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# Hemi-Slant submanifolds of lcK manifolds as warped products

**Umar Mohd Khan*** **and Viqar Azam Khan**
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
Email: umar.007.morpheus@gmail.com, viqarster@gmail.com

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**Abstract**

We study immersions of a hemi-slant submanifold of lcK manifolds as a warped product with the leaves of the holomorphic (respectively slant) distribution warped and establish characterisation theorems and estimations for the squared length of the second fundamental form in both cases.

## 1 Introduction

Vaisman introduced locally conformal Kähler(lcK) manifolds as a generalisation of Kähler manifolds [21, 33–38]. An lcK manifold is a Hermitian manifold that can be written as the union of Kähler manifolds such that the lcK metric is locally conformal to these Kähler metrics. LcK manifolds are characterised by the existence of a globally defined closed 1-form $\omega$, called the *Lee form*, such that the fundamental 2-form of the lcK metric satisfies $d\Omega = \Omega \wedge \omega$. The Lee form and its associated Lee Vector field play an important part in the geometry of lcK manifolds.

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*Corresponding Author

From an extrinsic geometric standpoint, holomorphic and totally real submanifolds are important objects of study in the setting of almost Hermitian manifolds. Bejancu [5, 6] defined CR submanifolds as a generalisation of holomorphic and totally real submanifolds which were further studied by Chen [11, 12]. Later, Chen [13, 14] extended the class of holomorphic and totally real submanifolds by introducing the notion of slant submanifolds. The concept was further generalised to pointwise slant submanifolds [20] by the same author. The study of CR submanifolds and slant submanifolds was later generalised by several authors to semi-slant submanifolds, hemi-slant submanifolds(also called pseudo-slant submanifolds) and bi-slant submanifolds, in various ambient manifolds.

Papaghiuc [28] studied semi-slant submanifolds in almost Hermitian manifolds. Cabrerizo et al.

[9, 10] studied semi-slant submanifolds in Sasakian manifolds. Slant and semi-slant submanifolds in almost product Riemannian manifolds were studied in [2, 24, 29]. Hemi-slant submanifolds were also studied in nearly Kenmotsu manifolds [4], LCS-manifolds [3] and locally product Riemannian manifolds [31].

Bishop and O'Neill [7] while studying examples of manifolds with negative sectional curvature, defined warped product manifolds by homothetically warping the product metric on a product manifold. Warped products are a natural generalisation of Riemannian products and they have found extensive applications in relativity. Most notably the Schwarzschild metric describing the gravitational field outside a spherical mass under certain assumptions and the Robertsen Walker metric (FLRW metric) are examples of warped product metrics. A natural example of warped product manifolds are surfaces of revolution. Hiepko [22] gave a characterisation for a Riemannian manifold to be the warped product of its submanifolds, generalising the deRham decomposition theorem for product manifolds. Later on Nölker [27] and Chen [15, 16, 19] initiated the study of extrinsic geometry of warped product manifolds.

Chen [17,18] initiated the study of CR submanifolds immersed as warped products in Kähler manifolds. Bonanzinga and Matsumoto [8,25,26] continued the study in the setting of lcK manifolds. Nargis Jamal et al. [23] studied Generic warped products in lcK manifolds. Further studies of semi-slant and hemi-slant submanifolds of lcK manifolds were carried out in [1, 30, 32]. Generic submanifolds, CR-submanifolds and semi-slant submanifolds immeresd as warped products in lcK manifolds were studied by [1, 23].

We continue the study by considering hemi-slant submanifolds in an lcK manifold. In particular we give characterisation theorems and establish estimations for the length of the second fundamental forms of hemi-slant submanifolds immersed as warped products in an lcK manifold.

## 2 Preliminaries

**Definition 2.1.** *A Hermitian Manifold $(\widetilde{M}^{2n}, J, g)$ is said to be a locally conformal Kähler (l.c.K.) manifold if there exists an open cover $\{U_i\}_{i\in I}$ of $\widetilde{M}^{2n}$ and a family $\{f_i\}_{i\in I}$ of $C^\infty$ functions $f_i : U_i \to \mathbb{R}$ such that for each $i \in I$, the metric*

$$g_i = e^{-f_i} g|_{U_i} \tag{2.1}$$

*on $U_i$ is a Kähler metric.*

Given an l.c.K. manifold $(\widetilde{M}^{2n}, J, g)$, let $U, V$ denote smooth sections of $T\widetilde{M}^{2n}$, then the local 1-forms $df_i$ glue up to a globally defined closed 1-form $\omega$, called the *Lee form*, and it satisfies the following equation

$$d\Omega = \Omega \wedge \omega \tag{2.2}$$

where $\Omega(U, V) = g(JU, V)$ is the fundamental 2-form associated to $(J, g)$.

Denote by $\Theta$ the global closed 1-form defined as $\Theta = \omega \circ J$. Then, $\Theta$ is called the *anti Lee form*.

Denote by $B$ and $A$ the vector fields equivalent to $\omega$ and $\Theta$ respectively with respect to $g$, i.e. $\omega(U) = g(B, U)$ and $\Theta(U) = g(A, U)$.

$B$ and $A$ are respectively called the *Lee vector field* and the *anti Lee vector field*, and are related as

$$A = -JB \tag{2.3}$$

Let $\overline{\nabla}$ denote the Levi-Civita connection of $(\widetilde{M}^{2n}, g)$ and $\widetilde{\nabla}_i$ denote the Levi-Civita connection of the local metrics $g_i$ for all $i \in I$. Then $\widetilde{\nabla}_i$ glue up to a globally defined torsion-free linear connection $\widetilde{\nabla}$ on $\widetilde{M}^{2n}$ given by

$$\widetilde{\nabla}_U V = \overline{\nabla}_U V - \frac{1}{2}\{\omega(U)V + \omega(V)U - g(U, V)B\} \tag{2.4}$$

where $U, V \in T\widetilde{M}^{2n}$ and satisfying

$$\widetilde{\nabla} g = \omega \otimes g \tag{2.5}$$

$\widetilde{\nabla}$ is called the *Weyl connection* of the l.c.K. manifold $(\widetilde{M}^{2n}, J, g)$. As $g_i$ are Kähler metrics, the almost complex structure $J$ is parallel with respect to the Weyl connection, i.e. $\widetilde{\nabla} J = 0$. This gives

$$\overline{\nabla}_U JV = J\overline{\nabla}_U V + \frac{1}{2}\{\Theta(V)U - \omega(V)JU - g(U,V)A + \Omega(U,V)B\} \quad (2.6)$$

Now as $\omega$ is a closed form on $\widetilde{M}^{2n}$, we have

$$(\overline{\nabla}_U \omega)V = (\overline{\nabla}_V \omega)U \quad (2.7)$$

Hence using (2.6) and (2.7) we have

$$(\overline{\nabla}_U \Theta)V = U\omega(JV) - \omega(\overline{\nabla}_U JV) + \frac{1}{2}\Theta(V)\omega(U) - \frac{1}{2}\omega(V)\Theta(U) + g(JU,V)||B||^2$$

as $\omega(A) = g(B,A) = 0$ from (2.3) and $\omega(B) = g(B,B) = ||B||^2$
Thus, we have

$$(\overline{\nabla}_U \Theta)V = (\overline{\nabla}_U \omega)V + \frac{1}{2}\Theta(V)\omega(U) - \frac{1}{2}\omega(V)\Theta(U) + g(JU,V)||B||^2 \quad (2.8)$$

Let $M^m$ be a Riemannian manifold isometrically immersed in an l.c.K. manifold $(\widetilde{M}^{2n}, J, g)$. Let $U, V, W$ denote smooth sections of $TM^m$ and $\xi, \eta$ denote smooth sections of $T^{\perp}M^m$.

