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Commutativity theorems in prime rings with generalized derivations and anti-automorphisms

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Abstract

The objective of this paper is to study the commutativity of prime rings satisfying certain central differential identities with anti-automorphisms. Several known results have been generalized as well as improved.

1 Introduction

Throughout the text, $\mathfrak{A}$ represents a prime ring with centre $\mathcal{Z}(\mathfrak{A})$, extended centroid $\mathcal{C}$ and maximal right ring of quotients $\mathcal{Q}_{mr}(\mathfrak{A})$. A bijective map $\tau : \mathfrak{A} \to \mathfrak{A}$ is called an anti-automorphism if it is additive and $(\mathfrak{u}\mathfrak{v})^\tau = \mathfrak{v}^\tau \mathfrak{u}^\tau$ holds for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. An involution '$*$' on $\mathfrak{A}$ is an anti-automorphism of peroid 1 or 2. An anti-automorphism τ of $\mathfrak{A}$ is said to be of the first kind if it acts as the identity map on $\mathcal{Z}(\mathfrak{A})$ and of the second kind, otherwise. We remark that τ is of the first kind if and only if τ^{-1} is of the first kind. For $x, y \in \mathfrak{A}$, we denote $xy + yx$ by $x \circ y$, $xy - yx$ by $[x, y]$, $x^\tau y - yx$ by ${}_\tau[x, y]$ and $xy - yx^\tau$ by $[x, y]_\tau$.

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An additive map $\psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ is said to be a derivation if $\psi(uv) = \psi(u)v + u\psi(v)$ holds for all $u, v \in \mathfrak{A}$. An additive map $\psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ is known as left (resp. right) multiplier if $\psi(uv) = \psi(u)v$ (resp. $\psi(uv) = u\psi(v)$) holds for all $u, v \in \mathfrak{A}$. Moreover, $\psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ is called a multiplier if it is both left as well as right multiplier. An additive map $\Psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ is called a generalized derivation if there exists a derivation $\psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ such that $\Psi(uv) = \Psi(u)v + u\psi(v)$ holds for all $u, v \in \mathfrak{A}$. Throughout the text, $(\Psi, \psi) : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ denotes a generalized derivation $\Psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ with $\psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ as associated derivation. We remark that if $\mathfrak{A}$ is a prime ring and $(\Psi, \psi) : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ a generalized derivation, then there exists a unique derivation $\psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ associated with Ψ. Moreover, the concept of generalized derivation includes both the concepts of derivations and left multipliers. Hence, the notion of generalized derivation is a natural generalization of the notions of derivation and left multiplier. A map $\Phi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ is called centralizing (resp. commuting) on $\mathrm{S} \subseteq \mathfrak{A}$ if $[\Phi(\mathfrak{u}), \mathfrak{u}] \in \mathcal{C}$ (resp. $[\Phi(\mathfrak{u}), \mathfrak{u}] = 0$) holds for all $\mathfrak{u} \in \mathrm{S}$.

The relationship between the commutativity of the ring $\mathfrak{A}$ and certain specific types of maps on $\mathfrak{A}$ has been extensively studied over the last few decades. The first remarkable result in this direction is due to Divinsky [9], who proved that a simple artinian ring is commutative if it admits a commuting nontrivial automorphism. E. C. Posner [22], showed that a prime ring must be commutative if it admits a nonzero derivation. Motivated by these two results, numerous authors have established the commutativity of rings, more often that of prime and semiprime rings, satisfying certain differential identities and $*$-differential identities on some appropriate subsets of the ring in consideration [see bibliography].

Continuing the same line of investigation, in this paper we study the commutativity of prime rings satisfying certain central differential identities involving anti-automorphisms. In fact, our results improve, generalize and unify some recent results proved by several authors viz.; [[2], Theorems 1, 6 and 7; [17], Theorem 1.11 and [1], Theorem 4].

2 Preliminary results

We facilitate our discussion with the following lemmas which play crucial role in the proofs of our main results.

