ISSN: 0304-9787
The Aligarh Bulletin of Mathematics
Volume 42, Number 1 (2023), 107-123

Some multi figurate numbers in terms of generalized fibonacci and lucas numbers

Ahmet Emin

Karabuk University, Faculty of Science
Department of Mathematics, 78050, Karabuk, Turkiye
Email: ahmetemin@karabuk.edu.tr

(Received: June 23, 2023 Accepted: July 22, 2023)

Abstract

The purpose of this work is to find some multi figurate numbers. Find positive solutions of various Pell equations and show that these solutions are in terms of generalized Fibonacci and Lucas numbers.

1 Introduction and Motivation

Figurate numbers or n-gonal numbers are a sequence of positive integers that stand for regular geometric forms. Polygonal numbers (i.e., triangular numbers, square numbers, heptagonal numbers) and centered polygonal numbers (i.e., centered triangular numbers, centered square numbers) are such figurate numbers. The study of figurate numbers is an interesting subject of number theory since contributions have been made by many famous mathematicians who have revolutionized their work since ancient times. For more information about figurate numbers, see [4] and [13].

Multi figurate numbers are numbers that can simultaneously be different figurate numbers. For example, 36 is simultaneously a triangular and a square number. That's why the number 36 is called a triangular square number. There are many such multi figurate numbers, for example, heptagonal triangular numbers, square centered square numbers, etc.

Keywords and phrases: Polygonal numbers, centered polygonal numbers, Pell equation, simple continued fraction.

2020 AMS Subject Classification: 11B37,11B39,11B99,11D04,11D09

There are many methods for determining multi figurate numbers. One of the methods used is to find the solution of Diophantine equations (Pell equations). Pell equation is in the form of $x^2 - dy^2 = N$ where d is not a square positive integer. If $N = 1$, then it's called a classical Pell equation and it always has a solution on $\mathbb{N}$. If $N \neq 1$, then the Pell equation might not have a natural number solution. For more details about Pell equation, see [1], [2], [7], [8], [9], [10] and [14].

In [4], E. Deza and M. M. Deza, gave the general definition of figurate numbers and the properties of these numbers in detail. They also explained how to find the indices of two or more figurate numbers that give the terms that are equal to each other. In [12], R. Keskin and M. G. Duman, developed an unusual method while investigating the natural number solutions of various Pell equations. Using the method they developed, they solved many Pell equations in terms of generalized Fibonacci and Lucas numbers. While investigating the solutions of the multi figurate numbers defined in [4], it was observed that the terms of the multi figurate numbers were not examined with regard to Generalized Fibonacci and Lucas numbers. Inspired by [4] and [12], the motivation of this work is to investigate multi figurate numbers whose terms are generalized Fibonacci and Lucas numbers.

In section 2, we will mention the fundamentals of figurate numbers. We divide section 3 in three subsections, namely triangular square numbers, square centered square numbers and triangular heptagonal numbers.

2 Preliminaries

Let μ and τ be two nonzero integers and let $\mu^2 - 4\tau > 0$. The integer sequences $U_n(\mu, \tau)$ which is called generalized Fibonacci sequence, is defined by

$$U_{n+1}(\mu, \tau) = \mu U_n(\mu, \tau) - \tau U_{n-1}(\mu, \tau) \tag{2.1}$$

with initial $U_0(\mu, \tau) = 0$ and $U_1(\mu, \tau) = 1$, for $n \geq 1$. Similarly, the integer sequence $V_n(\mu, \tau)$ which is called generalized Lucas sequence, is defined by

$$V_{n+1}(\mu, \tau) = \mu V_n(\mu, \tau) - \tau V_{n-1}(\mu, \tau) \tag{2.2}$$

with initial $V_0(\mu, \tau) = 2$ and $V_1(\mu, \tau) = \mu$, for $n \geq 1$. If one takes $\mu = 1$ and $\tau = -1$, then the sequences $U_n(1, -1)$ and $V_n(1, -1)$ are called Fibonacci and Lucas sequences indicated by F_n and L_n, respectively. If one takes $\mu = 2$ and $\tau = -1$, then the sequences $U_n(2, -1)$ and $V_n(2, -1)$ are called Pell and Pell-Lucas sequences indicated by P_n and Q_n, respectively.

Let α and β be the roots of the characteristic quadratic equation $x^2-\mu x+\tau=0$. Then generalized Fibonacci sequences can be symbolized in the Binet form which is,

$$U_n(\mu,\tau)=\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

for $n\geq 0$. Likewise Lucas sequences can be symbolized in the Binet form which is,

$$V_n(\mu,\tau)=\alpha^n+\beta^n$$

for $n\geq 0$. Obviously, $\alpha+\beta=\mu$, $\alpha-\beta=\sqrt{\mu^2-4\tau}$ and $\alpha\beta=\tau$. Recall that, from (2.1) and (2.2),

$U_n(1,-1)=F_n$, Fibonacci numbers [16, A000045],
$V_n(1,-1)=L_n$, Lucas numbers [16, A000032],
$U_n(2,-1)=P_n$, Pell numbers [16, A000129],
$V_n(2,-1)=Q_n$, Pell- Lucas numbers [16, A002203].