The Gauss and Weingarten formulae with respect to the Riemannian connection of $\widetilde{M}^{2n}$ are given as

$$\overline{\nabla}_U V = \nabla_U V + h(U,V) \quad (2.9)$$

$$\overline{\nabla}_U \xi = -\mathfrak{A}_\xi U + \nabla^{\perp}_U \xi \quad (2.10)$$

where $h$ is the second fundamental form, $\mathfrak{A}$ is the shape operator and $\nabla, \nabla^{\perp}$ are respectively the induced connections in the tangent bundle and the normal bundle of $M^m$ with respect to $\overline{\nabla}$.

The Gauss and Weingarten formulae with respect to the Weyl connection of $\widetilde{M}^{2n}$ are given as

$$\widetilde{\nabla}_U V = \hat{\nabla}_U V + \widetilde{h}(U,V) \quad (2.11)$$

$$\widetilde{\nabla}_U \xi = -\widetilde{\mathfrak{A}}_\xi U + \widetilde{\nabla}^{\perp}_U \xi \quad (2.12)$$

where $\widetilde{h}$ is the second fundamental form, $\widetilde{\mathfrak{A}}$ is the shape operator and $\hat{\nabla}, \widetilde{\nabla}^{\perp}$ are respectively the induced connections in the tangent bundle and the normal bundle of $M^m$ with respect to $\widetilde{\nabla}$.

Let $H$ denote the trace of $h$, then $H$ is called the mean curvature vector of $M^m$ in $(\widetilde{M}^{2n}, J, g)$ and is a smooth section of $T^{\perp}M^m$. We say $M^m$ is a totally umbilic submanifold of $(\widetilde{M}^{2n}, J, g)$, if $h(U, V) = g(U, V)H$. We say $M^m$ is a totally geodesic submanifold of $(\widetilde{M}^{2n}, J, g)$, if $h(U, V) = 0$.

Let $B^T, B^N$ denote the tangential and normal components of the Lee vector field $B$ and let $A^T, A^N$ denote the tangential and normal components of the anti Lee vector field $A$.

From (2.4), we have the following relations

$$\hat{\nabla}_U V = \nabla_U V - \frac{1}{2}\left\{\omega(U)V + \omega(V)U - g(U, V)B^T\right\} \tag{2.13}$$

$$\widetilde{h}(U, V) = h(U, V) + \frac{1}{2}g(U, V)B^N \tag{2.14}$$

$$\widetilde{\mathfrak{A}}_\xi U = \mathfrak{A}_\xi U + \frac{1}{2}\omega(\xi)U \tag{2.15}$$

$$\widetilde{\nabla}^{\perp}_U \xi = \nabla^{\perp}_U \xi - \frac{1}{2}\omega(U)\xi \tag{2.16}$$

Now define

$$JU = PU + FU \qquad J\xi = t\xi + f\xi \tag{2.17}$$

where $PU, t\xi$ and $FU, f\xi$ are respectively the tangential and normal parts. Then, we have

$$\begin{aligned} P^2 + tF &= -I & f^2 + Ft &= -I \\ FP + fF &= 0 & tf + Pt &= 0 \end{aligned} \tag{2.18}$$

Now from (2.3) and (2.17) we have

$$A^T = -PB^T - tB^N \qquad A^N = -FB^T - fB^N \tag{2.19}$$

Define the covariant differentiation of $P$, $F$, $t$ and $f$ with respect to the Levi-Civita connection of $\widetilde{M}^{2n}$ as

$$\begin{aligned} (\overline{\nabla}_U P)V &= \nabla_U PV - P\nabla_U V \\ (\overline{\nabla}_U F)V &= \nabla^{\perp}_U FV - F\nabla_U V \\ (\overline{\nabla}_U t)\xi &= \nabla_U t\xi - t(\nabla^{\perp}_U \xi) \\ (\overline{\nabla}_U f)\xi &= \nabla^{\perp}_U f\xi - f(\nabla^{\perp}_U \xi) \end{aligned} \tag{2.20}$$

Similarly, define the covariant differentiation of $P$, $F$, $t$ and $f$ with respect to the Weyl connection of $\widetilde{M}^{2n}$ as

$$\begin{aligned}(\widetilde{\nabla}_U P)V &= \hat{\nabla}_U PV - P\hat{\nabla}_U V\\ (\widetilde{\nabla}_U F)V &= \widetilde{\nabla}^{\perp}_U FV - F\hat{\nabla}_U V\\ (\widetilde{\nabla}_U t)\xi &= \hat{\nabla}_U t\xi - t\widetilde{\nabla}^{\perp}_U \xi\\ (\widetilde{\nabla}_U f)\xi &= \widetilde{\nabla}^{\perp}_U f\xi - f\widetilde{\nabla}^{\perp}_U \xi\end{aligned} \tag{2.21}$$

Then as $\widetilde{\nabla}J = 0$, using (2.13), (2.14), (2.15), (2.16) we have

$$\begin{aligned}(\overline{\nabla}_U P)V &= \mathfrak{A}_{FV}U + th(U,V) + \frac{1}{2}\{\Theta(V)U - \omega(V)PU\\ &\quad + g(PU,V)B^T - g(U,V)A^T\}\\ (\overline{\nabla}_U F)V &= fh(U,V) - h(U,PV) + \frac{1}{2}\{g(PU,V)B^N - g(U,V)A^N\\ &\quad - \omega(V)FU\}\\ (\overline{\nabla}_U t)\xi &= \mathfrak{A}_{f\xi}U - P\mathfrak{A}_{\xi}U + \frac{1}{2}\{g(FU,\xi)B^T - \omega(\xi)PU + \Theta(\xi)U\}\\ (\overline{\nabla}_U f)\xi &= -h(U,t\xi) - F\mathfrak{A}_{\xi}U + \frac{1}{2}\{g(FU,\xi)B^N - \omega(\xi)FU\}\end{aligned} \tag{2.22}$$

Define the covariant derivative of the second fundamental form $h$ of the Riemannian connection $\overline{\nabla}$ as

$$(\overline{\nabla}_U h)(V,W) = \nabla^{\perp}_U h(V,W) - h(\nabla_U V,W) - h(V,\nabla_U W) \tag{2.23}$$

Let $\overline{R}, R, R^{\perp}$ denote the curvature tensors associated to $\overline{\nabla}, \nabla, \nabla^{\perp}$ respectively. Then the Gauss, Codazzi and Ricci equations are respectively given by

$$\begin{aligned}g(\overline{R}(U,V)W,S) &= g(R(U,V)W,S) + g(h(V,S),h(U,W))\\ &\quad - g(h(U,S),h(V,W))\end{aligned} \tag{2.24}$$

$$(\overline{R}(U,V)W)^{\perp} = (\overline{\nabla}_U h)(V,W) - (\overline{\nabla}_V h)(U,W) \tag{2.25}$$

$$g(\overline{R}(U,V)\xi,\eta) = g(R^{\perp}(U,V)\xi,\eta) - g([A_{\xi},A_{\eta}]U,V) \tag{2.26}$$

Bishop and O'Neill [7] defined warped product as

**Definition 2.2.** *Let $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ be Riemmanian manifolds and let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the canonical projections. Let $\lambda : M_1 \to (0,\infty)$ be a smooth function. Then the warped product* *manifold* $(M, g) = M_1 \times {}_\lambda M_2$ *is defined as the manifold* $M_1 \times M_2$ *equipped with the Riemannian metric*

$$g = \pi_1^\star g_1 + \lambda^2 \pi_2^\star g_2 \tag{2.27}$$

Warped product manifolds are a generalization of the usual product of two Riemannian manifolds. In fact we have the following characterisation theorem.