Lemma 2.1. *Let $\mathfrak{A}$ be a prime ring with an anti-automorphism τ of the* *second kind. Then* $\mathfrak{A}$ *is a commutative integral domain if and only if* $uu^\tau \in \mathcal{Z}(\mathfrak{A})$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

Proof. If $\mathfrak{u}\mathfrak{u}^\tau \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$, then by [13], $[\mathfrak{u}^\tau, \mathfrak{u}] = 0$ for all $\mathfrak{u} \in \mathfrak{A}$. Now if $\mathfrak{A}$ is noncommutative, then by [[12], Lemma 2.8], τ is an involution of $\mathfrak{A}$. Hence by [[18], Lemma 2.1], $\mathfrak{A}$ is commutative, a contradiction. Thus $\mathfrak{A}$ must be commutative. The converse part holds trivially. □

Lemma 2.2. *Let* $\mathfrak{A}$ *be a prime ring with an anti-automorphism* τ *of the second kind. Then* $\mathfrak{A}$ *is a commutative integral domain if and only if* $[\mathfrak{u}, \mathfrak{u}]_\tau + \epsilon\mathfrak{u}^2 \in \mathcal{Z}(\mathfrak{A})$ *for all* $\mathfrak{u} \in \mathfrak{A}$ *or* ${}_\tau[\mathfrak{u}, \mathfrak{u}] + \epsilon\mathfrak{u}^2 \in \mathcal{Z}(\mathfrak{A})$ *for all* $\mathfrak{u} \in \mathfrak{A}$*, where* $\epsilon \in \mathcal{Z}(\mathfrak{A}) \cup \{-1, 1\}$ *is fixed.*

Proof. Suppose

$$[\mathfrak{u}, \mathfrak{u}]_\tau + \epsilon\mathfrak{u}^2 \in \mathcal{Z}(\mathfrak{A}) \tag{2.1}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Linearizing it, we get

$$[\mathfrak{u}, \mathfrak{v}]_\tau + [\mathfrak{v}, \mathfrak{u}]_\tau + \epsilon\mathfrak{u} \circ \mathfrak{v} \in \mathcal{Z}(\mathfrak{A}) \tag{2.2}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Now τ is given to be of the second kind. Hence there is $\eta \in \mathcal{Z}(\mathfrak{A})$ such that $\eta^\tau \neq \eta$. Replacing $\mathfrak{u}$ by $\eta\mathfrak{u}$ in (2.2), we get

$$\eta\mathfrak{u}\mathfrak{v} - \eta^\tau\mathfrak{v}\mathfrak{u}^\tau + \eta[\mathfrak{v}, \mathfrak{u}]_\tau + \epsilon\eta\mathfrak{u} \circ \mathfrak{v} \in \mathcal{Z}(\mathfrak{A}) \tag{2.3}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Also from (2.2), we have

$$\eta\mathfrak{u}\mathfrak{v} - \eta\mathfrak{v}\mathfrak{u}^\tau + \eta[\mathfrak{v}, \mathfrak{u}]_\tau + \epsilon\eta\mathfrak{u} \circ \mathfrak{v} \in \mathcal{Z}(\mathfrak{A}) \tag{2.4}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Hence from (2.3) and (2.4), we get $(\eta^\tau - \eta)\mathfrak{v}\mathfrak{u}^\tau \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Taking $\mathfrak{u} = \eta$ in the last relation, we conclude that $\mathfrak{A}$ is commutative. The converse part is obvious.