For further information, see [3], [5], [6], [12] and [15].

Pell equation is of the form $x^2-dy^2=N$ where d is square free positive integer. If $N\neq 1$, then the Pell equation may not always have a natural number solution. If $N=1$, then it's called a classical Pell equation and it always has a natural number solution. The least natural number solution of the Pell equation $x^2-dy^2=1$ is called the fundamental solution and denoted by (x_1,y_1) or $x_1+y_1\sqrt{d}$. The other natural number solutions of the Pell equation $x^2-dy^2=1$ are derived with the help of the fundamental solution (x_1,y_1). If (x_1,y_1) is the fundamental solution of $x^2-dy^2=1$, then all the other positive integer solution (x_n,y_n) of $x^2-dy^2=1$ are given by (see in [11])

$$x_n+y_n\sqrt{d}=\left(x_1+y_1\sqrt{d}\right)^n \tag{2.3}$$

for $n\geq 1$. There are many different methods available in order to find the fundamental solution. In this study, we will use the continued fraction expansion of $\sqrt{d}$, which is the most well-known method. Let the continued fraction expansion of $\sqrt{d}$ be $\left[a_0;\overline{a_1,a_2,...,a_{l-1},a_l=2a_0}\right]$, in which l is the length of the period of this expansion and $a_0=\lfloor d\rfloor$. The continued fraction algorithm is given by;

$$\alpha_0=\sqrt{d},\ a_k=\lfloor\alpha_k\rfloor,\alpha_{k+1}=\frac{1}{\alpha_k-a_k},\ k\geq 0,k\in\mathbb{Z}. \tag{2.4}$$

The m^{th} convergent of $\sqrt{d}$ is given by,

$$\frac{p_m}{q_m} = [a_0, a_1, a_2, ..., a_{m-1}, a_m] \tag{2.5}$$

where $(p_m, q_m) = 1$ and for $m \geq 0$.

A polygonal number is a sequence of natural numbers and for $n \in \mathbb{N}$, $m \geq 3$, the n^{th} m-gonal numbers is denoted by $S_m(n)$. Algebraically $S_m(n)$ is acquired as the sum of the first n elements of the sequence, which are 1, $1 + (m-2)$, $1 + 2(m-2)$,..., $1 + (n-1)(m-2)$. Precisely,

$$\begin{aligned} S_m(n) &= 1 + 1 + (m-2) + 1 + 2(m-2) + ... + 1 + (n-1)(m-2) \\ &= n + (m-2)(1 + 2 + ... + (n-1)) \\ &= n + (m-2)\frac{(n-1)n}{2} \\ &= \frac{(m-2)n^2 - (m-4)n}{2}, \end{aligned}$$

(see [4]). In particular,
if $m = 3$, then $S_3(n)$ is called triangular number and $S_3(n) = \frac{n(n+1)}{2}$,
if $m = 4$, then $S_4(n)$ is called square number and $S_4(n) = n^2$,
if $m = 5$, then $S_5(n)$ is called pentagonal number and $S_5(n) = \frac{n(3n-1)}{2}$,
if $m = 6$, then $S_6(n)$ is called hexagonal number and $= S_6(n) = n(2n-1)$,
if $m = 7$, then $S_7(n)$ is called heptagonal number and $S_7(n) = \frac{n(5n-3)}{2}$.

A centered polygonal number is a sequence of natural numbers and for $n \in \mathbb{N}$, $m \geq 3$, the n^{th} centered m-gonal numbers is denoted by $CS_m(n)$. Algebraically $CS_m(n)$ is acquired as the sum of the first n elements of the sequence, which are 1, m, $2m$,..., $(n-1)m$. One can calculate that,

$$\begin{aligned} CS_m(n) &= 1 + m + 2m + 3m + ... + (n-1)m \\ &= 1 + m(1 + 2 + ... + (n-1)) \\ &= 1 + m\frac{(n-1)n}{2} \\ &= \frac{mn^2 - mn + 2}{2}. \end{aligned}$$

(see [4]). Particularly,
if $m = 3$, then $CS_3(n)$ is called centered triangular number and $CS_3(n) = \frac{3n^2-3n+2}{2}$,

if $m = 4$, then $CS_4(n)$ is called centered square number and $CS_4(n) = 2n^2 - 2n + 1$,
if $m = 5$, then $CS_5(n)$ is called centered pentagonal number and $CS_5(n) = \frac{5n^2-5n+2}{2}$.

3 Some Multi Figurate Numbers

In this section, we will discuss some particular multi figurate numbers and obtain the main results of the paper.