**Theorem 2.1** ( [22]). *Let* $(M^m, g)$ *be a connected Riemannian manifold equipped with orthogonal, complementary, involutive distributions* $\mathcal{D}_1$ *and* $\mathcal{D}_2$*. Further let the leaves of* $\mathcal{D}_1$ *be totally geodesic and the leaves of* $\mathcal{D}_2$ *be extrinsic spheres in* $M^m$*, where by extrinsic spheres we mean totally umbilic submanifolds such that the mean curvature vector is parallel in the normal bundle. Then* $(M^m, g)$ *is locally a warped product* $(M, g) = M_1 \times {}_\lambda M_2$*, where* $M_1$ *and* $M_2$ *respectively denote the leaves of* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *and* $\lambda : M_1 \to (0, \infty)$ *is a smooth function such that grad*$(\ln \lambda)$ *is the mean curvature vector of* $M_2$ *in* $M$*.*

*Further, if* $(M^m, g)$ *is simply connected and complete, then* $(M^m, g)$ *is globally a warped product.*

For $(M_1^{n_1}, g_1)$, $(M_2^{n_2}, g_2)$ and $(M, g)$ denote respectively the Levi-Civita connections by $\nabla^1$, $\nabla^2$ and $\nabla$. Given any smooth function $\lambda : M_1 \to \mathbb{R}$, let grad$(\lambda)$ denote the lift of the gradient vector field of $\lambda$ to $(M, g)$.

**Theorem 2.2** ( [22]). *Given a warped product manifold* $(M, g) = M_1 \times {}_\lambda M_2$ *of Riemmanian manifolds* $(M_1^{n_1}, g_1)$ *and* $(M_2^{n_2}, g_2)$*, we have for all* $X, Y \in \mathcal{L}(M_1)$ *and* $Z, W \in \mathcal{L}(M_2)$*,*

$$\nabla_X Y = \nabla^1_X Y \tag{2.28}$$

$$\nabla_X Z = \nabla_Z X = X(\ln \lambda) Z \tag{2.29}$$

$$\nabla_Z W = \nabla^2_Z W - g(Z, W)\, grad(\ln \lambda) \tag{2.30}$$

It follows from Theorem 2.2 that $\mathcal{H} = -\text{grad}(\ln \lambda)$ is the mean curvature vector of $M_2$ in $M$. Let $M^m$ be a Riemannian manifold isometrically immersed in an l.c.K. manifold $(\widetilde{M}^{2n}, J, g)$.

$M^m$ is said to be a *hemi-slant submanifold* if it admits two orthogonal complementary distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, such that $\mathcal{D}^\perp$ is totally real, i.e.

$J\mathcal{D}^\perp \subseteq T^\perp M^m$ and $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq 0, \frac{\pi}{2}$, i.e. $P^2 Z = -\cos^2\theta Z$, for every smooth vector field $Z \in \mathcal{D}^\theta$.

The tangent bundle and the normal bundle of a hemi-slant submanifold admits an orthogonal decomposition as

$$TM^m = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \qquad T^\perp M^m = J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu \tag{2.31}$$

where $\mu$ is the orthogonal complementary distribution of $J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta$ in $T^\perp M^m$ and is an invariant subbundle of $T^\perp M^m$ with respect to $J$. It is easy to observe that,

$$\begin{array}{ccc} P\mathcal{D}^\perp = \{0\} & P\mathcal{D}^\theta = \mathcal{D}^\theta & F\mathcal{D}^\perp = J\mathcal{D}^\perp \\ t(J\mathcal{D}^\perp) = \mathcal{D}^\perp & t(F\mathcal{D}^\theta) = \mathcal{D}^\theta & t(\mu) = \{0\} \\ f(F\mathcal{D}^\perp) = \{0\} & f(F\mathcal{D}^\theta) = F\mathcal{D}^\theta & f(\mu) = \mu \end{array} \tag{2.32}$$

Let $M^m$ be a hemi-slant manifold isometrically immersed in an l.c.K. manifold $(\widetilde{M}^{2n}, J, g)$ such that the totally real distribution $\mathcal{D}^\perp$ and the slant distribution $\mathcal{D}^\theta$ are both involutive. Let $M_\perp^{n_1}$ and $M_\theta^{2n_2}$ respectively denote the leaves of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, where $n_1 = \dim_{\mathbb{R}} \mathcal{D}^\perp$ and $2n_2 = \dim_{\mathbb{R}} \mathcal{D}^\theta$. We say $M^m$ is a

- *mixed totally geodesic hemi-slant submanifold* if $h(\mathcal{D}^\perp, \mathcal{D}^\theta) = \{0\}$.
- *hemi-slant product submanifold* if $M^m$ can be expressed locally as $M_\perp \times M_\theta$.
- *hemi-slant warped product submanifold* if $M^m$ can be expressed locally as $M_\perp \times {}_\lambda M_\theta$ for some smooth function $\lambda : M_\perp \to (0, \infty)$.
- *warped product hemi-slant submanifold* if $M^m$ can be expressed locally as $M_\theta \times {}_\lambda M_\perp$ for some smooth function $\lambda : M_\theta \to (0, \infty)$.

From here on we use $X, Y, X_1$ to denote smooth vector fields in $\mathcal{L}(M_\perp)$ and $Z, W$ to denote smooth vector fields in $\mathcal{L}(M_\theta)$.

**Theorem 2.3.** *[30] Let $M^m$ be a hemi-slant submanifold of an l.c.K. manifold $\widetilde{M}^{2n}$. Then*

- *the totally real distribution $\mathcal{D}^\perp$ is involutive.*

- *the leaves of the totally real distribution* $\mathcal{D}^{\perp}$ *are totally geodesic in* $M^m$ *if and only if*

$$g(\mathfrak{A}_{JX}Z - \mathfrak{A}_{FZ}X, Y) = \frac{1}{2}\omega(PZ)g(X, Y) \tag{2.33}$$

- *the leaves of the totally real distribution* $\mathcal{D}^{\perp}$ *are totally umbilic in* $M^m$ *if and only if*

$$g(\mathfrak{A}_{JX}Z - \mathfrak{A}_{FZ}X, Y) = \left(\frac{1}{2}\omega(PZ) + g(\mathcal{H}, PZ)\right) g(X, Y) \tag{2.34}$$

*for some smooth vector field* $\mathcal{H} \in \mathcal{D}^{\theta}$.

