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Nonexistence of global solutions for a system of Kirchhoff-type equations with variable exponents

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Abstract

In this work, we deal with the system of Kirchhoff-type equations with variable exponents. Under suitable conditions on variable exponents, we prove the nonexistence of solutions with negative initial energy.

1 Introduction

1.1 Setting of the problem:

In this work, we consider the following system of Kirchhoff-type equations with variable exponents

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$$\begin{cases} u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+|u_t|^{m(x)-1}u_t=f_1(u,v), & in\ \Omega,\ t>0,\\ v_{tt}-M\left(\|\nabla v\|^2\right)\Delta v+|v_t|^{r(x)-1}v_t=f_2(u,v), & in\ \Omega,\ t>0,\\ u(x,t)=v(x,t)=0, & on\ \partial\Omega,\ t\geq 0,\\ u(x,0)=u_0(x),v(x,0)=v_0(x), & \multirow{2}{*}{$in\ \Omega,$}\\ u_t(x,0)=u_1(x),v_t(x,0)=v_1(x) & \end{cases} \tag{1.1}$$

where Ω be a bounded and regular domain of R^n, $n\geq 1$, with a smooth boundary $\partial\Omega$, and

$$M(s)=1+s^{\gamma}\qquad (\gamma\geq 1).$$

The source terms $f_1(u,v)$ and $f_2(u,v)$ as follows

$$\begin{cases} f_1(u,v)=a|u+v|^{2(p(\cdot)+1)}(u+v)+b|u|^{p(\cdot)}u|v|^{p(\cdot)+2},\\ f_2(u,v)=a|u+v|^{2(p(\cdot)+1)}(u+v)+b|u|^{p(\cdot)+2}|v|^{p(\cdot)}v, \end{cases} \tag{1.2}$$

with a, b positive constans; and $m(\cdot)$, $r(\cdot)$, $p(\cdot)$ are given continuous functions on $\overline{\Omega}$ satisfying some conditions to be specified later and the log-Hölder continuity condition given for any function $p(\cdot)$ by

$$|p(x)-p(y)|\leq -\frac{A}{\log|x-y|}, for\ all\ x,y\in\Omega,\ with\ |x-y|<\delta, \tag{1.3}$$

where $0<\delta<1$ and $A>0$. By the definition of $f_1(u,v)$ and $f_2(u,v)$, one can easily verify that

$$uf_1(u,v)+vf_2(u,v)=2(p(x)+2)F(u,v),\ \forall(u,v)\in R^2, \tag{1.4}$$

where

$$F(u,v)=\frac{1}{2(p(x)+2)}\left[a|u+v|^{2(p(x)+2)}+2b|uv|^{p(x)+2}\right].$$

The exponents $m(\cdot), r(\cdot)$ and $p(\cdot)$ are measurable functions on Ω satisfiying

$$\begin{cases} 2\leq m_1\leq m(x)\leq m_2\leq m^*,\\ 2\leq r_1\leq r(x)\leq r_2\leq r^*,\\ 2\leq p_1\leq p(x)\leq p_2\leq p^*, \end{cases}$$

where

$$\begin{cases} m_1=ess\inf_{x\in\Omega}m(x),\ m_2=ess\sup_{x\in\Omega}m(x),\\ r_1=ess\inf_{x\in\Omega}r(x),\ r_2=ess\sup_{x\in\Omega}r(x),\\ p_1=ess\inf_{x\in\Omega}p(x),\ p_2=ess\sup_{x\in\Omega}p(x), \end{cases}$$

and

$$\begin{cases} 2 < m^*, r^*, p^* < \infty & if\ n \leq 2, \\ 2 < m^*, r^*, p^* < \frac{2n}{n-2} & if\ n > 2. \end{cases}$$

1.2 Literature overview:

Pişkin [14] considered the single Kirchhoff-type equation with variable exponents

$$u_{tt} + M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u.$$

He proved the blow up of solutions.