3.1 Triangular Square Numbers

In this part of the article, it will be investigated exactly which numbers are both triangular numbers and square numbers. While investigating these numbers, Pell equation, Pell and Pell-Lucas sequences will be used. Firstly, $S_3(u)$ and $S_4(v)$ equations are equated to obtain the diophantine equation

$$\frac{u(u+1)}{2} = v^2. \tag{3.1}$$

We can rewrite equation (3.1) as

$$(2u+1)^2 - 2(2v)^2 = 1. \tag{3.2}$$

If we take $x = 2u + 1$ and $y = 2v$, then we obtain the Pell equation

$$x^2 - 2y^2 = 1. \tag{3.3}$$

To find the positive solutions of the Pell equation (3.3), the following theorem in [8] should be given.

Theorem 3.1. *Let l be the period length of continued fraction expansion of $\sqrt{d}$. When l is even, the positive solutions of the Pell equation $x^2 - dy^2 = 1$ are $x = p_{jl-1},\ y = q_{jl-1},\ j = 1, 2, 3, ...$ When l is odd, the positive solutions of the Pell equation $x^2 - dy^2 = 1$ are $x = p_{2jl-1}, y = q_{2jl-1},\ j = 1, 2, 3, ...$*

In order to find the fundamental solution of the Pell equation (3.3), the following Lemma 3.1 and Lemma 3.2 will be given. Also, these lemmas can be found in many basic textbooks.

Lemma 3.1. *The continued fraction expansion of* $\sqrt{2}$ *is* $\left[1;\overline{2}\right]$.

Proof. We use the continued fraction algorithm from (2.4).
Let $d=\sqrt{2}$, $\alpha_0=\sqrt{2}$ and $a_0=\left\lfloor\sqrt{2}\right\rfloor=1$,
$\alpha_1=\frac{1}{\alpha_0-a_0}=\frac{1}{\sqrt{2}-1}=\sqrt{2}+1$. So $a_1=\left\lfloor\sqrt{2}+1\right\rfloor=2$,
$\alpha_2=\frac{1}{\alpha_1-a_1}=\frac{1}{\left(\sqrt{2}+1\right)-2}=\sqrt{2}+1$. So $a_2=\left\lfloor\sqrt{2}+1\right\rfloor=2$.
If it continues in the same way, it seen that $a_n=2$. Hence we get $\sqrt{2}=[1;2,2,...,2,2\,]=\left[1;\overline{2}\right]$. □

Lemma 3.2. *The fundamental solution of the Pell equation* $x^2-2y^2=1$ *is* $(x_1,y_1)=(3,2)$.

Proof. From Lemma 3.1 we know that $\sqrt{2}=\left[1;\overline{2}\right]$ and from here the period length l is 1, that is, l is odd. From Theorem 3.1 the fundamental solution of $x^2-2y^2=1$ is $x_1=p_{2l-1}$ and $y_1=q_{2l-1}$. So $(x_1,y_1)=(p_1,q_1)$. From (2.5) we have $\frac{p_1}{q_1}=a_0+\frac{1}{a_1}=1+\frac{1}{2}=\frac{3}{2}$. Thus,$(x_1,y_1)=(p_1,q_1)=(3,2)$. Indeed, $x_1^2-2y_1^2=3^2-2.2^2=9-8=1$. □

Now, a method for finding all positive integer solutions of the Pell equation $x^2-2y^2=1$ will be given.

From equation (2.3), if (x_1,y_1) is the minimum solution of $x^2-dy^2=1$, then the other solutions of this equation are (x_n,y_n), in which $x_n+y_n\sqrt{d}=\left(x_1+y_1\sqrt{d}\right)^n$. From lemma 3.2 and equation (2.3) we can obtain some solutions of $x^2-2y^2=1$. $(x_1,y_1)=(3,2)$, and so

$$
\begin{aligned}
&x_2+y_2\sqrt{2}=\left(3+2\sqrt{2}\right)^2=17+12\sqrt{2}\Rightarrow(x_2,y_2)=(17,12)\,,\\
&x_3+y_3\sqrt{2}=\left(3+2\sqrt{2}\right)^3=99+70\sqrt{2}\Rightarrow(x_3,y_3)=(99,70)\,,\\
&x_4+y_4\sqrt{2}=\left(3+2\sqrt{2}\right)^4=577+408\sqrt{2}\Rightarrow(x_4,y_4)=(577,408)\,,\\
&x_5+y_5\sqrt{2}=\left(3+2\sqrt{2}\right)^5=3363+2378\sqrt{2}\Rightarrow(x_5,y_5)=(3363,2378)\,,\\
&x_6+y_6\sqrt{2}=\left(3+2\sqrt{2}\right)^6=19\,601+13860\sqrt{2}\Rightarrow(x_6,y_6)=(19\,601,\;13860)\,.
\end{aligned}
$$

If some solutions of this equation are examined carefully, it will be seen that its solutions are related to Pell and Pell-Lucas Numbers.

$$
\begin{aligned}
&(x_1,y_1)=(3,2)=\left(\tfrac{6}{2},\;2\right)=\left(\tfrac{Q_2}{2},P_2\right),\\
&(x_2,y_2)=(17,\;12)=\left(\tfrac{34}{2},\;12\right)=\left(\tfrac{Q_4}{2},P_4\right),
\end{aligned}
$$