**Theorem 2.4.** *[30] Let* $M^m$ *be a hemi-slant submanifold of an l.c.K. manifold* $\widetilde{M}^{2n}$. *Then*

- *the slant distribution* $\mathcal{D}^{\theta}$ *is involutive if and only if*

$$g(\mathfrak{A}_{FPZ}X, W) + g(\nabla^{\perp}_{W}FZ, JX) = g(\mathfrak{A}_{FPW}X, Z) + g(\nabla^{\perp}_{Z}FW, JX) \tag{2.35}$$

- *the leaves of the slant distribution* $\mathcal{D}^{\theta}$ *are totally geodesic in* $M^m$ *if and only if*

$$\omega(\mathcal{D}^{\perp}) = \{0\} \text{ and } g(\mathfrak{A}_{FPZ}X, W) + g(\nabla^{\perp}_{W}FZ, JX) = 0 \tag{2.36}$$

- *the leaves of the slant distribution* $\mathcal{D}^{\theta}$ *are totally umbilic in* $M^m$ *if and only if*

$$g(\mathfrak{A}_{FPZ}X, W) + g(\nabla^{\perp}_{W}FZ, JX) = \sin^2\theta \left(\frac{1}{2}\omega(X) + g(\mathcal{H}, X)\right) g(Z, W) \tag{2.37}$$

*for some smooth vector field* $\mathcal{H} \in \mathcal{D}^{\perp}$.

**Notations:** Let $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ be the totally real and slant distributions on a hemi-slant submanifold $M^m$ of an lcK manifold $\widetilde{M}^{2n}$ such that both distributions are involutive and let $M_{\perp}$ and $M_{\theta}$ respectively denote the leaves of the distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ respectively. Then $\mathcal{D}^{\perp}(p,q) = T_{(p,q)}(M_{\perp} \times \{q\})$ and $\mathcal{D}^{\theta}(p,q) = T_{(p,q)}(\{p\} \times M_{\theta})$. Let $\mathcal{L}(M_{\perp})$ and $\mathcal{L}(M_{\theta})$ respectively denote the set of lifts of vector fields from $M_{\perp}$ and $M_{\theta}$ to $M$. Then $X \in$

$\mathcal{L}(M_\perp)$ if and only if $X|_{\{p\}\times M_\theta}$ is constant for every $p \in M_\perp$. Similarly, $Z \in \mathcal{L}(M_\theta)$ if and only if $Z|_{M_\perp\times\{q\}}$ is constant for every $q \in M_\theta$. Also, if $\pi_\perp : M_\perp \times M_\theta \to M_\perp$ and $\pi_\theta : M_\perp \times M_\theta \to M_\theta$ are the canonical projections, we have $d\pi_\perp(\mathcal{L}(M_\perp)) = TM_\perp$ and $d\pi_\theta(\mathcal{L}(M_\theta)) = TM_\theta$. It is clear that a general vector field in $\mathcal{D}^\perp$ (respectively $\mathcal{D}^\theta$) need not be in $\mathcal{L}(M_\perp)$ (respectively $\mathcal{L}(M_\theta)$).

## 3 Hemi-Slant Warped Product Submanifolds of l.c.K. manifolds

**Lemma 3.1.** *Given a hemi-slant warped product submanifold $M = M_\perp \times {}_\lambda M_\theta$ in an lcK manifold $(\widetilde{M}^{2n}, J, g)$, we have for all $X, Y, X_1 \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$,*

$$g(h(X,Z),JY) = g(h(Y,Z),JX) \tag{3.1}$$
$$g(h(X,Z),FW) = g(h(X,W),FZ) \tag{3.2}$$
$$g(h(Z,W),JX) = g(h(X,Z),FW) + \frac{1}{2}g(Z,W)g(JB,X) \tag{3.3}$$
$$X(\ln\lambda) = \frac{1}{2}g(B,X) \tag{3.4}$$
$$\begin{aligned} g(h(X,Y),JX_1) &= g(h(X,X_1),JY) - \frac{1}{2}g(X,Y)g(B,JX_1) \\ &\quad + \frac{1}{2}g(X,X_1)g(B,JY) \end{aligned} \tag{3.5}$$

*Proof.* For all $X, Y, X_1 \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$, we have using (2.6) and (2.29),

$$\begin{aligned} g(h(X,Z),JY) &= g(\overline{\nabla}_Z X, JY) = -g(J\overline{\nabla}_Z X, Y) = -g(\overline{\nabla}_Z JX, Y) = g(\mathfrak{A}_{JX}Z, Y) \\ &= g(h(Y,Z),JX) \end{aligned}$$

which implies (3.1). Similarly,

$$\begin{aligned} g(h(X,Z),FW) &= g(\overline{\nabla}_X Z, JW) - g(\overline{\nabla}_X Z, PW) \\ &= -g(J\overline{\nabla}_X Z, W) - g(\nabla_X Z, PW) \\ &= -g(\overline{\nabla}_X JZ, W) - X(\ln\lambda)g(Z,PW) \end{aligned}$$

$$= -X(\ln\lambda)g(PZ,W) + g(\mathfrak{A}_{FZ}X,W) - X(\ln\lambda)g(Z,PW)$$

which implies (3.2). Repeating the above calculation, we have

$$\begin{aligned} g(h(X,Z),FW) &= -g(J\overline{\nabla}_Z X,W) - g(\nabla_Z X,PW) \\ &= -g(\overline{\nabla}_Z JX,W) - \frac{1}{2}g(JB,X)g(Z,W) \\ &\quad - \frac{1}{2}g(B,X)g(JZ,W) - X(\ln\lambda)g(Z,PW) \\ &= g(\mathfrak{A}_{JX}Z,W) - \frac{1}{2}g(JB,X)g(Z,W) \\ &\quad - \frac{1}{2}g(B,X)g(PZ,W) - X(\ln\lambda)g(Z,PW) \end{aligned}$$

Using (3.2) and comparing symmetric and skew symmetric terms in $Z$ and $W$ we have,

$$g(h(X,Z),FW) = g(h(Z,W),JX) - \frac{1}{2}g(JB,X)g(Z,W)$$

which proves (3.3) and

$$0 = \left(X(\ln\lambda) - \frac{1}{2}g(B,X)\right)g(PZ,W)$$

which proves (3.4). Finally,

$$\begin{aligned} g(h(X,Y),JX_1) &= g(\overline{\nabla}_X Y,JX_1) \\ &= -g(J\overline{\nabla}_X Y,X_1) \\ &= -g(\overline{\nabla}_X JY,X_1) - \frac{1}{2}g(X,X_1)g(JB,Y) + \frac{1}{2}g(X,Y)g(JB,X_1) \\ &= g(\mathfrak{A}_{JY}X,X_1) - \frac{1}{2}g(X,X_1)g(JB,Y) + \frac{1}{2}g(X,Y)g(JB,X_1) \end{aligned}$$

which gives (3.5). □

**Remark 3.1.** *Given a hemi-slant warped product submanifold $M_\perp \times {}_\lambda M_\theta$ of an l.c.K manifold $\widetilde{M}^{2n}$, let $\{X_i\}_{i=1}^p$ and $\{Z_j, \beta PZ_j\}_{j=1}^q$ respectively be local* *orthonormal frames of $TM_\perp$ and $TM_\theta$. Then a local orthonormal frame of $\widetilde{M}^{2n}$ is*

$$\left\{\widehat{X_i} = X_i\right\} \cup \left\{\widehat{Z_j} = \frac{Z_j}{\lambda}, \widehat{PZ_j} = \frac{\beta PZ_j}{\lambda}\right\} \cup \left\{\widehat{JX_i} = JX_i\right\}$$
$$\cup \left\{\widehat{FZ_j} = \frac{\alpha FZ_j}{\lambda}, \widehat{FPZ_j} = \frac{\alpha\beta FPZ_j}{\lambda}\right\} \cup \left\{\widehat{\xi_k}, \widehat{J\xi_k}\right\}$$