In [9], Messaoudi and Talahmeh studied the following system of nonlinear wave equations with variable exponents

$$\begin{cases} u_{tt} + \Delta u + |u_t|^{m(x)-1} u_t = f_1(u, v), \\ v_{tt} + \Delta v + |v_t|^{r(x)-1} v_t = f_2(u, v). \end{cases}$$

They obtained the blow up of solutions with negative initial energy. Also, Messaoudi et al. [11] proved the existence and stability of solutions the same system.

Recently, problems with variable exponents have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability have been found ([2, 3, 6, 12, 13, 17]). The problems with variable exponents arise in many branches of sciences such as image processing, electrorheological fluids and nonlinear elasticity theory [7, 8, 16].

In this paper, we prove the blow up of the solutions (1.1). To the best of our knowledge, there is no result the blow up of the fourth order system with variable exponents. The rest of our work is organized as follows: In section 2, we give some lemmas, definition and theorem. In section 3, we state and prove our main result.

2 Preliminaries

In this part, we state some results about the variable exponents Lebesgue space and Sobolev space $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, (see [5, 8, 15]).

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of R^n. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to R;\ u\ measurable\ in\ \Omega : \varrho_{p(\cdot)}(\lambda u) < \infty, for\ some\ \lambda > 0\right\},$$

where

$$\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx$$

is a modular. Equipped with the following Luxembourg-type norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space.

We also define the variable-exponent Sobolev space $W^{1,q(\cdot)}(\Omega)$ as

$$W^{1,q(\cdot)}(\Omega) = \left\{u \in L^{q(\cdot)}(\Omega) \ \textit{such that}\ \nabla u \ \textit{exists and}\ \ |\nabla u| \in L^{q(\cdot)}(\Omega)\right\}.$$

Lemma 2.1. *[8] (Poincarě Inequality). Let Ω be a bounded domain of R^n and $q(\cdot)$ satisfies (1.3), and $1 \le q_1 \le q(x) \le q_2 < \infty$, where*

$$q_1 = ess \inf_{x\in\Omega} q(x), \ \ q_2 = ess \sup_{x\in\Omega} q(x).$$

Then

$$\|u\|_{q(\cdot)} \le C \|\nabla u\|_{q(\cdot)}, \ \textit{for all}\ u \in W_0^{1,q(\cdot)}(\Omega),$$

where the positive constant C depending on q_1, q_2 and Ω only.

Lemma 2.2. *[8]. If $1 < q_1 \le q(x) \le q_2 < \infty$ holds, then*

$$\min\left\{\|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2}\right\} \le \varrho_{q(\cdot)}(w) \le \max\left\{\|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2}\right\},$$

for any $w \in L^{q(\cdot)}(\Omega)$.

Lemma 2.3. *[9]. There exist two constants c_0 and c_1 such that*
$\frac{c_0}{2(p(x)+2)}\left[|u|^{2(p(x)+2)} + |v|^{2(p(x)+2)}\right] \le F(u,v) \le \frac{c_1}{2(p(x)+2)}$
$\left[|u|^{2(p(x)+2)} + |v|^{2(p(x)+2)}\right].$

Corollary 2.1. *[9]. There exist two constants c_0 and c_1 such that*

$$c_0\left[\varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(v)\right] \le \int_{\Omega} F(u,v)\, dx \le c_1\left[\varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(v)\right]. \quad (2.1)$$