$(x_3, y_3) = (99,\ 70) = \left(\frac{198}{2},\ 70\right) = \left(\frac{Q_6}{2}, P_6\right),$
$(x_4, y_4) = (577,\ 408) = \left(\frac{1154}{2},\ 408\right) = \left(\frac{Q_8}{2}, P_8\right),$
$(x_5, y_5) = (3363,\ 2378) = \left(\frac{6726}{2},\ 2378\right) = \left(\frac{Q_{10}}{2}, P_{10}\right),$
$(x_6, y_6) = (19601,\ 13860) = \left(\frac{39202}{2},\ 13860\right) = \left(\frac{Q_{12}}{2}, P_{12}\right).$

Now we can give the following Theorem which gives us the solutions of equation $x^2 - 2y^2 = 1$ in terms of Pell and Pell-Lucas numbers.

Theorem 3.2. *All positive integer solutions of the equation $x^2 - 2y^2 = 1$ are given by*

$$(x_n, y_n) = \left(\frac{Q_{2n}}{2}, P_{2n}\right)$$

for $n \geq 1$.

Proof. If the Pell and Pell-Lucas sequences are written in the form of the Binet's formula, we get, for $n \geq 1$,

$$P_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \text{ and } Q_{2n} = \alpha^{2n} + \beta^{2n}$$

where $\alpha = \frac{2+\sqrt{8}}{2} = 1 + \sqrt{2}$ and $\beta = \frac{2-\sqrt{8}}{2} = 1 - \sqrt{2}$. Obviously, we can obtain $\alpha + \beta = 2,\ \alpha - \beta = \sqrt{8}$ and $\alpha\beta = -1$.

$$\begin{aligned}
x^2 - 2y^2 &= \left(\frac{Q_{2n}}{2}\right)^2 - 2(P_{2n})^2 \\
&= \left(\frac{\alpha^{2n} + \beta^{2n}}{2}\right)^2 - 2\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)^2 \\
&= \left(\frac{\alpha^{2n} + \beta^{2n}}{2}\right)^2 - 2\left(\frac{\alpha^{2n} - \beta^{2n}}{\sqrt{8}}\right)^2 \\
&= \left(\frac{\alpha^{4n} + 2\alpha^{2n}\beta^{2n} + \beta^{4n}}{4}\right) - 2\left(\frac{\alpha^{4n} - 2\alpha^{2n}\beta^{2n} + \beta^{4n}}{8}\right) \\
&= \frac{4\alpha^{2n}\beta^{2n}}{4} \\
&= (\alpha\beta)^{2n} \\
&= 1.
\end{aligned}$$

□

Using relations $x = 2u + 1$ and $y = 2v$, we have

$$u_n = \tfrac{x_n - 1}{2} = \tfrac{\frac{Q_{2n}}{2} - 1}{2} \text{ and } v_n = \tfrac{y_n}{2} = \tfrac{P_{2n}}{2},$$

$n \in \mathbb{N}$. One gets the sequence (1,1), (8,6), (49,35), (288,204), (1681,1189), ... which consist of all the positive integer solutions (u_n, v_n) of the diophantine equation (3.1).

Finally, the following corollary will tell us for which (u_n, v_n) terms the diophantine equation $S_3(u) = S_4(v)$ is valid.

Corollary 3.1. *Let u and v be a positive integer. Then $S_3(u) = S_4(v)$ if and only if*

$$(u_n, v_n) = \left(\frac{Q_{2n} - 2}{4}, \frac{P_{2n}}{2}\right)$$

for $n \geq 1$.

3.2 Square Centered Square Numbers

In this subsection, we consider exactly which numbers are both square numbers and centered square numbers. While investigating these numbers, Pell equation, Pell and Pell-Lucas sequences will be used.

Firstly, $CS_4(v)$ and $S_4(u)$ equations are equated to obtain the diophantine equation

$$1 + 4\frac{v(v-1)}{2} = u^2. \tag{3.4}$$

If both sides of the equation (3.4) multiplied by 16, then we have,

$$(4u)^2 - 8(2v-1)^2 = 8. \tag{3.5}$$

If we take $x = 4u$ and $y = 2v - 1$, then we obtain the Pell equation

$$x^2 - 8y^2 = 8. \tag{3.6}$$

Now we can give the following lemma which gives us the fundamental solution of equation (3.6), see in [13].

Lemma 3.3. *The fundamental solution of the Pell equation $x^2 - 8y^2 = 8$ is $(x_1, y_1) = (4, 1)$.*

Proof. $y_1 = 1$ is the minimal positive integer number. So, $x_1^2 - 8y_1^2 = 8 \Rightarrow x_1^2 - 8.1^2 = 8 \Rightarrow x_1 = 4$. Hence, $(x_1, y_1) = (4, 1)$ is the fundamental solution of $x^2 - 8y^2 = 8$. □

In the following theorem, a method to find all natural number solutions of the Pell equation $x^2 - 8y^2 = 8$ will be given.