*where $\alpha = \csc\theta$, $\beta = \sec\theta$ and*

$\left\{\widehat{X_i} : 1 \le i \le n_1\right\}$ *is an orthonormal basis of $\mathcal{D}^\perp$*

$\left\{\widehat{Z_j}, \widehat{PZ_j} : 1 \le j \le n_2\right\}$ *is an orthonormal basis of $\mathcal{D}^\theta$*

$\left\{\widehat{JX_i} : 1 \le i \le n_1\right\}$ *is an orthonormal basis of $J\mathcal{D}^\perp$*

$\left\{\widehat{FZ_j}, \widehat{FPZ_j} : 1 \le j \le n_2\right\}$ *is an orthonormal basis of $F\mathcal{D}^\theta$*

$\left\{\widehat{\xi_k}, \widehat{J\xi_k} : 1 \le k \le \dfrac{n - n_1 - 2n_2}{2}\right\}$ *is an orthonormal basis of $\mu$*

*However, while $Z_j, \beta PZ_j \in \mathcal{L}(M_\theta)$ we have $\widehat{Z_j}, \widehat{PZ_j} \notin \mathcal{L}(M_\theta)$ in general, as $\lambda$ is a function on $M_\perp$. Also, note that*

$$J\left(\widehat{Z_j}\right) = J\left(\frac{Z_j}{\lambda}\right) = \frac{PZ_j}{\lambda} + \frac{FZ_j}{\lambda} = \cos\theta\widehat{PZ_j} + \sin\theta\widehat{FZ_j}$$
$$J\left(\widehat{PZ_j}\right) = J\left(\sec\theta\frac{PZ_j}{\lambda}\right) = \frac{\sec\theta P^2 Z_j}{\lambda} + \frac{\sec\theta FPZ_j}{\lambda}$$
$$= -\cos\theta\widehat{Z_j} + \sin\theta\widehat{FPZ_j}$$

We now give a characterisation for hemi-slant warped product submanifolds of l.c.K. manifolds.

**Theorem 3.1.** *Let $M^m$ be a hemi-slant submanifold of an l.c.K. manifold $\widetilde{M}^{2n}$. Then the following are equivalent*

1. *$M^m$ is a hemi-slant warped product submanifold $M_\perp \times {}_\lambda M_\theta$ of $\widetilde{M}^{2n}$*

2. *For every $X, Y \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$ we have*

$$g(\mathfrak{A}_{JX}Z - \mathfrak{A}_{FZ}X, Y) = \frac{1}{2}\omega(PZ)g(X,Y)$$
$$g(\mathfrak{A}_{FPZ}X, W) + g(\nabla^\perp_W FZ, JX) = \sin^2\theta\left(\frac{1}{2}\omega(X) - X(\ln\lambda)\right)g(Z,W) \tag{3.6}$$

*for some smooth function* $\lambda : M_\perp \to (0,\infty)$.

3. *For every* $X \in \mathcal{L}(M_\perp)$ *and* $Z \in \mathcal{L}(M_\theta)$ *we have*

$$\nabla_X Z = \nabla_Z X = \frac{1}{2}\omega(X)Z \tag{3.7}$$

*Also, in this case we have the mean curvature vector* $\mathcal{H}$ *of* $M_\theta$ *in* $M^m$ *is*

$$\mathcal{H} = -grad(\ln\lambda) = -\frac{1}{2}B|_{\mathcal{D}^\perp} \tag{3.8}$$

*where* $B|_{\mathcal{D}^\perp}$ *is the component of* $B$ *along* $\mathcal{D}^\perp$.

*Proof.* (1)⇔(2) This follows from Theorem 2.3, Theorem 2.4 and the fact that $\nabla \ln\lambda \in \mathcal{L}(M_\perp)$ which implies for all $X \in \mathcal{L}(M_\perp)$ and $Z \in \mathcal{L}(M_\theta)$

$$\begin{aligned} g(\nabla_Z(\nabla \ln\lambda), X) &= ZX(\ln\lambda) - g(\nabla \ln\lambda, \nabla_Z X) \\ &= [Z,X](\ln\lambda) - \nabla_Z X(\ln\lambda) \quad (\text{as } Z(\ln\lambda) = 0) \\ &= -\nabla_X Z(\ln\lambda) \\ &= g(Z, \nabla_X(\nabla \ln\lambda)) \\ &= 0 \end{aligned}$$

as $\mathcal{D}^\perp$ is totally geodesic. Also, (3.8) follows from Lemma 3.1 (3.4).
(1)⇔(3) Let $M = M_\perp \times {}_\lambda M_\theta$ be a hemi-slant warped product submanifold. Then (3.7) and (3.8) follow from (2.29) and Lemma 3.1 (3.4).

Conversely, let $M^m$ be a hemi-slant submanifold of an l.c.K. manifold $\widetilde{M}^{2n}$ such that (3.7) holds. Then for all $X, Y \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$ we have

$$\begin{aligned} g([X,Y], Z) &= g(\nabla_X Y - \nabla_Y X, Z) \\ &= -g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \\ &= 0 \end{aligned}$$

which implies $\mathcal{D}^\perp$ is involutive.

$$g(\nabla_X Y, Z) = -g(\nabla_X Z, Y) = 0$$

which implies leaves of $\mathcal{D}^\perp$ are totally geodesic in $M$.

$$\begin{aligned} g([Z,W],X) &= g(\nabla_Z W - \nabla_W Z, X) \\ &= -g(\nabla_Z X, W) + g(\nabla_W X, Z) \\ &= -\frac{1}{2}\omega(X)g(Z,W) + \frac{1}{2}\omega(X)g(W,Z) = 0 \end{aligned}$$

which implies $\mathcal{D}^\theta$ is involutive.

$$\begin{aligned} g(\nabla_Z W, X) &= -g(\nabla_Z X, W) \\ &= -\frac{1}{2}\omega(X)g(Z,W) \\ &= -\frac{1}{2}g(Z,W)g(B^T, X) \end{aligned}$$

which implies leaves of $\mathcal{D}^\theta$ are totally umbilical in $M$ with mean curvature vector $-\frac{1}{2}B|_{\mathcal{D}^\perp}$.

$$g\left(\nabla_Z B|_{\mathcal{D}^\perp}, X\right) = \frac{1}{2}\omega\left(B|_{\mathcal{D}^\perp}\right) g(Z,X) = 0$$

which implies $B|_{\mathcal{D}^\perp}$ is parallel in the normal bundle of $M_\theta$ in $M$.

Hence by Theorem 2.1 we have $M = M_\perp \times {}_\lambda M_\theta$ is a hemi-slant warped product submanifold. □

We conclude our study of hemi-slant warped product submanifolds of l.c.K. manifolds by giving an inequality for the norm of the second fundamental form.