Similarly we can prove that ${}_\tau[\mathfrak{u}, \mathfrak{u}] + \epsilon\mathfrak{u}^2 \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$ if and only if $\mathfrak{A}$ is commutative. □

Corollary 2.1 ([1], Lemma 4)**.** *Let* $\mathfrak{A}$ *be a* 2*-torsion free prime ring with an involution '*∗*' of the second kind. Then* $[\mathfrak{u}, \mathfrak{u}^*]_* \in \mathcal{Z}(\mathfrak{A})$ *for all* $\mathfrak{u} \in \mathfrak{A}$ *if and only if* $\mathfrak{A}$ *is commutative.*

Proof. Suppose $[\mathfrak{u}, \mathfrak{u}^*]_* \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Then replacing $\mathfrak{u}$ by $\mathfrak{u}^*$ in the last relation, we have ${}_*[\mathfrak{u}, \mathfrak{u}] \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Applying Lemma 2.2, we deduce that $\mathfrak{A}$ is commutative. □

3 Main results

Theorem 3.1. *Let $\mathfrak{A}$ be a prime ring with an anti-automorphism τ of the second kind and let $(\Psi, \psi) : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ be a nonzero generalized derivation satisfying any one of the following conditions:*

(i) $\Psi([\mathfrak{u}, \mathfrak{u}]_\tau) \in \mathcal{C}$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

(ii) $\Psi({}_\tau[\mathfrak{u}, \mathfrak{u}]) \in \mathcal{C}$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

Then $\mathfrak{A}$ is a commutative integral domain.

Proof. (i) Suppose

$$\Psi([\mathfrak{u}, \mathfrak{u}]_\tau) \in \mathcal{C} \tag{3.1}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Linearizing this, we get

$$\Psi([\mathfrak{u}, \mathfrak{v}]_\tau) + \Psi([\mathfrak{v}, \mathfrak{u}]_\tau) \in \mathcal{C} \tag{3.2}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. By the given hypothesis, τ is of the second kind. Hence there exists $\eta \in \mathcal{Z}(\mathfrak{A})$ such that $\eta^\tau \neq \eta$. Replacing $\mathfrak{v}$ by $\eta\mathfrak{v}$ in (3.2), we find that

$$\eta\Psi([\mathfrak{u}, \mathfrak{v}]_\tau) + \psi(\eta)[\mathfrak{u}, \mathfrak{v}]_\tau + \eta\Psi(\mathfrak{v}\mathfrak{u}) + \psi(\eta)\mathfrak{v}\mathfrak{u} - \eta^\tau\Psi(\mathfrak{u}\mathfrak{v}^\tau) - \psi(\eta^\tau)\mathfrak{u}\mathfrak{v}^\tau \in \mathcal{C} \tag{3.3}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Now we divide the proof into the following cases:
Case I. When $\psi(\eta) = 0$. Putting $\mathfrak{v} = \mathfrak{u}$, (3.3) yields

$$\eta\Psi(\mathfrak{u}^2) - \eta^\tau\Psi(\mathfrak{u}\mathfrak{u}^\tau) - \psi(\eta^\tau)\mathfrak{u}\mathfrak{u}^\tau \in \mathcal{C} \tag{3.4}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Replacing $\mathfrak{u}$ by $\mathfrak{u} + \mathfrak{v}$ in the previous relation, we get

$$\eta\Psi(\mathfrak{u} \circ \mathfrak{v}) - \eta^\tau\Psi(\mathfrak{u}\mathfrak{v}^\tau + \mathfrak{v}\mathfrak{u}^\tau) - \psi(\eta^\tau)(\mathfrak{u}\mathfrak{v}^\tau + \mathfrak{v}\mathfrak{u}^\tau) \in \mathcal{C} \tag{3.5}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Replacing $\mathfrak{v}$ by $\eta\mathfrak{v}$ in (3.5), we see that

$$\eta^2\Psi(\mathfrak{u}\circ\mathfrak{v}) - (\eta^\tau)^2\Psi(\mathfrak{u}\mathfrak{v}^\tau) - \eta\eta^\tau\Psi(\mathfrak{v}\mathfrak{u}^\tau) - \eta^\tau\psi(\eta^\tau)\mathfrak{u}\mathfrak{v}^\tau - \psi(\eta^\tau)(\eta^\tau\mathfrak{u}\mathfrak{v}^\tau + \eta\mathfrak{v}\mathfrak{u}^\tau) \in \mathcal{C} \tag{3.6}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. From (3.5) and (3.6), we get