Definition 2.1. *(Weak solution). A pair of functions* (u,v) *is said to be weak solution of (1.1) on* $[0,T]$, $T>0$, *if*

$$\begin{aligned}(u,v) &\in L^{\infty}\left((0,T), H_0^1(\Omega)\right),\\ u_t &\in L^{\infty}\left((0,T), L^2(\Omega)\right)\cap L^{m(\cdot)+1}(\Omega\times(0,T)),\\ v_t &\in L^{\infty}\left((0,T), L^2(\Omega)\right)\cap L^{r(\cdot)+1}(\Omega\times(0,T))\end{aligned}$$

with

$$u(\cdot,0)=u_0,\ v(\cdot,0)=v_0,\ u_t(\cdot,0)=u_1,\ v_t(\cdot,0)=v_1$$

and (u,v) *satisfies*

$$\begin{aligned}&\int_{\Omega}u_t\phi-\int_{\Omega}u_1\phi+\int_0^t\int_{\Omega}M\left(\|\nabla u\|^2\right)\nabla u\nabla\phi\\ &+\int_0^t\int_{\Omega}|u_t|^{m(\cdot)-1}u_t\phi=\int_0^t\int_{\Omega}f_1\phi,\\ &\int_{\Omega}v_t\psi-\int_{\Omega}v_1\psi+\int_0^t\int_{\Omega}M\left(\|\nabla v\|^2\right)\nabla v\nabla\psi\\ &+\int_0^t\int_{\Omega}|v_t|^{r(\cdot)-1}u_t\psi=\int_0^t\int_{\Omega}f_2\psi,\end{aligned}$$

for all $\phi\in H_0^1(\Omega)\cap L^{m(\cdot)+1}(\Omega)$, $\psi\in H_0^1(\Omega)\cap L^{r(\cdot)+1}(\Omega)$, *and all* $t\in[0,T]$.

We state the following theorem which can be obtained by exploiting the Faedo-Galerkin method and using the similar arguments as in [4,10,11].

Theorem 2.1. *(Local existence). Suppose that* $p(\cdot)$, $m(\cdot)$, $r(\cdot)\in C\left(\overline{\Omega}\right)$, *satisfy (1.3) and, for all* $x\in\Omega$,

$$\begin{cases} p(x)\geq 0, & \text{if } n=1,2,\\ p(x)=0, & \text{if } n=3,\end{cases}\tag{2.2}$$

$$\begin{cases} m(x)\geq 2, & \text{if } n=1,2,\\ 2\leq m(x)\leq 6, & \text{if } n=3,\end{cases}\tag{2.3}$$

$$\begin{cases} r(x)\geq 0, & \text{if } n=1,2\\ 2\leq r(x)\leq 6, & \text{if } n=3,\end{cases}\tag{2.4}$$

and (u_0, u_1),$(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then (1.1) has a unique weak local solution*

$$(u, v) \in L^\infty\left((0, T), H_0^1(\Omega)\right),$$
$$u_t \in L^\infty\left((0, T), L^2(\Omega)\right) \cap L^{m(\cdot)+1}(\Omega \times (0, T)),$$
$$v_t \in L^\infty\left((0, T), L^2(\Omega)\right) \cap L^{r(\cdot)+1}(\Omega \times (0, T)),$$

for $T > 0$.

3 Nonexistence of global solutions

In this part we state and prove our main result. For this purpose, we define energy functional of (1.1) as

$$\begin{aligned} E(t) &= \frac{1}{2}\left[\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2\right] \\ &\quad + \frac{1}{2(\gamma+1)}\left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right] - \int_\Omega F(u, v)\, dx. \end{aligned} \tag{3.1}$$

Lemma 3.1. $E(t)$ *energy functional is nonincreasing function.*

Proof. Multiplying the first equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, using integration by parts and summing up the product results, we get

$$E'(t) = -\int_\Omega |u_t|^{m(x)+1}\, dx - \int_\Omega |v_t|^{r(x)+1}\, dx \leq 0. \tag{3.2}$$