**Theorem 3.2.** *Let $M = M_\perp \times {}_\lambda M_\theta$ be a hemi-slant warped product submanifold in an lcK manifold $(\widetilde{M}^{2n}, J, g)$. Then the norm of the second fundamental form satisfies the inequality*

$$\begin{aligned} ||h||^2 \geq \frac{(n_1 + n_2 - 1)}{2} \|B|_{J\mathcal{D}^\perp}\|^2 + 2g\left(H_{\mathcal{D}^\perp}|_{J\mathcal{D}^\perp}, B|_{J\mathcal{D}^\perp}\right) \\ + g\left(H_{\mathcal{D}^\theta}|_{J\mathcal{D}^\perp}, B|_{J\mathcal{D}^\perp}\right) - K \end{aligned} \tag{3.9}$$

*where $n_1 = \dim_{\mathbb{R}} \mathcal{D}^\perp$, $2n_2 = \dim_{\mathbb{R}} \mathcal{D}^\theta$, $H_{\mathcal{D}^\perp}$ and $H_{\mathcal{D}^\theta}$ are respectively the components of the mean curvature vector $H$ of $M$ in $\widetilde{M}^{2n}$ along $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$*

*and given any orthonormal basis* $\left\{\widehat{X_i} : 1 \leq i \leq n_1\right\}$ *of* $\mathcal{D}^{\perp}$,
$K = 2\sum_i g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_i}\right) g\left(B, \widehat{JX_i}\right)$ .
*If equality holds then we have*

- *Image*$(h) \subseteq (J\mathcal{D}^{\perp} \oplus F\mathcal{D}^{\theta})$, *and*
- $M_\theta$ *is totally umbilical in* $\widetilde{M}^{2n}$ *(with mean curvature vector* $\mathcal{H} = -\frac{1}{2}B|_{\mathcal{D}^{\perp}}$*)* *if and only if,* $M$ *is mixed-totally geodesic in* $\widetilde{M}^{2n}$.

*Proof.*

$$\begin{aligned}
||h||^2 =& \left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp})\big|_{J\mathcal{D}^{\perp}} \right\|^2 + \left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\theta})\big|_{J\mathcal{D}^{\perp}} \right\|^2 + \left\| h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta})\big|_{J\mathcal{D}^{\perp}} \right\|^2 \\
&+ \left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp})\big|_{F\mathcal{D}^{\theta}} \right\|^2 + \left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\theta})\big|_{F\mathcal{D}^{\theta}} \right\|^2 + \left\| h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta})\big|_{F\mathcal{D}^{\theta}} \right\|^2 \\
&+ \left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp})\big|_{\mu} \right\|^2 + \left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\theta})\big|_{\mu} \right\|^2 + \left\| h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta})\big|_{\mu} \right\|^2
\end{aligned}$$

From (3.5) and Remark 3.1 we have

$$\begin{aligned}
\left\| h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp})\big|_{J\mathcal{D}^{\perp}} \right\|^2 =& \sum_i g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_i}\right)^2 + \sum_{i\neq j\neq k} g\left(h\left(\widehat{X_i}, \widehat{X_j}\right), \widehat{JX_k}\right)^2 \\
&+ \sum_{i\neq j}\left\{g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_j}\right)^2 + 2g\left(h\left(\widehat{X_i}, \widehat{X_j}\right), \widehat{JX_i}\right)^2\right\} \\
\geq& \frac{2}{\lambda^6}\sum_{i\neq j} g\left(h\left(X_i, X_j\right), JX_i\right)^2 \\
=& \frac{2}{\lambda^6}\sum_{i\neq j}\left\{g\left(h\left(X_i, Z_i\right), JX_j\right) + \frac{1}{2}g\left(X_i, X_i\right) g\left(B, JX_j\right)\right\}^2 \\
=& 2\sum_{i\neq j} g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_j}\right)^2 \\
&+ \frac{1}{2}\sum_{i\neq j} g\left(\widehat{X_i}, \widehat{X_i}\right)^2 g\left(B, \widehat{JX_j}\right)^2 \\
&+ 2\sum_{i\neq j} g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_j}\right) g\left(\widehat{X_i}, \widehat{X_i}\right) g\left(B, \widehat{JX_j}\right) \\
\geq& \frac{p-1}{2}\left\| B\big|_{J\mathcal{D}^{\perp}} \right\|^2 + 2g\left(\sum_i h\left(\widehat{X_i}, \widehat{X_i}\right)\Big|_{J\mathcal{D}^{\perp}}, B\big|_{J\mathcal{D}^{\perp}}\right)
\end{aligned}$$

$$
\begin{aligned}
&- 2\sum_{i} g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_i}\right) g\left(B, \widehat{JX_i}\right) \\
&= \frac{p-1}{2}\left\|B\big|_{J\mathcal{D}^\perp}\right\|^2 + 2g\left(H_{\mathcal{D}^\perp}|_{J\mathcal{D}^\perp}, B|_{J\mathcal{D}^\perp}\right) - K
\end{aligned}
$$

From (3.3) and Remark 3.1 we have

$$
\begin{aligned}
g\left(h\left(\widehat{Z_p}, \widehat{Z_q}\right), \widehat{JX_i}\right) &= \frac{1}{\lambda^2} g\left(h\left(Z_p, Z_q\right), JX_i\right) \\
&= \frac{1}{\lambda^2}\left\{g\left(h\left(X_i, Z_p\right), FZ_q\right) + \frac{1}{2}\lambda^2 \delta_{pq} g\left(JB, X_i\right)\right\} \\
&= \sin\theta g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FZ_q}\right) + \frac{1}{2}\delta_{pq} g\left(JB, \widehat{X_i}\right) \\
g\left(h\left(\widehat{Z_p}, \widehat{PZ_q}\right), \widehat{JX_i}\right) &= \frac{\sec\theta}{\lambda^2} g\left(h\left(Z_p, PZ_q\right), JX_i\right) \\
&= \frac{\sec\theta}{\lambda^2} g\left(h\left(X_i, Z_p\right), FPZ_q\right) \\
&= \sin\theta g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FPZ_q}\right) \\
g\left(h\left(\widehat{PZ_p}, \widehat{Z_q}\right), \widehat{JX_i}\right) &= \sin\theta g\left(h\left(\widehat{X_i}, \widehat{PZ_p}\right), \widehat{FZ_q}\right) \\
g\left(h\left(\widehat{PZ_p}, \widehat{PZ_q}\right), \widehat{JX_i}\right) &= \frac{\sec^2\theta}{\lambda^2} g\left(h\left(PZ_p, PZ_q\right), JX_i\right) \\
&= \frac{\sec^2\theta}{\lambda^2}\left\{g\left(h\left(X_i, PZ_p\right), FPZ_q\right)\right. \\
&\quad \left. + \frac{1}{2}\lambda^2\cos^2\theta\delta_{pq} g\left(JB, X_i\right)\right\} \\
&= \sin\theta g\left(h\left(\widehat{X_i}, \widehat{PZ_p}\right), \widehat{FPZ_q}\right) + \frac{1}{2}\delta_{pq} g\left(JB, \widehat{X_i}\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left\|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\big|_{J\mathcal{D}^\perp}\right\|^2 \\
&= \sum_{i,p,q}\left\{g\left(h\left(\widehat{Z_p}, \widehat{Z_q}\right), \widehat{JX_i}\right)^2 + g\left(h\left(\widehat{Z_p}, \widehat{PZ_q}\right), \widehat{JX_i}\right)^2\right. \\
&\left. + g\left(h\left(\widehat{PZ_p}, \widehat{Z_q}\right), \widehat{JX_i}\right)^2 + g\left(h\left(\widehat{PZ_p}, \widehat{PZ_q}\right), \widehat{JX_i}\right)^2\right\} \\
&= \sin^2\theta \sum_{i,p,q}\left\{g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FZ_q}\right)^2 + g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FPZ_q}\right)^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+g\left(h\left(\widehat{X_i},\widehat{PZ_p}\right),\widehat{FZ_q}\right)^2+g\left(h\left(\widehat{X_i},\widehat{PZ_p}\right),\widehat{FPZ_q}\right)^2\Big\}\\
&-\frac{1}{2}\sum_{i,p}g\left(JB,\widehat{X_i}\right)^2+\sum_{i,p}\Big\{g\left(h\left(\widehat{Z_p},\widehat{Z_p}\right),\widehat{JX_i}\right)g\left(JB,\widehat{X_i}\right)\\
&+g\left(h\left(\widehat{PZ_p},\widehat{PZ_p}\right),\widehat{JX_i}\right)g\left(JB,\widehat{X_i}\right)\Big\}\\
&=\sin^2\theta\left\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)|_{F\mathcal{D}^\theta}\right\|^2-\frac{2q}{4}\left\|B|_{J\mathcal{D}^\perp}\right\|^2\\
&-g\left(\sum_p\left\{h\left(\widehat{Z_p},\widehat{Z_p}\right)+h\left(\widehat{PZ_p},\widehat{PZ_p}\right)\right\}\Big|_{J\mathcal{D}^\perp},B|_{J\mathcal{D}^\perp}\right)\\
&=\sin^2\theta\left\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)|_{F\mathcal{D}^\theta}\right\|^2-g\left(H_{\mathcal{D}^\theta}|_{J\mathcal{D}^\perp},B|_{J\mathcal{D}^\perp}\right)-\frac{q}{2}\left\|B|_{J\mathcal{D}^\perp}\right\|^2
\end{aligned}
$$

