfying

$$a_p \geq a_{p-1} \geq ... \geq a_q, \quad p \geq q,$$

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \quad and \quad N_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Then $P(z)$ does not vanish in

$$|z| < \min\left[1, \frac{|a_0|}{M_p + N_q - a_q + a_p + |a_n|}\right].$$

These results proved to be, each in its own way, enabling the growth of sophisticated techniques and critical practices are foundational in the development of the geometry of the zeros of univariate complex polynomials.

2 Main Results

The main purpose of this paper is to present some more generalizations of the above results by relaxing the hypothesis. Here we shall first prove the following generalization of Theorem E.

Theorem 2.1

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_p z^p + a_{p-1} z^{p-1} + ... + a_q z^q + a_{q-1} z^{q-1} + ... + a_1 z + a_0$ be a polynomial of degree n such that for some $k \geq 1$ and $0 < \sigma \leq 1$

$$k a_p \geq a_{p-1} \geq ... \geq \sigma a_q, \quad p \geq q,$$

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \quad and \quad N_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{(M_p + N_q + (k-1)|a_p| + ka_p - \sigma a_q + (1-\sigma)|a_q| + |a_0|)}{|a_n|}.$$

Remark 2.2

Theorem 2.1 is applicable to the situations where Theorems C and E gives no information. To see this consider the polynomial

$$P(z) = 20z^6 + 22z^5 + 14z^4 + 15z^3 + 16z^2 + 15z + 10.$$

Here $a_n = 20, \quad n = 6, \quad a_p = 14, \quad p = 4, \quad q = 2, \quad a_q = 16, \quad a_0 = 10, \quad M_p = 10, \quad N_q = 6.$

Then the Theorems C and E are not applicable. Setting $k = \frac{15}{14}$ and $\sigma = \frac{8}{9}$, we get by Theorem 2.1 all the zeros of $P(z)$ lie in

$$\begin{aligned}|z| &\leq \frac{M_p + N_q + (k-1)|a_p| + ka_p + (1-\sigma)|a_q| - \sigma a_q + |a_0|}{|a_n|} \\ &= \frac{16 + 1 + 15 - 12.44 + 10}{20} \\ &= 1.48.\end{aligned}$$

By Cauchy's Theorem all the zeros of $P(z)$ lie in

$$|z| \leq 2.1.$$

Hence, the bound obtained from Theorem 2.1 is better than Cauchy's bound.

Taking $p = n$ and $q = 0$ in Theorem 2.1, we get the following result.

Corollary 2.3

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_p z^p + a_{p-1} z^{p-1} + ... + a_q z^q + a_{q-1} z^{q-1} + ... + a_1 z + a_0$ be a polynomial of degree n such that for some $k \geq 1$ and $0 < \sigma \leq 1$,

$$ka_n \geq a_{n-1} \geq ... \geq \sigma a_0,$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{(k-1)|a_n| + ka_n - \sigma a_0 + (2-\sigma)|a_0|}{|a_n|}.$$

Remark 2.4

1 If we assume $a_0 > 0$ in Corollary 2.3, it reduces to Theorem D and in addition to the above condition if $k = 1, \quad \sigma = 1$, then it reduces to Theorem B.

2 Taking $k = 1$ and $\sigma = 1$ in Theorem 2.1, it reduces to Theorem E.

3 Taking $k = 1, \ \sigma = 1$ and $q = 0$ in Theorem 2.1, it reduces to a result of M. A. Shah [12][Theorem 1].

4 For $k = 1, \ p = n, \quad q = 0$ and $\sigma = 1$, Theorem 2.1 reduces to Theorem C.

Applying Theorem 2.1 to the polynomial $P(tz)$, we get the following result:

Corollary 2.5

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_p z^p + a_{p-1} z^{p-1} + ... + a_q z^q + a_{q-1} z^{q-1} + ... + a_1 z + a_0$ be a polynomial of degree n such that for some $t > 0, \quad k \geq 1$ and $0 < \sigma \leq 1$,

$$kt^p a_p \geq t^{p-1} a_{p-1} \geq ... \geq \sigma t^q a_q, \quad p \geq q,$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \sum_{j=p+1}^{n} \frac{|ta_j - a_{j-1}|}{|a_n| t^{n-j+1}} + \sum_{j=1}^{q} \frac{|ta_j - a_{j-1}|}{|a_n| t^{n-j+1}} + \frac{(k-1)t^p |a_p| + kt^p a_p - t^q a_q + (1-\sigma) t^q |a_q| + |a_0|}{t^n |a_n|}.$$

Our next observation is the zero free region of a polynomial. It asserts that none of the zeros of a polynomial under restricted coefficients lie in a given disk. In fact, we prove the following result, which in particular yields Theorem F as a special case.

Theorem 2.6

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_p z^p + a_{p-1} z^{p-1} + ... + a_q z^q + a_{q-1} z^{q-1} + ... + a_1 z + a_0$ be a polynomial of degree n such that for some $k \geq 1$ and $0 < \sigma \leq 1$,

$$ka_p \geq a_{p-1} \geq ... \geq \sigma a_q, \quad p \geq q,$$

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \quad and \quad N_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Then $P(z)$ does not vanish in

$$|z| < \min\left[1, \frac{|a_0|}{M_p + N_q + (k-1)|a_p| + ka_p + (1-\sigma)|a_q| - \sigma a_q + |a_n|}\right].$$

Remark 2.7

Consider the polynomial

$$P(z) = 18z^8 + 20z^7 + 21z^6 + 18z^5 + 19z^4 + 17z^3 + 18z^2 + 15z + 12.$$

Here $a_n = 18, \quad n = 8, \quad a_p = 18, \quad p = 5, \quad q = 2, \quad a_q = 18, \quad a_0 = 12, \quad M_p = 6, \quad N_q = 6.$
Then the Theorem F is not applicable. Setting $k = \frac{19}{18}$ and $\sigma = \frac{7}{9}$, we get by Theorem 2.6 that $P(z)$ does not vanish in

$$\begin{aligned} |z| &< \min\left[1, \frac{|a_0|}{M_p + N_q + (k-1)|a_p| + ka_p + (1-\sigma)|a_q| - \sigma a_q + |a_n|}\right] \\ &= \min\left[1, \frac{12}{6+6+1+19+4-14+18}\right], \end{aligned}$$

i.e.,

$$|z| < \min(1, 0.3),$$

i.e.,

$$|z| < 0.3.$$

Takin g $p = n$ and $q = 0$ in Theorem 2.6, we get the following result:

Corollary 2.8

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n satisfying

$$ka_n \geq a_{n-1} \geq ... \geq \sigma a_0,$$

then $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{2k|a_n| + (1-2\sigma)|a_0|}.$$

Remark 2.9

1 If we assume $a_0 > 0$ in Corollary 2.8 , then it reduces to a result of M .H. Gulzar [7] and in addition to above condition if $k = 1$ and $\sigma = 1$, it reduces to a result of B. A. Zargar [15].

2 By taking $k = 1$ and $\sigma = 1$ in Theorem 2.6, it reduces to Theorem F.

3 By taking $k = 1$, $\sigma = 1$ and $q = 0$ in Theorem 2.6, it reduces to a result of M. A. Shah [13][Theorem 2].

Taking $p = n$, $k = 1$, $\sigma = 1$ and $q = 0$ in Theorem 2.6, we get the following result.

Corollary 2.10

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n satisfying

$$a_n \geq a_{n-1} \geq ... \geq a_0,$$

then $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0}.$$

3 Proofs of the Theorems

Proof of Theorem 2.1

In order to prove the theorem we consider the polynomial

$$\begin{aligned} G(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p) \\ &\quad z^{p+1} + (a_p - a_{p-1}) + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \ldots (a_q - \\ &\quad a_0)z + a_0 = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + \\ &\quad (a_{p+1} - a_p)z^{p+1} + (a_p - ka_p + ka_p - a_{p-1})z^p + \cdots + (a_{q+1}\sigma a_q + \sigma a_q) \\ &\quad z^{q+1} + (a_q - a_{q-1})z^q + \ldots (a_1 - a_0)z + a_0. \end{aligned}$$