□

Lemma 3.2. *[9]. Assume that (2.2) holds. Then, we have the following inequalities:*

$$\left[\varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(v)\right]^{\frac{s}{2(p_1+2)}} \leq C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(v)\right], \tag{3.3}$$

$$\|u\|_{2(p_1+2)}^s \leq C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \|u_t\|_{2(p_1+2)}^{2(p_1+2)} + \|v_t\|_{2(p_1+2)}^{2(p_1+2)}\right], \tag{3.4}$$

$$\|v\|_{2(p_1+2)}^s \leq C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \|u_t\|_{2(p_1+2)}^{2(p_1+2)} + \|v_t\|_{2(p_1+2)}^{2(p_1+2)}\right], \tag{3.5}$$

$$\int_{\Omega}|u|^{m(x)+1}\,dx\leq c_1\left[\left(\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right)^{\frac{m_1+1}{2(p_1+2)}}+\left(\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right)^{\frac{m_2+1}{2(p_1+2)}}\right], \tag{3.6}$$

$$\int_{\Omega}|u|^{r(x)+1}\,dx\leq c_2\left[\left(\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right)^{\frac{r_1+1}{2(p_1+2)}}+\left(\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right)^{\frac{r_2+1}{2(p_1+2)}}\right], \tag{3.7}$$

for any $u,v\in H_0^1(\Omega)$ *and* $2\leq s\leq 2(p_1+2)$. *Where* $C>1$, $c_1>0$,$c_2>0$ *are constants and*

$$H(t)=-E(t).$$

Theorem 3.1. *Assume that (1.3),(2.2), (2.3) and (2.4) hold. Assume further that*

$$2(p_1+1)\geq\max\{m_2+1,r_2+1\} \tag{3.8}$$

and

$$E(0)<0.$$

Then the solution of problem (1.1) blows up in finite time.

Proof. Then $E(0)<0$ and (3.2) gives $H(t)\geq H(0)>0$. By the definition $H(t)$ and (2.1), we get

$$\begin{aligned} H(t) &= -\frac{1}{2}\left[\|u_t\|^2+\|v_t\|^2+\|\nabla u\|^2+\|\nabla v\|^2\right]\\ &\quad -\frac{1}{2(\gamma+1)}\left[\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right]+\int_{\Omega}F(u,v)\,dx\\ &\leq \int_{\Omega}F(u,v)\,dx\\ &\leq c_1\left[\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right]. \end{aligned} \tag{3.9}$$

We define

$$\Psi(t)=H^{1-\alpha}(t)+\varepsilon\int_{\Omega}(uu_t+vv_t)\,dx \tag{3.10}$$

for ε small to be chosen later and

$$0<\alpha\leq\min\left\{\frac{p_1+2}{2(p_1+2)},\frac{2(p_1+2)-(m_2+1)}{2m_2(p_1+2)},\frac{2(p_1+2)-(r_2+1)}{2r_2(p_1+2)}\right\}. \tag{3.11}$$

Differentiating $\Psi(t)$ with respect to t, and using (1.1) and (1.4), we have

$$\begin{aligned}\Psi'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &\quad -\varepsilon\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\varepsilon\left(\|\nabla u\|_{2(\gamma+1)}^{2(\gamma+1)}+\|\nabla v\|_{2(\gamma+1)}^{2(\gamma+1)}\right)\\ &\quad +2\varepsilon\int_\Omega (p(x)+2)F(u,v)\,dx\\ &\quad -\varepsilon\int_\Omega u|u_t|^{m(x)-1}u_t dx-\varepsilon\int_\Omega v|v_t|^{r(x)-1}v_t dx. \end{aligned} \tag{3.12}$$