Combining we have (3.9).

If equality holds in (3.9), then the only non-zero components of $||h||$ are $\left\|h(\mathcal{D}^\perp,\mathcal{D}^\perp)|_{J\mathcal{D}^\perp}\right\|^2$, $\left\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)|_{F\mathcal{D}^\theta}\right\|^2$ and $\left\|h(\mathcal{D}^\theta,\mathcal{D}^\theta)|_{J\mathcal{D}^\perp}\right\|^2$. Also, from the above computations we have, $\left\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)|_{F\mathcal{D}^\theta}\right\|^2 = 0$ if and only if $\left\|h(\mathcal{D}^\theta,\mathcal{D}^\theta)|_{J\mathcal{D}^\perp}\right\|^2 = 0$. Hence, the result follows. □

# 4 Warped Product Hemi-Slant Submanifolds of l.c.K. manifolds

**Lemma 4.1.** *Given a warped product hemi-slant submanifold $M = M_\theta \times {}_\lambda M_\perp$ in an lcK manifold $(\widetilde{M}^{2n}, J, g)$, we have for all $X, Y \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$,*

$$g(h(X,Z),JY) = g(h(Y,Z),JX) \tag{4.1}$$

$$g(h(X,Z),FW) = g(h(X,W),FZ) \tag{4.2}$$

$$g(h(Z,W),JX) = g(h(X,Z),FW) + \frac{1}{2}g(Z,W)g(JB,X) \tag{4.3}$$

$$g(B,X) = 0 \tag{4.4}$$

$$\begin{aligned} g(h(X,Y),JX_1) = {}& g(h(X,X_1),JY) - \frac{1}{2}g(X,Y)g(B,JX_1)\\ &+ \frac{1}{2}g(X,X_1)g(B,JY) \end{aligned} \tag{4.5}$$

*Proof.* For all $X, Y \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$, we have using (2.6) and (2.29),

$$\begin{aligned} g(h(X,Z),JY) &= g(\overline{\nabla}_Z X, JY) = -g(J\overline{\nabla}_Z X, Y) \\ &= -g(\overline{\nabla}_Z JX, Y) = g(\mathfrak{A}_{JX} Z, Y) \\ &= g(h(Y,Z),JX) \end{aligned}$$

which implies (4.1). Similarly,

$$\begin{aligned} g(h(X,Z),FW) &= g(\overline{\nabla}_X Z, JW) - g(\overline{\nabla}_X Z, PW) \\ &= -g(J\overline{\nabla}_X Z, W) - g(\nabla_X Z, PW) \\ &= -g(\overline{\nabla}_X JZ, W) \\ &= g(\mathfrak{A}_{FZ} X, W) \end{aligned}$$

which implies (4.2). Repeating the above calculation, we have

$$\begin{aligned} g(h(X,Z),FW) &= g(\overline{\nabla}_Z X, JW - PW) \\ &= -g(J\overline{\nabla}_Z X, W) - g(\nabla_Z X, PW) \\ &= -g(\overline{\nabla}_Z JX, W) - \frac{1}{2} g(JB,X) g(Z,W) - \frac{1}{2} g(B,X) g(JZ,W) \\ &= g(\mathfrak{A}_{JX} Z, W) - \frac{1}{2} g(JB,X) g(Z,W) - \frac{1}{2} g(B,X) g(PZ,W) \end{aligned}$$

Using (4.2) and comparing symmetric and skew symmetric terms in $Z$ and $W$ we have,

$$g(h(X,Z),FW) = g(h(Z,W),JX) - \frac{1}{2} g(JB,X) g(Z,W)$$

which shows (4.3) and

$$0 = \frac{1}{2} g(B,X) g(PZ,W)$$

which shows (4.4). Finally,

$$g(h(X,Y),JX_1) = g(\overline{\nabla}_X Y, JX_1)$$

$$
\begin{aligned}
&= -g(J\overline{\nabla}_X Y, X_1)\\
&= -g(\overline{\nabla}_X JY, X_1) - \frac{1}{2}g(X, X_1)g(JB, Y)\\
&\ + \frac{1}{2}g(X, Y)g(JB, X_1)\\
&= g(\mathfrak{A}_{JY}X, X_1) - \frac{1}{2}g(X, X_1)g(JB, Y) + \frac{1}{2}g(X, Y)g(JB, X_1)
\end{aligned}
$$

which gives (4.5). □

**Remark 4.1.** *Given a warped product hemi-slant submanifold $M_\theta \times {}_\lambda M_\perp$ of an l.c.K manifold $\widetilde{M}^{2n}$, let $\{X_i\}_{i=1}^{p}$ and $\{Z_j, \beta PZ_j\}_{j=1}^{q}$ respectively be local orthonormal frames of $TM_\perp$ and $TM_\theta$. Then a local orthonormal frame of $\widetilde{M}^{2n}$ is*

$$
\left\{\widehat{X_i} = \frac{X_i}{\lambda}\right\} \cup \left\{\widehat{Z_j} = Z_j, \widehat{PZ_j} = \beta PZ_j\right\} \cup \left\{\widehat{JX_i} = \frac{JX_i}{\lambda}\right\}
$$
$$
\cup \left\{\widehat{FZ_j} = \alpha FZ_j, \widehat{FPZ_j} = \alpha\beta FPZ_j\right\} \cup \left\{\widehat{\xi_k}, \widehat{J\xi_k}\right\}
$$

*where $\alpha = \csc\theta$, $\beta = \sec\theta$ and*

$\left\{\widehat{X_i} : 1 \le i \le n_1\right\}$ *is an orthonormal basis of $\mathcal{D}^\perp$*

$\left\{\widehat{Z_j}, \widehat{PZ_j} : 1 \le j \le n_2\right\}$ *is an orthonormal basis of $\mathcal{D}^\theta$*

$\left\{\widehat{JX_i} : 1 \le i \le n_1\right\}$ *is an orthonormal basis of $J\mathcal{D}^\perp$*

$\left\{\widehat{FZ_j}, \widehat{FPZ_j} : 1 \le j \le n_2\right\}$ *is an orthonormal basis of $F\mathcal{D}^\theta$*

$\left\{\widehat{\xi_k}, \widehat{J\xi_k} : 1 \le k \le \dfrac{n - n_1 - 2n_2}{2}\right\}$ *is an orthonormal basis of $\mu$*

*However, while $X_i \in \mathcal{L}(M_\perp)$ we have $\widehat{X_i} \notin \mathcal{L}(M_\perp)$ in general, as $\lambda$ is a function on $M_\theta$.*

We now give a characterisation for warped product hemi-slant submanifolds of l.c.K. manifolds.