Theorem 3.3. *Let (x_m, y_m) be any solution of $x^2 - 8y^2 = 8$. Then the other solutions of this equation are (x_n, y_n), where*

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}^n \begin{pmatrix} x_m \\ y_m \end{pmatrix}$$

for $n \geq 1$.

Proof. By using the method of mathematical induction on n, we can prove this theorem. Let (x_m, y_m) be any solution of $x^2 - 8y^2 = 8$. That is ${x_m}^2 - 8{y_m}^2 = 8$. For $n = 1$, we get

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} 3x_m + 8y_m \\ x_m + 3y_m \end{pmatrix}.$$

We have, $x_1 = 3x_m + 8y_m$ and $y_1 = x_m + 3y_m$. Also,

$$\begin{aligned} {x_1}^2 - 8{y_1}^2 &= (3x_m + 8y_m)^2 - 8(x_m + 3y_m)^2 \\ &= (3x_m)^2 + 2\,(3x_m 8y_m) + (8y_m)^2 - 8(x_m)^2 - 2.8.\,(x_m 3y_m) - 8(3y_m)^2 \\ &= {x_m}^2\left(3^2 - 8\right) - 8{y_m}^2\left(3^2 - 8\right) \\ &= {x_m}^2 - 8{y_m}^2 \\ &= 8. \end{aligned}$$

Now we assume that (x_{n-1}, y_{n-1}) is a solution of $x^2 - 8y^2 = 8$. That is, ${x_{n-1}}^2 - 8{y_{n-1}}^2 = 8$.

$$\begin{aligned}\begin{pmatrix}x_n\\y_n\end{pmatrix} &= \begin{pmatrix}3&8\\1&3\end{pmatrix}^n \begin{pmatrix}x_m\\y_m\end{pmatrix}\\ &= \begin{pmatrix}3&8\\1&3\end{pmatrix}\begin{pmatrix}3&8\\1&3\end{pmatrix}^{n-1} \begin{pmatrix}x_m\\y_m\end{pmatrix}\\ &= \begin{pmatrix}3&8\\1&3\end{pmatrix}\begin{pmatrix}x_{n-1}\\y_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}3x_{n-1}+8y_{n-1}\\x_{n-1}+3y_{n-1}\end{pmatrix}.\end{aligned}$$

So, we get $x_n = 3x_{n-1} + 8y_{n-1}$ and $y_n = x_{n-1} + 3y_{n-1}$. If we substitute these two equations in the Pell equation $x^2 - 8y^2 = 8$, then we have

$$\begin{aligned}{x_n}^2 - 8{y_n}^2 &= (3x_{n-1} + 8y_{n-1})^2 - 8(x_{n-1} + 3y_{n-1})^2\\ &= (3x_{n-1})^2 + 2\,(3x_{n-1}8y_{n-1}) + (8y_{n-1})^2\\ &- 8(x_{n-1})^2 - 2.8.\,(x_{n-1}3y_{n-1}) - 8(3y_{n-1})^2\\ &= x_{n-1}^2\left(3^2-8\right) - 8y_{n-1}^2\left(3^2-8\right)\\ &= x_{n-1}^2 - 8y_{n-1}^2\\ &= 8.\end{aligned}$$

Thus, (x_n, y_n) is solution of $x^2 - 8y^2 = 8$. □

The following theorem is given in [13] and therefore we skip its proof. The statement of this theorem is important because it serves to find two consecutive positive integer solutions of this equation. In other words, this theorem states that there is no other integer solution between two consecutive integers.

Theorem 3.4. *The consecutive solutions* (x_{n-2}, y_{n-2}) , (x_{n-1}, y_{n-1})*and* (x_n, y_n) *satisfy the recurrence relations,* $x_n = 6x_{n-1} - x_{n-2}$ *and* $y_n = 6y_{n-1} - y_{n-2}$*, for* $n \geq 3$.

From lemma 3.3, we know that the fundamental solution of $x^2 - 8y^2 = 8$ is $(4, 1)$. From the proof of theorem 3.3, we can obtain some solutions of $x^2 - 8y^2 = 8$.

$x_2 = 3x_1 + 8y_1 = 3.4 + 8.1 = 20,$
$y_2 = x_1 + 3y_1 = 4 + 3.1 = 7,$

$x_3 = 3x_2 + 8y_2 = 3.20 + 8.7 = 116,$
$y_3 = x_2 + 3y_2 = 20 + 3.7 = 41,$

$x_4 = 3x_3 + 8y_3 = 3.116 + 8.41 = 676,$
$y_4 = x_3 + 3y_3 = 116 + 3.41 = 239,$

$x_5 = 3x_4 + 8y_4 = 3.676 + 8.239 = 3940,$
$y_5 = x_4 + 3y_4 = 676 + 3.239 = 1393,$

$x_6 = 3x_5 + 8y_5 = 3.3940 + 8.1393 = 22964,$
$y_6 = x_5 + 3y_5 = 3940 + 3.1393 = 8119.$

If some solutions of this equation are examined carefully, it will be seen that its solutions are related to Pell and Pell-Lucas Numbers.