By using the definition of the $H(t)$, it follows that

$$\begin{aligned}-\varepsilon q_1(1-\xi)H(t) &= \frac{\varepsilon q_1(1-\xi)}{2}\left[\|u_t\|^2+\|v_t\|^2\right]\\ &\quad +\frac{\varepsilon q_1(1-\xi)}{2}\left[\|\nabla u\|^2+\|\nabla v\|^2\right]\\ &\quad +\frac{\varepsilon q_1(1-\xi)}{2}\left[\|\nabla u\|_{2(\gamma+1)}^{2(\gamma+1)}+\|\nabla v\|_{2(\gamma+1)}^{2(\gamma+1)}\right]\\ &\quad -\varepsilon q_1(1-\xi)\int_\Omega F(u,v)\,dx, \end{aligned}$$

where $2<\eta<2(p_1+2)$. Adding and subtracting $-\varepsilon q_1(1-\xi)H(t)$ from the right-hand side of (3.12), we obtain

$$\begin{aligned}\Psi'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\left(1+\frac{q_1(1-\xi)}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &\quad +\varepsilon\left(\frac{q_1(1-\xi)}{2}-1\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon q_1(1-\xi)H(t)\\ &\quad +\varepsilon\left(\frac{q_1(1-\xi)}{2(\gamma+1)}-1\right)\left[\|\nabla u\|_{2(\gamma+1)}^{2(\gamma+1)}+\|\nabla v\|_{2(\gamma+1)}^{2(\gamma+1)}\right]\\ &\quad +\varepsilon\left(2(p_1+2)-q_1(1-\xi)\right)\int_\Omega F(u,v)\,dx\\ &\quad -\varepsilon\int_\Omega\left[u|u_t|^{m(x)-1}u_t+v|v_t|^{r(x)-1}v_t\right]dx. \end{aligned} \tag{3.13}$$

By using (2.1), we have

$$\begin{aligned}\Psi'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\beta\Big[H(t)+\|u_t\|_2^2+\|v_t\|_2^2\\ &\quad +\|\nabla u\|^2+\|\nabla v\|^2+\|\nabla u\|_{2(\gamma+1)}^{2(\gamma+1)}+\|\nabla v\|_{2(\gamma+1)}^{2(\gamma+1)}+\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\Big]\\ &\quad -\varepsilon\int_\Omega\left[u|u_t|^{m(x)-1}u_t+v|v_t|^{r(x)-1}v_t\right]dx \end{aligned} \tag{3.14}$$

where

$$\beta=\min\left\{\begin{array}{c} q_1(1-\xi),1+\frac{q_1(1-\xi)}{2},\left(\frac{q_1(1-\xi)}{2}-1\right),\left(\frac{q_1(1-\xi)}{2(\gamma+1)}-1\right), \\ \left(\frac{\eta}{2}+1\right),\ c_0\left(2\left(p_1+2\right)-q_1\left(1-\xi\right)\right)\end{array}\right\}>0.$$

To estimate the last term in (3.14), we use the Young inequality, we have

$$\int_\Omega |u_t|^{m(x)}|u|\,dx\le\frac{1}{m_1+1}\int_\Omega \delta_1^{m(x)+1}|u|^{m(x)+1}dx+\frac{m_2}{m_1+1}\int_\Omega \delta_1^{-\frac{m(x)+1}{m(x)}}|u_t|^{m(x)+1}dx. \tag{3.15}$$

Similarly, we have

$$\int_\Omega |v_t|^{r(x)}|v|\,dx\le\frac{1}{r_1+1}\int_\Omega \delta_2^{r(x)+1}|v|^{r(x)+1}dx+\frac{r_2}{r_1+1}\int_\Omega \delta_2^{-\frac{r(x)+1}{r(x)}}|v_t|^{r(x)+1}dx, \tag{3.16}$$

where $\delta_1,\delta_2>0$ are constants depending on the time t and specified later. Let us choose δ_1 and δ_2 so that

$$\delta_1^{-\frac{m(x)+1}{m(x)}}=k_1H^{-\alpha}(t)\ \ and\ \ \delta_2^{-\frac{r(x)+1}{r(x)}}=k_2H^{-\alpha}(t),$$

for a large constant k_1 and k_2 to be specified later, and substituting in (3.15) and (3.16), respectively, we get

$$\int_\Omega |u_t|^{m(x)}|u|\,dx\le\frac{k_1^{-m_1}}{m_1+1}\int_\Omega |u|^{m(x)+1}H^{\alpha m(x)}(t)\,dx+\frac{m_2k_1}{m_1+1}H^{-\alpha}(t)\,H'(t), \tag{3.17}$$

and

$$\int_\Omega |v_t|^{r(x)}|v|\,dx\le\frac{k_2^{-r_1}}{r_1+1}\int_\Omega |v|^{r(x)+1}H^{\alpha r(x)}(t)\,dx+\frac{r_2k_1}{r_1+1}H^{-\alpha}(t)\,H'(t). \tag{3.18}$$