$$\eta^\tau(\eta-\eta^\tau)\Psi(\mathfrak{u}\mathfrak{v}^\tau)-\eta^\tau\psi(\eta^\tau)\mathfrak{u}\mathfrak{v}^\tau-(\eta-\eta^\tau)\psi(\eta^\tau)\mathfrak{v}\mathfrak{u}^\tau\in\mathcal{C}$$

for all $\mathfrak{u},\mathfrak{v}\in\mathfrak{A}$. Setting $\mathfrak{u}=\eta$ in the last relation, we have $[\Psi(\mathfrak{v}),\mathfrak{v}]=0$ for all $\mathfrak{u}\in\mathfrak{A}$. By [[14], Theorem 2], there exist $\lambda\in\mathcal{C}$ and an additive map $\mu:\mathfrak{A}\to\mathcal{C}$ such that $\Psi(\mathfrak{u})=\lambda\mathfrak{u}+\mu(\mathfrak{u}\ )$ for all $\mathfrak{u}\in\mathfrak{A}$. Hence $\Psi(\mathfrak{u})-\lambda\mathfrak{u}\in\mathcal{C}$ for all $\mathfrak{u}\in\mathfrak{A}$. Applying [[11], Lemma 3], we infer that $\Psi(\mathfrak{u})=\lambda\mathfrak{u}$ for all $\mathfrak{u}\in\mathfrak{A}$. Therefore from (3.1), we have $\lambda[\mathfrak{u},\mathfrak{u}]_\tau\in\mathcal{C}$ for all $\mathfrak{u}\in\mathfrak{A}$. Consequently, $[\mathfrak{u},\mathfrak{u}]_\tau\in\mathcal{C}$ for all $\mathfrak{u}\in\mathfrak{A}$. By Lemma 2.1, $\mathfrak{A}$ is commutative.

Case II. When $\psi(\eta)\neq 0$. Putting $\mathfrak{v}=\mathfrak{u}$ in (3.3), we see that

$$2\psi(\eta)u^2-\psi(\eta+\eta^\tau)uu^\tau+\eta\Psi(\mathfrak{u}^2)-\eta^\tau\Psi(\mathfrak{u}\mathfrak{u}^\tau)\in\mathcal{C} \tag{3.7}$$

for all $\mathfrak{u}\in\mathfrak{A}$. Replacing $\mathfrak{u}$ by $\mathfrak{u}+\mathfrak{v}$ in the previous relation, we get

$$2\psi(\eta)(\mathfrak{u}\circ\mathfrak{v})-\psi(\eta+\eta^\tau)(\mathfrak{u}\mathfrak{v}^\tau+\mathfrak{v}\mathfrak{u}^\tau)+\eta\Psi(\mathfrak{u}\circ\mathfrak{v})-\eta^\tau\Psi(\mathfrak{u}\mathfrak{v}^\tau+\mathfrak{v}\mathfrak{u}^\tau)\in\mathcal{C} \tag{3.8}$$

for all $\mathfrak{u}\in\mathfrak{A}$. Using $\eta\mathfrak{u}$ in place of $\mathfrak{u}$ in (3.8), we get

$$\begin{aligned}2\eta\psi(\eta)(\mathfrak{u}\circ\mathfrak{v})-\psi(\eta \quad + \quad & \eta^\tau)(\eta\mathfrak{u}\mathfrak{v}^\tau+\eta^\tau\mathfrak{v}\mathfrak{u}^\tau)\\ & +\eta^2\Psi(\mathfrak{u}\circ\mathfrak{v})+\eta\psi(\eta)(\mathfrak{u}\circ\mathfrak{v})-\eta\eta^\tau\Psi(\mathfrak{u}\mathfrak{v}^\tau)\\ & -\eta^\tau\psi(\eta)\mathfrak{u}\mathfrak{v}^\tau-(\eta^\tau)^2\Psi(\mathfrak{v}\mathfrak{u}^\tau)-\eta^\tau\psi(\eta^\tau)\mathfrak{v}\mathfrak{u}^\tau\in\mathcal{C}\end{aligned} \tag{3.9}$$