Which implies

$$||G(z)|a_n||z|^{n+1} - \big[|(a_n - _{n-1})z^n + (a_{n-1} - a_{n-2})z^{n+1} + \cdots + (a_{p+1} - a_p) z^{p+1} + (a_p - ka_p - a_{p-1})z^p + \cdots + (a_{q+1} - \sigma a_q + \sigma a_q - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - a_0)z + a_0|\big] \geq |a_n||z|^{n+1}z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{p+1} - a_p||z|^{p+1} + |(1-k)a_p||z|^p + |ka_p - a_{p-1})||z|^p + \cdots + |a_{q+1} - \sigma a_q||z|^{p+1} + |(\sigma - 1)a_q||z|^{q+1} + |a_q - a_{q-1}||z|^q + \cdots + |a_1 - a_0||z| + |a_0|\big]$$

$$= |z|^n \left[|a_n||z| - \left(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \ldots \frac{|a_{p+1} - a_p|}{|z|^{n-p-1}} + \frac{|a^{p+1} - a^p|}{|z|^{n-p-1}} + \frac{|(1-k)a_p|}{|z|^{n-p}} + \frac{|ka^p - a^{p-1}|}{|z|^{n-p}} + \cdots + \frac{|a_{q+1} - \sigma a_q|}{|z|^{n-q-1}} + \frac{|(\sigma - 1)a_q|}{|z|^{n-q-1}} + \frac{|a_q - a_{q-1}|}{|z|^{n-q}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\right)\right].$$

Now for $|z| > 1$ so that $\frac{1}{|z|^{n-j}} < 1,\ 0 \leq j \leq n$, $k \geq 1$ and $0 < \sigma \leq 1$, we have

$$|G(z)| \geq |z|^n \big[|a_n||z| - |(|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| + |(1-k)a_p|ka_p - a_{p-1} + \cdots + |a_{q+1} - \sigma a_a|(\sigma - 1)a_q - a_{q-1}| + \cdots + |a_1 - a_0 + |a_0|)\big] = |z|^n \big[|a_n - (|a_n - a_{n-1}| + |a^{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| - a_p| + (k-1)|a_p| + ka_p - a_p - a^{p-1} + a_{p-1} - a_{p-2} + \cdots + a_{q+1} - a_{q+1} - a_{q+1} + a_{q+1} - \sigma a_q + (1-\sigma)|a_q| + |a_q - a_{q-1}| + \ldots |a_1 - a_0| + a - 0|)\big]$$

$$= |z|^n \left[|a_n||z| - \left(\sum_{j=p+1}^{n} |a_j - a_{j-1}| + \sum_{j=1}^{q} |a_j - a_{j-1}| + (k-1)|a_p|| + ka_p - \sigma a_p + (1-\sigma)|a_q| + |a_0|\right)\right] = |z|^n \big[|a^n||z| - (M_p + N_q + (k-1)|a_p| + ka_p - \sigma a_q + (1-\sigma)\ |a_q| + |a_0|)\big] > 0,$$

if

$$|z||a_n| > (M_p + N_q + (k-1)|a_p| + ka_p - \sigma a_q + (1-\sigma)|a_q| + |a_0|).$$

That is, if

$$|z| > \frac{(M_p + N_q + (k-1)|a_p| + ka_p - \sigma a_q + (1-\sigma)|a_q| + |a_0|)}{|a_n|}.$$

Thus all the zeros of $G(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \leq \frac{(M_p + N_q + (k-1)|a_p| + ka_p - \sigma a_q + (1-\sigma)|a_q| + |a_0|)}{|a_n|}.$$

But the zeros of $G(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are the zeros of $G(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{(M_p + N_q + (k-1)|a_p| + ka_p - \sigma a_q + (1-\sigma)|a_q| + |a_0|)}{|a_n|}.$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.6