Combining (3.14), (3.17) and (3.18) gives

$$\begin{aligned}\Psi'(t)\ \ \ge\ \ &\left[(1-\alpha)-\varepsilon\frac{m_2k_1}{m_1+1}-\varepsilon\frac{r_2k_1}{r_1+1}\right]H^{-\alpha}(t)\,H'(t)\\ &+\varepsilon\beta\Big[H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|^2+\|\nabla v\|^2\\ &+\|\nabla u\|_{2(\gamma+1)}^{2(\gamma+1)}+\|\nabla v\|_{2(\gamma+1)}^{2(\gamma+1)}+\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\ \Big]\\ &-\frac{\varepsilon k_1^{-m_1}}{m_1+1}\int_\Omega |u|^{m(x)+1}H^{\alpha m(x)}(t)\,dx\\ &-\frac{\varepsilon k_2^{-r_1}}{r_1+1}\int_\Omega |v|^{r(x)+1}H^{\alpha r(x)}(t)\,dx.\end{aligned} \tag{3.19}$$

From (3.6), (3.7) and (3.9), we have

$$\int_\Omega |u|^{m(x)+1} H^{\alpha m(x)}(t)\,dx$$
$$\leq C'\left[\left(\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right)^{\frac{m_1+1}{2(p_1+2)}+\alpha m_2}+\left(\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right)^{\frac{m_2+1}{2(p_2+2)}+\alpha m_2}\right]. \tag{3.20}$$

We then use lemma 3.2, for

$$s=(m_2+1)+2\alpha m_2(p_1+2)\leq 2(p_1+2),$$

and

$$s=(m_1+1)+2\alpha m_2(p_1+2)\leq 2(p_1+2),$$

to do deduce, from (3.20), that

$$\int_\Omega |u|^{m(x)+1} H^{\alpha m(x)}(t)\,dx \leq C\left[\|\nabla u\|^2+\|\nabla v\|^2+\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right]. \tag{3.21}$$

Similarly

$$\int_\Omega |v|^{r(x)+1} H^{\alpha r(x)}(t)\,dx \leq C\left[\|\nabla u\|^2+\|\nabla v\|^2+\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right]. \tag{3.22}$$

Combining (3.21), (3.22) and (3.19), give

$$\begin{aligned}\Psi'(t) \geq\ & \left[(1-\alpha)-\varepsilon\frac{m_2k_1}{m_1+1}-\varepsilon\frac{r_2k_2}{r_1+1}\right]H^{-\alpha}(t)\,H'(t)\\ &+\varepsilon\left(\beta-\frac{k_1^{-m_1}}{m_1+1}C-\frac{k_2^{-r_2}}{r_2+1}C\right)\\ &\times\left[H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|^2+\|\nabla v\|^2+\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot)}(v)\right].\end{aligned} \tag{3.23}$$

Let us choose k_1, k_2 large enough so that

$$\gamma=\beta-\frac{k_1^{-m_1}}{m_1+1}C-\frac{k_2^{-r_2}}{r_2+1}C>0,$$

and picking ε small enough such that

$$(1-\alpha)-\varepsilon\frac{m_2k_1}{m_1+1}-\varepsilon\frac{r_2k_2}{r_1+1}\geq 0,$$

and

$$\Psi(0) = H^{1-\alpha}(0) + \varepsilon \int_\Omega (u_0 u_1 + v_0 v_1)\, dx + \frac{\varepsilon}{2}\left[\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right] > 0.$$