for all $\mathfrak{u},\mathfrak{v}\in\mathfrak{A}$. Also from (3.8), we have

$$2\eta\psi(\eta)(\mathfrak{u}\circ\mathfrak{v})-\eta\psi(\eta+\eta^\tau)(\mathfrak{u}\mathfrak{v}^\tau+\eta\mathfrak{v}\mathfrak{u}^\tau)+\eta^2\Psi(\mathfrak{u}\circ\mathfrak{v})-\eta\eta^\tau\Psi(\mathfrak{u}\mathfrak{v}^\tau+\mathfrak{v}\mathfrak{u}^\tau)\in\mathcal{C} \tag{3.10}$$

for all $\mathfrak{u},\mathfrak{v}\in\mathfrak{A}$. From (3.9) and (3.10), we have

$$\begin{aligned}(\eta-\eta^\tau)\psi(\eta+\eta^\tau)\mathfrak{v}\mathfrak{u}^\tau+\eta\psi(\eta)(\mathfrak{u}\circ\mathfrak{v}) \quad + \quad & \eta^\tau(\eta-\eta^\tau)\Psi(\mathfrak{v}\mathfrak{u}^\tau)\\ & -\eta^\tau\psi(\eta)\mathfrak{u}\mathfrak{v}^\tau-\eta^\tau\psi(\eta^\tau)\mathfrak{v}\mathfrak{u}^\tau\in\mathcal{C}\end{aligned} \tag{3.11}$$

for all $\mathfrak{u},\mathfrak{v}\in\mathfrak{A}$. Now substituting $\eta\mathfrak{v}$ in place of $\mathfrak{v}$ in (3.11), we get

$$\begin{aligned}\eta(\eta \quad - \quad & \eta^\tau)\psi(\eta+\eta^\tau)\mathfrak{v}\mathfrak{u}^\tau+\eta^2\psi(\eta)(\mathfrak{u}\circ\mathfrak{v})+\eta\eta^\tau(\eta-\eta^\tau)\Psi(\mathfrak{v}\mathfrak{u}^\tau)\\ & +\eta^\tau(\eta^\tau-\eta)\psi(\eta)\mathfrak{v}\mathfrak{u}^\tau-(\eta^\tau)^2\psi(\eta)\mathfrak{u}\mathfrak{v}^\tau-\eta\eta^\tau\psi(\eta^\tau)\mathfrak{v}\mathfrak{u}^\tau\in\mathcal{C}\end{aligned} \tag{3.12}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Also from (3.11), we have

$$\begin{aligned}\eta(\eta-\eta^{\tau})\psi(\eta+\eta^{\tau})\mathfrak{v}\mathfrak{u}^{\tau} \quad + \quad & \eta^2\psi(\eta)(\mathfrak{u}\circ\mathfrak{v})+\eta\eta^{\tau}(\eta-\eta^{\tau})\Psi(\mathfrak{v}\mathfrak{u}^{\tau}) \\ & -\eta\eta^{\tau}\psi(\eta)\mathfrak{u}\mathfrak{v}^{\tau}-\eta\eta^{\tau}\psi(\eta^{\tau})\mathfrak{v}\mathfrak{u}^{\tau}\in\mathcal{C}\end{aligned} \tag{3.13}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. From (3.12) and (3.13), we have

$$\eta^{\tau}(\eta-\eta^{\tau})\psi(\eta)\mathfrak{v}\mathfrak{u}^{\tau}-\eta^{\tau}(\eta^{\tau}-\eta)\psi(\eta)\mathfrak{u}\mathfrak{v}^{\tau}\in\mathcal{C} \tag{3.14}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Consequently, $\mathfrak{v}\mathfrak{u}^{\tau}-\mathfrak{u}\mathfrak{v}^{\tau}\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Setting $\mathfrak{u}=\eta\mathfrak{u}$, in the last relation and using it again, we get $\mathfrak{u}\mathfrak{v}^{\tau}\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Thus $\mathfrak{A}$ is commutative.