$(x_1, y_1) = (4, 1) = \left(4.1, \frac{2}{2}\right) = \left(4P_1, \frac{Q_1}{2}\right),$
$(x_2, y_2) = (20, 7) = \left(4.5, \frac{14}{2}\right) = \left(4P_3, \frac{Q_3}{2}\right),$
$(x_3, y_3) = (116, 41) = \left(4.29, \frac{82}{2}\right) = \left(4P_5, \frac{Q_5}{2}\right),$
$(x_4, y_4) = (676, 239) = \left(4.169, \frac{478}{2}\right) = \left(4P_7, \frac{Q_7}{2}\right),$
$(x_5, y_5) = (3940, 1393) = \left(4.985, \frac{2786}{2}\right) = \left(4P_9, \frac{Q_9}{2}\right),$
$(x_6, y_6) = (22964, 8119) = \left(4.5741, \frac{16238}{2}\right) = \left(4P_{11}, \frac{Q_{11}}{2}\right).$

Now we can use the following Theorem which provides us the solutions of equation (3.6) in terms of Pell and Pell-Lucas numbers.

Theorem 3.5. *All positive integer solutions of the equation* $x^2 - 8y^2 = 8$ *are given by*

$$(x_n, y_n) = \left(4P_{2n-1}, \frac{Q_{2n-1}}{2}\right)$$

for $n \geq 1$.

Proof. If the Pell and Pell-Lucas sequences are written in the form of the Binet's formula, for $n \geq 1$, we get

$$P_{2n-1} = \frac{\alpha^{2n-1}-\beta^{2n-1}}{\alpha-\beta} \text{ and } Q_{2n-1} = \alpha^{2n-1} + \beta^{2n-1}$$

where $\alpha = \frac{2+\sqrt{8}}{2} = 1 + \sqrt{2}$ and $\beta = \frac{2-\sqrt{8}}{2} = 1 - \sqrt{2}$. Obviously, we can obtain $\alpha + \beta = 2,\ \alpha - \beta = \sqrt{8}$ and $\alpha\beta = -1$.

$$\begin{aligned}
x^2 - 8y^2 &= (4P_{2n-1})^2 - 8\left(\frac{Q_{2n-1}}{2}\right)^2 \\
&= \left(4\frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta}\right)^2 - 8\left(\frac{Q_{2n-1}}{2}\right)^2 \\
&= \left(4\frac{\alpha^{2n-1} - \beta^{2n-1}}{\sqrt{8}}\right)^2 - 8\left(\frac{\alpha^{2n-1} + \beta^{2n-1}}{2}\right)^2 \\
&= 2\left(\alpha^{4n-2} - 2\alpha^{2n-1}\beta^{2n-1} + \beta^{4n-2}\right) - 2\left(\alpha^{4n-2} + 2\alpha^{2n-1}\beta^{2n-1} + \beta^{4n-2}\right) \\
&= -8\alpha^{2n-1}\beta^{2n-1} \\
&= -8(\alpha\beta)^{2n-1} \\
&= 8.
\end{aligned}$$

□

Using the relations $x = 4u$ and $y = 2v - 1$, we have

$$u_n = \frac{x_n}{4} = \frac{4P_{2n-1}}{4} = P_{2n-1} \text{ and } v_n = \frac{y_n+1}{2} = \frac{\frac{Q_{2n-1}}{2}+1}{2},$$

$n \in \mathbb{N}$. One gets the sequence $(1,1), (5,4), (29,21), (169,120), (985,607), \ldots$ which are all natural number solutions (u_n, v_n) of the diophantine equation (3.4).

Finally, the following corollary will tell us for which (u_n, v_n) terms the diophantine equation $S_4(u) = CS_4(v)$ is satisfied.

Corollary 3.2. *Let u and v be a positive integer. Then $S_4(u) = CS_4(v)$ if and only if*

$$(u_n, v_n) = \left(P_{2n-1}, \frac{Q_{2n-1} + 2}{4}\right)$$

for $n \geq 1$.

3.3 Triangular Heptagonal Numbers

In this final subsection, it will be proved exactly which numbers are both triangular numbers and heptagonal numbers (i.e. triangular heptagonal numbers). While researching these numbers, Pell equation, Fibonacci and Lucas sequences will be used. Note that, we use Fibonacci and Lucas sequences for the first time in this article.

Firstly, $S_3\left(v\right)$ and $S_7\left(u\right)$ equations are equated to obtain the diophantine equation

$$\frac{1}{2}v\left(v+1\right)=\frac{1}{2}u\left(5u-3\right). \tag{3.7}$$

Heptagonal triangular numbers match the natural number solutions of the diophantine equation (3.7). From equation (3.7) we find,

$$(10u-3)^2-5(2v+1)^2=4. \tag{3.8}$$

If we take $x=10u-3$ and $y=2v+1$, then we obtain the Pell equation

$$x^2-5y^2=4. \tag{3.9}$$

In the following Theorem 3.6 and Theorem 3.7, a method for finding all positive integer solutions of the Pell equation $x^2-dy^2=4$ will be given from [7], [12], [1] and [2] respectively.