Hence (3.23) takes the form

$$\begin{aligned}\Psi'(t) &\geq \gamma\varepsilon\left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(v)\right] \\ &\geq \gamma\varepsilon\left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{2(p_1+2)}^{2(p_1+2)} + \|v\|_{2(p_1+2)}^{2(p_1+2)}\right].\end{aligned} \tag{3.24}$$

Consequently, we get

$$\Psi(t) \geq \Psi(0) > 0, \ \textit{for all } t \geq 0.$$

On the other hand, thanks to the Hölder inequality and the embedding $L^{2(p_1+2)}(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned}\left|\int_\Omega u u_t dx\right| &\leq \|u\|_2 \|u_t\|_2 \\ &\leq C\|u\|_{2(p_1+2)} \|u_t\|_2,\end{aligned}$$

which implies

$$\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\alpha}} \leq C\|u\|_{2(p_1+2)}^{\frac{1}{1-\alpha}} \|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Similarly

$$\left|\int_\Omega v v_t dx\right|^{\frac{1}{1-\alpha}} \leq C\|v\|_{2(p_1+2)}^{\frac{1}{1-\alpha}} \|v_t\|_2^{\frac{1}{1-\alpha}}.$$

Young's inequality gives

$$\begin{aligned}&\left|\int_\Omega u u_t dx + \int_\Omega v v_t dx\right|^{\frac{1}{1-\alpha}} \\ \leq\ & C\left[\|u\|_{2(p_1+2)}^{\frac{\mu}{1-\alpha}} \|u_t\|_2^{\frac{\theta}{1-\alpha}} + \|v\|_{2(p_1+2)}^{\frac{\mu}{1-\alpha}} \|v_t\|_2^{\frac{\theta}{1-\alpha}}\right],\end{aligned} \tag{3.25}$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\alpha)$,to get $\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(p_1+2)$, by (3.20).

Therefore (3.25), becomes

$$\begin{aligned}&\left|\int_{\Omega}uu_t dx+\int_{\Omega}vv_t dx\right|^{\frac{1}{1-\alpha}}\\ \leq\ & C\left[\|u\|_{2(p_1+2)}^{s}+\|v\|_{2(p_1+2)}^{\frac{\mu}{1-\alpha}}+\|u_t\|_2^2+\|v_t\|_2^2\right],\end{aligned}$$

where

$$s=\frac{2}{1-2\alpha}\leq 2\left(p_1+2\right).$$

By using (3.4) and (3.5), we have

$$\begin{aligned}&\left|\int_{\Omega}uu_t dx+\int_{\Omega}vv_t dx\right|^{\frac{1}{1-\alpha}}\\ \leq\ & C\left[H\left(t\right)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|^2+\|\nabla v\|^2+\varrho_{p(\cdot)}\left(u\right)+\varrho_{p(\cdot)}\left(v\right)\right],\end{aligned} \tag{3.26}$$

for all $t\geq 0$. Thus,

$$\begin{aligned}\Psi^{\frac{1}{1-\alpha}}\left(t\right) &= \left[H^{1-\alpha}\left(t\right)+\varepsilon\int_{\Omega}uu_t+vv_t dx\right]^{\frac{1}{1-\alpha}}\\ &\leq 2^{\frac{1}{1-\alpha}}\left[H\left(t\right)+\left|\int_{\Omega}uu_t dx+\int_{\Omega}vv_t dx\right|^{\frac{1}{1-\alpha}}\right]\\ &\leq C\left[H\left(t\right)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|^2+\|\nabla v\|^2+\varrho_{p(\cdot)}\left(u\right)+\varrho_{p(\cdot)}\left(v\right)\right]\end{aligned} \tag{3.27}$$

where $(a+b)^p\leq 2^{p-1}\left(a^p+b^p\right)$ is used. By combining of (3.24) and (3.27), we arrive

$$\Psi'\left(t\right)\geq\xi\Psi^{\frac{1}{1-\sigma}}\left(t\right), \tag{3.28}$$

where ξ is a positive constant. A simple integration of (3.28) over $(0,t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}\left(t\right)\geq\frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}\left(0\right)-\frac{\xi\sigma t}{1-\sigma}},$$

which implies that the solution blows up in a finite time T^*, with

$$T^*\leq\frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}\left(0\right)}.$$