(ii) Using similar arguments as presented in (i), we can prove that if $\Psi({}_{\tau}[\mathfrak{u},\mathfrak{u}])\in\mathcal{C}$ for all $\mathfrak{u}\in\mathfrak{A}$, then $\mathfrak{A}$ is commutative.

□

Corollary 3.1 ([2], Theorem 1)**.** *Let $\mathfrak{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind and let $(\Psi,\psi):\mathfrak{A}\to\mathfrak{A}$ be a generalized derivation such that $\Psi([\mathfrak{u},\mathfrak{u}^*]_*)\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}\in\mathfrak{A}$. Then either $\mathfrak{A}$ is a commutative integral domain or $\Psi=0$.*

Proof. Suppose $\Psi([\mathfrak{u},\mathfrak{u}^*]_*)\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}\in\mathfrak{A}$. Replacing $\mathfrak{u}$ by $\mathfrak{u}^*$, we get $\Psi({}_*[\mathfrak{u},\mathfrak{u}])\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}\in\mathfrak{A}$. Applying Theorem 3.1 (ii), we get the desired conclusion.

□

Corollary 3.2 ([17], Theorem 1.11)**.** *Let $\mathfrak{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind and let $\psi:\mathfrak{A}\to\mathfrak{A}$ be a derivation such that $\psi([\mathfrak{u},\mathfrak{u}^*]_*)\pm[\mathfrak{u},\mathfrak{u}^*]_*\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}\in\mathfrak{A}$. Then either $\mathfrak{A}$ is a commutative integral domain or $\psi=0$.*

Proof. Suppose $\psi([\mathfrak{u},\mathfrak{u}^*]_*)\pm[\mathfrak{u},\mathfrak{u}^*]_*\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}\in\mathfrak{A}$. Replacing $\mathfrak{u}$ by $\mathfrak{u}^*$, we get $\psi({}_*[\mathfrak{u},\mathfrak{u}])\pm{}_*[\mathfrak{u},\mathfrak{u}]\in\mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}\in\mathfrak{A}$. Hence $(\psi\pm I)({}_*[\mathfrak{u},\mathfrak{u}])\in\mathcal{C}$ for all $\mathfrak{u}\in\mathfrak{A}$. Applying Lemma 2.2 and Theorem 3.1, we get the desired conclusion.

□

Theorem 3.2. *Let $\mathfrak{A}$ be a prime ring with an anti-automorphism τ of the second kind and let $(\Psi, \psi) : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ be a generalized derivation such that $[\mathfrak{u}, \Psi(\mathfrak{u})]_\tau \pm [\mathfrak{u}, \mathfrak{u}^\tau] \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Then $\mathfrak{A}$ is a commutative integral domain.*

Proof. Suppose

$$[\mathfrak{u}, \Psi(\mathfrak{u})]_\tau + [\mathfrak{u}, \mathfrak{u}^\tau] \in \mathcal{C} \tag{3.15}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Linearizing (3.15), we have

$$[\mathfrak{u}, \Psi(\mathfrak{v})]_\tau + [\mathfrak{v}, \Psi(\mathfrak{u})]_\tau + [\mathfrak{u}, \mathfrak{v}^\tau] + [\mathfrak{v}, \mathfrak{u}^\tau] \in \mathcal{C} \tag{3.16}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. By the given hypothesis, τ is of the second kind. Hence there exists $\eta \in \mathcal{Z}(\mathfrak{A})$ such that $\eta^\tau \neq \eta$. Substituting $\eta\mathfrak{u}$ in place of $\mathfrak{u}$ in (3.16), we have

$$\eta\mathfrak{u}\Psi(\mathfrak{v}) - \eta^\tau\Psi(\mathfrak{v})\mathfrak{u}^\tau + \eta[\mathfrak{v}, \Psi(\mathfrak{u})]_\tau + \psi(\eta)[\mathfrak{v}, \mathfrak{u}]_\tau + \eta[\mathfrak{u}, \mathfrak{v}^\tau] + \eta^\tau[\mathfrak{v}, \mathfrak{u}^\tau] \in \mathcal{C} \tag{3.17}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Now we divide the proof into the following two cases:
Case I. When $\psi(\eta) = 0$. In this situation (3.17) reduces to