**Theorem 4.1.** *Let $M^m$ be a hemi-slant submanifold of an l.c.K. manifold $\widetilde{M}^{2n}$. Then the following are equivalent*

1. *$M^m$ is a warped product hemi-slant submanifold $M_\theta \times {}_\lambda M_\perp$ of $\widetilde{M}^{2n}$*

2. *For every* $X, Y \in \mathcal{L}(M_\perp)$ *and* $Z, W \in \mathcal{L}(M_\theta)$ *we have*

$$\begin{aligned} g(\mathfrak{A}_{JX}Z - \mathfrak{A}_{FZ}X, Y) &= \left(\frac{1}{2}\omega(JZ) - Z(\ln\lambda)\right) g(X,Y) \\ \omega(\mathcal{D}^\perp) = \{0\} \text{ and } & g(\mathfrak{A}_{FPZ}X, W) + g(\nabla^\perp_W FZ, JX) = 0 \end{aligned} \tag{4.6}$$

*for some smooth function* $\lambda : M_\theta \to (0, \infty)$.

3. *For every* $X \in \mathcal{L}(M_\perp)$ *and* $Z \in \mathcal{L}(M_\theta)$ *we have*

$$\omega(\mathcal{D}^\perp) = \{0\} \text{ and } \nabla_X Z = \nabla_Z X = Z(\ln\lambda)X \tag{4.7}$$

*Also, in this case we have the mean curvature vector* $\mathcal{H}$ *of* $M_\perp$ *in* $M^m$ *is*

$$\mathcal{H} = -grad(\ln\lambda) \tag{4.8}$$

*Proof.* (1)⇔(2) This follows from Theorem 2.3, Theorem 2.4 and the fact that $\operatorname{grad}\ln\lambda \in \mathcal{L}(M_\theta)$ which implies for all $X \in \mathcal{L}(M_\perp)$ and $Z \in \mathcal{L}(M_\theta)$

$$\begin{aligned} g(\nabla_X(\operatorname{grad}\ln\lambda), Z) &= XZ(\ln\lambda) - g(\operatorname{grad}\ln\lambda, \nabla_X Z) \\ &= [X,Z](\ln\lambda) - \nabla_X Z(\ln\lambda) \quad (\text{as } X(\ln\lambda) = 0) \\ &= -\nabla_Z X(\ln\lambda) \\ &= g(X, \nabla_Z(\operatorname{grad}(\ln\lambda))) \\ &= 0 \end{aligned}$$

as $\mathcal{D}^\theta$ is totally geodesic.
(1)⇔(3) Let $M = M_\theta \times {}_\lambda M_\perp$ be a warped product hemi-slant submanifold. Then (4.7) and (4.8) follow from (2.29).

Conversely, let $M^m$ be a hemi-slant submanifold of an l.c.K. manifold $\widetilde{M}^{2n}$ such that (4.7) holds. Then for all $X, Y \in \mathcal{L}(M_\perp)$ and $Z, W \in \mathcal{L}(M_\theta)$ we have

$$\begin{aligned} g([X,Y], Z) &= g(\nabla_X Y - \nabla_Y X, Z) \\ &= -g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \\ &= -Z(\ln\lambda) g(X,Y) + Z(\ln\lambda) g(Y,X) = 0 \end{aligned}$$

which implies $\mathcal{D}^{\perp}$ is involutive.

$$\begin{aligned} g(\nabla_X Y, Z) &= -g(\nabla_X Z, Y) \\ &= -Z(\ln\lambda)g(X,Y) \\ &= -g(X,Y)g(\mathrm{grad}(\ln\lambda), Z) \end{aligned}$$

which implies leaves of $\mathcal{D}^{\perp}$ are totally umbilical in $M$ with mean curvature vector $-\mathrm{grad}(\ln\lambda)$.

$$\begin{aligned} g([Z,W],X) &= g(\nabla_Z W - \nabla_W Z, X) \\ &= -g(\nabla_Z X, W) + g(\nabla_W X, Z) \\ &= 0 \end{aligned}$$

which implies $\mathcal{D}^{\theta}$ is involutive.

$$g(\nabla_Z W, X) = -g(\nabla_Z X, W) = 0$$

which implies leaves of $\mathcal{D}^{\theta}$ are totally geodesic in $M$.

$$g\left(\nabla_X \mathrm{grad}(\ln\lambda), Z\right) = \mathrm{grad}(\ln\lambda)(\ln\lambda)g(X,Z) = 0$$

which implies $\mathrm{grad}(\ln\lambda)$ is parallel in the normal bundle of $M_{\perp}$ in $M$.

Hence by Theorem 2.1 we have $M = M_{\theta} \times {}_{\lambda}M_{\perp}$ is a warped product hemi-slant submanifold. □

We conclude our study of warped product hemi-slant submanifolds of l.c.K. manifolds by giving an inequality for the norm of the second fundamental form.

**Theorem 4.2.** *Let $M = M_{\perp} \times {}_{\lambda}M_{\theta}$ be a warped product hemi-slant submanifold in an lcK manifold $(\widetilde{M}^{2n}, J, g)$. Then the norm of the second fundamental form satisfies the inequality*

$$\begin{aligned} ||h||^2 \geq \frac{n_1+n_2-1}{2}\|B|_{J\mathcal{D}^{\perp}}\|^2 + 2g\left(H_{\mathcal{D}^{\perp}}|_{J\mathcal{D}^{\perp}}, B|_{J\mathcal{D}^{\perp}}\right) \\ + g\left(H_{\mathcal{D}^{\theta}}|_{J\mathcal{D}^{\perp}}, B|_{J\mathcal{D}^{\perp}}\right) - K \end{aligned} \tag{4.9}$$

*where* $n_1 = \dim_{\mathbb{R}} \mathcal{D}^\perp$, $2n_2 = \dim_{\mathbb{R}} \mathcal{D}^\theta$, $H_{\mathcal{D}^\perp}$ *and* $H_{\mathcal{D}^\theta}$ *are respectively the components of the mean curvature vector* $H$ *of* $M$ *in* $\widetilde{M}^{2n}$ *along* $\mathcal{D}^\perp$ *and* $\mathcal{D}^\theta$ *and given any orthonormal basis* $\left\{\widehat{X_i} : 1 \leq i \leq n_1\right\}$ *of* $\mathcal{D}^\perp$,
$K = 2\sum_i g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{JX_i}\right) g\left(B, \widehat{JX_i}\right)$ .
*If equality holds then we have*

- *Image*$(h) \subseteq (J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta)$, *and*
- $M_\theta$ *is totally geodesic in* $\widetilde{M}^{2n}$, *if and only if,* $M$ *is mixed-totally geodesic in* $\widetilde{M}^{2n}$.

The proof follows on the same lines as that of Theorem 3.2.

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