This completes the proof of the theorem. □

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References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces,* Academic Press, New York, 2003.

[2] L. Alkhalifa, H. Dridi and K. Zennir, *Blow-Up of Certain Solutions to Nonlinear Wave Equations in the Kirchhoff-Type Equation with Variable Exponents and Positive Initial Energy,* Journal of Function Spaces, (2021), 1-9.

[3] S. Antontsev, J. Ferreira and E. Pişkin, *Existence and Blow up of Petrovsky Equation Solutions with Strong Damping and Variable Exponents,* Electronic Journal of Differential Equations, 2021(2021), 1-18.

[4] S.N. Antontsev, J. Ferreira, E. Pişkin and S.M.S. Cordeiro, *Existence and non-existence of solutions for Timoshenko-type equations with variable exponents,* Nonlinear Anal.: Real Worl Appl., 61(2021), 1-13.

[5] S. Antontsev and S. Shmarev, *Evolution PDEs with nonstandard growth conditions: Existence, uniqueness, localization, blow-up,* Atlantis Studies in Differential Equations, 2015.

[6] O. Bouhoufani and I. Hamchi, *Coupled System of Nonlinear Hyperbolic Equations with Variable-Exponents: Global Existence and Stability,* Mediterr. J. Math. 17(2020), 1-15.

[7] Y. Chen, S. Levine and M. Rao, *Variable Exponent, Linear Growth Functionals in Image Restoration,* SIAM journal on Applied Mathematics, 66(4)(2006), 1383-1406.

[8] L. Diening, P. Harjulehto, P. Hasto and M. Ruzicka, *Lebesque and Sobolev spaces with variable exponents,* Berlin: Springer, 2011.

[9] S.A. Messaoudi and A.A. Talahmeh, *Blow up of negative initial-energy solutions of a system of nonlinear wave equations with variable-exponent nonlinearities,* Discrete and Continuous Dynamical Systems Series S, 15(5)(2022), 1233-1245.

[10] S.A. Messaoudi, O. Bouhoufani, I. Hamchi and M. Alahyone, *Existence and blow up in a system of wave equations with nonstandard nonlinearities,* Electronic Journal of Differential Equations, 2021(2021), 1-33.

[11] S.A. Messaoudi, A.A. Talahmeh, M.M. Al-Gharabli and M.Alahyone, *On the existence and stability of a nonlinear wave system with variable exponents,* Asymptotic Analysis 128(2)(2022), 211-238.

[12] S.A. Messaoudi, A.A. Talahmeh and J.H. Al-Shail, *Nonlinear damped wave equation: Existence and blow-up,* Comput. Math. Appl. 74(12)(2017), 3024-3041.

[13] S.H. Park and J.R. Kang, *Blow-up of solutions for a viscoelastic wave equation with variable exponents,* Math. Methods Appl. Sci. 42(6)(2019), 2083-2097.

[14] E. Pişkin, *Finite time blow up of solutions of the Kirchhoff-type equation with variable exponents,* International Journal of Nonlinear Analysis and Applications, 11(1)(2020), 37-45.

[15] E. Pişkin, B. Okutmuştur, *An Introduction to Sobolev Spaces,* Bentham Science, 2021.

[16] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory,* Springer Science & Business Media, 2000.

[17] M. Shahrouzi, *On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities,* Comput. Math. Appl. 75(11)(2018), 3946-3956.