Theorem 3.6. *Let $x_1+y_1\sqrt{d}$ be the fundamental solution of $x^2-dy^2=4$. Then all positive integer solutions to the equation $x^2-dy^2=4$ are given by*

$$x_n+y_n\sqrt{d}=\frac{\left(x_1+y_1\sqrt{d}\right)^n}{2^{n-1}}$$

with $n\geq 1$, see in [7] and [12].

Theorem 3.7. *If (x_1,y_1) is the fundamental solution of the Pell equation $x^2-Dy^2=4$, then*

$$x_n=\frac{x_1x_{n-1}+Dy_1y_{n-1}}{2} \text{ and } y_n=\frac{y_1x_{n-1}+x_1y_{n-1}}{2}$$

with $n\geq 2$, see in [1] and [2].

Now we can give the following lemma 4 which gives us the fundamental solution of equation (3.9).

Lemma 3.4. *The fundamental solution of the Pell equation $x^2-5y^2=4$ is $(x_1,y_1)=(3,1)$.*

Proof. $y_1 = 1$ is the minimal positive integer number. So, $x_1^2 - 5y_1^2 = 4 \Rightarrow x_1^2 - 5.1^2 = 4$ and here $x_1 = 4$. Hence, $(x_1, y_1) = (4, 1)$ is the fundamental solution of $x^2 - 5y^2 = 4$. $\square$

From lemma 3.4, we know that the fundamental solution of $x^2 - 5y^2 = 4$ is $(3, 1)$. Also from theorem 3.7 we can obtain some solutions of $x^2 - 5y^2 = 4$.

$x_2 = \frac{x_1x_1+5y_1y_1}{2} = \frac{3.3+5.1.1}{2} = 7,$
$y_2 = \frac{y_1x_1+x_1y_1}{2} = \frac{1.3+3.1}{2} = 3,$

$x_3 = \frac{x_1x_2+5y_1y_2}{2} = \frac{3.7+5.1.3}{2} = 18,$
$y_3 = \frac{y_1x_2+x_1y_2}{2} = \frac{1.7+3.3}{2} = 8,$

$x_4 = \frac{x_1x_3+5y_1y_3}{2} = \frac{3.18+5.1.8}{2} = 47,$
$y_4 = \frac{y_1x_3+x_1y_3}{2} = \frac{1.18+3.8}{2} = 21,$

$x_5 = \frac{x_1x_4+5y_1y_4}{2} = \frac{3.47+5.1.21}{2} = 123,$
$y_5 = \frac{y_1x_4+x_1y_4}{2} = \frac{1.47+3.21}{2} = 55.$

These solutions can also be found by using Theorem 3.6. But it may take more time to make this calculations. If some solutions of this equation are observed carefully, it will be seen that its solutions are related to Lucas and Fibonacci Numbers.

$(x_1, y_1) = (3, 1) = (L_2, F_2),$
$(x_2, y_2) = (7, 3) = (L_4, F_4),$
$(x_3, y_3) = (18, 8) = (L_6, F_6),$
$(x_4, y_4) = (47, 21) = (L_8, F_8),$
$(x_5, y_5) = (123, 55) = (L_{10}, F_{10}).$

One can make a lucky gues that this harmony is always true. In fact, we can give the following Theorem which gives us the solutions of equation (3.9) in terms of Lucas and Fibonacci numbers.

The following theorem is given in [10]. But we will give the proof of the following theorem using the binet's formula.

Theorem 3.8. *All positive integer solutions of the equation* $x^2 - 5y^2 = 4$ *are given by*

$$(x_n, y_n) = (L_{2n}, F_{2n}),$$

for $n \geq 1$.

Proof. If the Lucas and Fibonacci sequences are written in the form of the Binet's formula, we get,

$$F_{2n} = \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} \text{ and } L_{2n} = \alpha^{2n} + \beta^{2n}$$

for $n \geq 1$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Obviously, we can obtain $\alpha + \beta = 1,\ \alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$.

$$\begin{aligned}
x^2 - 5y^2 &= (L_{2n})^2 - 5(F_{2n})^2 \\
&= \left(\alpha^{2n} + \beta^{2n}\right)^2 - 5\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)^2 \\
&= \left(\alpha^{2n} + \beta^{2n}\right)^2 - 5\left(\frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}}\right)^2 \\
&= \alpha^{4n} + 2\alpha^{2n}\beta^{2n} + \beta^{4n} - 5\left(\frac{\alpha^{4n} - 2\alpha^{2n}\beta^{2n} + \beta^{4n}}{5}\right) \\
&= 4\alpha^{2n}\beta^{2n} \\
&= 4(\alpha\beta)^{2n} \\
&= 4.
\end{aligned}$$