$$\eta\mathfrak{u}\Psi(\mathfrak{v}) - \eta^\tau\Psi(\mathfrak{v})\mathfrak{u}^\tau + \eta[\mathfrak{v}, \Psi(\mathfrak{u})]_\tau + \eta[\mathfrak{u}, \mathfrak{v}^\tau] + \eta^\tau[\mathfrak{v}, \mathfrak{u}^\tau] \in \mathcal{C} \tag{3.18}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Also from (3.16), we have

$$\eta[\mathfrak{u}, \Psi(\mathfrak{v})]_\tau + \eta[\mathfrak{v}, \Psi(\mathfrak{u})]_\tau + \eta[\mathfrak{u}, \mathfrak{v}^\tau] + \eta[\mathfrak{v}, \mathfrak{u}^\tau] \in \mathcal{C} \tag{3.19}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. From (3.18) and (3.19), we have $(\eta^\tau - \eta)\Psi(\mathfrak{v})\mathfrak{u}^\tau - (\eta^\tau - \eta)[\mathfrak{v}, \mathfrak{u}^\tau] \in \mathcal{C}$ for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Consequently,

$$\Psi(\mathfrak{v})\mathfrak{u} - [\mathfrak{v}, \mathfrak{u}] \in \mathcal{C} \tag{3.20}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Taking $\mathfrak{u} = \eta$ in (3.20), we have $\Psi(\mathfrak{v}) \in \mathcal{C}$ for all $\mathfrak{v} \in \mathfrak{A}$. Applying [[11], Lemma 3], we infer that either $\mathfrak{A}$ is commutative or $\Psi = 0$. If the latter case prevails, then (3.15) gives us $[\mathfrak{u}, \mathfrak{u}^\tau] \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Hence $[[\mathfrak{u}^\tau, \mathfrak{u}], \mathfrak{u}] = 0$ for all $\mathfrak{u} \in \mathfrak{A}$. By [12] and [13], $\mathfrak{u}\mathfrak{u}^\tau \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Invoking Lemma 2.1, we conclude that $\mathfrak{A}$ is commutative.
Case II. When $\psi(\eta) \neq 0$. Using $\eta\mathfrak{u}$ instead of $\mathfrak{u}$ in (3.15), we have

$$\eta^2\mathfrak{u}\Psi(\mathfrak{u}) + \eta\psi(\eta)u^2 - \eta\eta^\tau\Psi(\mathfrak{u})\mathfrak{u}^\tau - \eta^\tau\psi(\eta)\mathfrak{u}\mathfrak{u}^\tau + \eta\eta^\tau[\mathfrak{u}, \mathfrak{u}^\tau] \in \mathcal{C} \tag{3.21}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Also from (3.15), we have

$$\eta\eta^{\tau}\mathfrak{u}\Psi(\mathfrak{u}) - \eta\eta^{\tau}\Psi(\mathfrak{u})\mathfrak{u}^{\tau} + \eta\eta^{\tau}[\mathfrak{u}, \mathfrak{u}^{\tau}] \in \mathcal{C} \tag{3.22}$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.21) and (3.22), we get

$$\eta(\eta - \eta^{\tau})\mathfrak{u}\Psi(\mathfrak{u}) + \eta\psi(\eta)u^2 - \eta^{\tau}\psi(\eta)\mathfrak{u}\mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.23}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Utilizing $\eta\mathfrak{u}$ in place of $\mathfrak{u}$ in (3.23), we have

$$(\eta - \eta^{\tau})\eta^3\mathfrak{u}\Psi(\mathfrak{u}) + (\eta - \eta^{\tau})\eta^2\psi(\eta)\mathfrak{u}^2 + \eta^3\psi(\eta)u^2 - \eta(\eta^{\tau})^2\psi(\eta)\mathfrak{u}\mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.24}$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.23), we have