Consider the reciprocal polynomial

$$S(z) = z^n P(\frac{1}{z}) = a_0 z^n + a_1 z^{n-1} + ... + a_q z^{n-q} + ... + a_p z^{n-p} + ...+ \\ a_{n-1} z + a_n.$$

Let

$$H(z) = (1-z)S(z) \\ = -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + ... + (a_q - a_{q+1})z^{n-q} + ... \\ + (a_p - a_{p+1})z^{n-p} + ... + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n.$$

This gives

$$|H(z)| \geq |a_0||z|^{n+1} - \big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{q-1} - a_q| \\ |z|^{n-q+1} + |a_q - \sigma a_q + \sigma a_q - a_{q+1}||z|^{n-q} + ... + |a_{p-1} - ka_p + ka_p - a_p| \\ |z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + ... + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\big] \\ = |z|^n \Bigg[|a_0||z| - \Bigg(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + ... + \frac{|a_{q-1} - a_q|}{|z|^{q-1}} + \frac{|(1-\sigma)a_q|}{|z|^q} \\ + \frac{|\sigma a_q - a_{q+1}|}{|z|^q} + ... + \frac{|(k-1)a_p|}{|z|^{p-1}} + \frac{|a_{p-1} - ka_p|}{|z|^{p-1}} + \frac{|a_p - a_{p+1}|}{|z|^p} + ...+ \\ \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\Bigg)\Bigg].$$

Now for $|z| > 1$ so that $\frac{1}{|z|^{n-j}} < 1,\ 0 \leq j \leq n$, $k \geq 1$ and $0 < \sigma \leq 1$, we have

$$|H(z)| \geq |z|^n \left[|a_0||z| - (|a_0 - a_1| + |a_1 - a_2| + ... + |a_{q-1} - a_q| + |(1-\sigma)a_q| \right.$$
$$+|\sigma a_q - a_{q+1}| + ... + |a_{p-1} - ka_p| + |(k-1)a_p| + |a_p - a_{p+1}| + ...+$$
$$\left. |a_{n-1} - a_n| + |a_n|)\right]$$
$$= |z|^n \left[|a_0||z| - \left(\sum_{j=p+1}^{n} |a_j - a_{j-1}| + \sum_{j+1}^{q} |a_j - a_{j-1}| + (1-\sigma)|a_q| + a_{q+1} \right.\right.$$
$$-\sigma a_q + a_{q+2} - a_{q+1} + ... + a_{p-2} - a_{p-3} + a_{p-1} - a_{p-2} + ka_p - a_{p-1}$$
$$\left.\left. +(k-1)|a_p| + |a_n| \right)\right]$$
$$= |z|^n \left[|a_0||z| - (M_p + N_q + ka_p + (k-1)|a_p| + (1-\sigma)|a_q| - \sigma a_q + |a_n|)\right]$$
$$> 0,$$

if

$$|z| > \frac{M_p + N_q + ka_p + (k-1)|a_p| + (1-\sigma)|a_q| - \sigma a_q + |a_n|}{|a_0|},$$

where

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \quad and \quad N_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Thus all the zeros of $H(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{M_p + N_q + ka_p + (k-1)|a_p| + (1-\sigma)|a_q| - \sigma a_q + |a_n|}{|a_0|}.$$

Hence all the zeros of $H(z)$ and hence of $S(z)$ lie in

$$|z| \leq \max \left[1, \frac{|a_0|}{M_p + N_q + ka_p + (k-1)|a_p| + (1-\sigma)|a_q| - \sigma a_q + |a_n|}\right].$$

Therefore all the zeros of $P(z)$ lie in

$$|z| \geq \min \left[1, \frac{|a_0|}{M_p + N_q + ka_p + (k-1)|a_p| + (1-\sigma)|a_q| - \sigma a_q + |a_n|}\right].$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < \min \left[1, \frac{|a_0|}{M_p + N_q + ka_p + (k-1)|a_p| + (1-\sigma)|a_q| - \sigma a_q + |a_n|}\right].$$

This completes the proof of Theorem 2.6.

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