□

Using relations $x = 10u - 3$ and $y = 2v + 1$, we have

$$u_n = \frac{x_n+3}{10} = \frac{L_{2n}+3}{10} \text{ and } v_n = \frac{y_n-1}{2} = \frac{F_{2n}-1}{2},$$

$n \in \mathbb{N}$. One gets the sequence $(1,1)$, $(\frac{21}{10}, \frac{7}{2})$, $(5,10)$, $(\frac{126}{10}, 27)$, $(\frac{325}{10}, \frac{143}{2})$, $(\frac{846}{10}, 188)$, $(221, 493)$, ... which are the positive solutions (u_n, v_n) of the diophantine equation (3.7). But not all elements of this sequence are positive integers. From [16, A046193 and A039835] the positive integer solutions of the diophantine equation (3.7) are given by $(1,1)$, $(5,10)$, $(221,493)$, $(1513,3382)$, $(71065,158905)$, ...

In order to determine triangular heptagonal numbers in term of Fibonacci and Lucas numbers, we need the following lemma, (see in [6]).

Lemma 3.5. *(Parity Lemma) Suppose μ is positive integer and $|\tau| = 1$. If μ is even: $V_n(\mu, \tau)$ is even if $2 \mid n$. If μ is odd: $V_n(\mu, \tau) \equiv U_n(\mu, \tau) \pmod{2}$, and $V_n(\mu, \tau)$ and $U_n(\mu, \tau)$ are even if $3 \mid n$.*

From Lemma 3.5, the following corollary can be given.

Corollary 3.3. *Suppose $|\tau| = 1$. If $\mu = 1$, then*

$$3 \,|n \Leftrightarrow 2 \,|F_n \Leftrightarrow 2 \,|L_n \,.$$

Finally, from Theorem 3.8, Lemma 3.5 and Corollary 3.3 we can give the following corollary which will tell us for which indices of triangular numbers are also heptagonal.

Corollary 3.4. *Let u and v be a positive integer. Then*

$$S_7(u) = S_3(v)$$

if and only if u_n and v_n are given by

$$(u_n, v_n) = \left(\frac{L_{2n} + 3}{10}, \frac{F_{2n} - 1}{2} \right)$$

where $n \geq 1$, $L_{2n} \equiv 7 \ (Mod\,10)$, F_{2n} is odd and $2n \not\equiv 0\,(Mod\,3)$.

Acknowledgements

The author is thankful to the anonymous referee for his/her valuable suggestions.

References

[1] A. Chandoul, *The Pell Equation $x^2 - Dy^2 = \mp k^2$*, Advances in Pure Mathematics, (1)(2011), 16-22.

[2] A. Tekcan, *The Pell Equation $x^2 - Dy^2 = \mp 4$*, Applied Mathematical Sciences, (1)(2007), 363-369.

[3] D. Kalman, R. Mena, *The Fibonacci numbers—exposed*. Mathematics magazine, (76)(2003), 167-181.

[4] E. Deza, M. M. Deza, Figurate Numbers, *World Scientific Publishing Co. Pte. Ltd., Singapore, 2012*.

[5] G.G. Karadeniz, *On Pell, Pell-Lucas, and balancing numbers*, Journal of Inequalities and Applications, (1)(2018), 1-16.

[6] J.P. Jones, *Representation of solutions of Pell equations using Lucas sequences*, In Annales Mathematicae et Informaticae (30)(2003), 75-86.

[7] J.P. Robertson, Solving the generalized Pell equation $x^2 - Dy^2 = N$, (2003).

[8] K.H. Rosen, Elementary number theory and its applications, *Pearson Education, London, 2011*.

[9] P. Kaplan, K.S. Williams, *Pell's Equations* $x^2 - my^2 = -1, -4$ *and Continued Fractions*, Journal of Number Theory, (23)(1986), 169-182.

[10] P. Olajos, *Properties of balancing, cobalancing and generalized balancing numbers*, Ann. Math. Inform. (37)(2010), 125-138.

[11] Q, Sun, P.Z. Yuan, *On the diophantine equation* $(ax^n - 1)/(ax - 1) \; = y^2$ *and* $(ax^n + 1)/(ax + 1) \; = y^2$, Journal of Sichuan University, (26)(1989), 20-24.

[12] R. Keskin, M.G. Duman, *Positive integer solutions of some Pell equations*, Palestine Journal of Mathematics, (8)(2019), 213-226.

[13] S. J. Schlicker, *Numbers Simultaneously Polygonal and centered Polygonal*, Mathematics Magazine, (84)(2011), 339-350.

[14] T. Andreescu, D. Andrica, T. Cucurezeanu, An introduction to Diophantine equations: a problem-based approach, *New York: Birkhäuser, 2010*.

[15] T. Koshy, Fibonacci and Lucas Numbers with Applications, *Volume 2. John Wiley and Sons, 2019*.

[16] "The on-line encyclopedia of integer sequences® (OEIS®)," Oeis.org. [Online]. Available: http://oeis.org. [Accessed: 24-May-2023].