$$(\eta - \eta^{\tau})\eta^3\mathfrak{u}\Psi(\mathfrak{u}) + \psi(\eta)\eta^3u^2 - \psi(\eta)\eta^2\eta^{\tau}\mathfrak{u}\mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.25}$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.24) and (3.25), we have

$$(\eta - \eta^{\tau})\eta^2\psi(\eta)\mathfrak{u}^2 - \eta\eta^{\tau}(\eta^{\tau} - \eta)\psi(\eta)\mathfrak{u}\mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.26}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Consequently, $\eta\mathfrak{u}^2 - \eta^{\tau}\mathfrak{u}\mathfrak{u}^{\tau} \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Replacing $\mathfrak{u}$ by $\eta\mathfrak{u}$ in the last relation and using it again, we get $\mathfrak{u}\mathfrak{u}^{\tau} \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Hence by Lemma 2.1, $\mathfrak{A}$ is commutative.

By using similar arguments we can prove that $\mathfrak{A}$ is commutative if $[\mathfrak{u}, \Psi(\mathfrak{u})]_{\tau} - [\mathfrak{u}, \mathfrak{u}^{\tau}] \in \mathcal{C}$ holds for all $\mathfrak{u} \in \mathfrak{A}$.

□

Corollary 3.3 ([2], Theorem 6)**.** *Let $\mathfrak{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind and let $(\Psi, \psi) : \mathfrak{A} \to \mathfrak{A}$ be a generalized derivation such that $[\mathfrak{u}, \Psi(\mathfrak{u})]_* \pm [\mathfrak{u}, \mathfrak{u}^*] \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Then $\mathfrak{A}$ is a commutative integral domain.*

Theorem 3.3. *Let $\mathfrak{A}$ be a prime ring with an anti-automorphism τ of the second kind and let $(\Psi, \psi) : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ be a generalized derivation such that ${}_{\tau}[\mathfrak{u}, \Psi(\mathfrak{u})] \pm \mathfrak{u} \circ \mathfrak{u}^{\tau} \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Then $\mathfrak{A}$ is a commutative integral domain.*

Proof. Suppose

$$_{\tau}[\mathfrak{u}, \Psi(\mathfrak{u})] + \mathfrak{u} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.27}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Linearizing it, we get

$$_{\tau}[\mathfrak{u}, \Psi(\mathfrak{v})] +_{\tau} [\mathfrak{v}, \Psi(\mathfrak{u})] + \mathfrak{u} \circ \mathfrak{v}^{\tau} + +\mathfrak{v} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.28}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. By the given hypothesis, τ is of the second kind. Hence there exists $\eta \in \mathcal{Z}(\mathfrak{A})$ such that $\eta^{\tau} \neq \eta$. Substituting $\eta\mathfrak{u}$ in place of $\mathfrak{u}$ in (3.28), we have

$$\eta^{\tau}\mathfrak{u}^{\tau}\Psi(\mathfrak{v}) - \eta\Psi(\mathfrak{v})\mathfrak{u} + \eta_{\tau}[\mathfrak{v}, \Psi(\mathfrak{u})] + \psi(\eta)_{\tau}[\mathfrak{v}, \mathfrak{u}] + \eta\mathfrak{u} \circ \mathfrak{v}^{\tau} + \eta^{\tau}\mathfrak{v} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.29}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Also from (3.28), we have

$$\eta\mathfrak{u}^{\tau}\Psi(\mathfrak{v}) - \eta\Psi(\mathfrak{v})\mathfrak{u} + \eta_{\tau}[\mathfrak{v}, \Psi(\mathfrak{u})] + \eta\mathfrak{u} \circ \mathfrak{v}^{\tau} + \eta\mathfrak{v} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.30}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. From (3.29) and (3.30), we have

$$(\eta^{\tau} - \eta)\mathfrak{u}^{\tau}\Psi(\mathfrak{v}) + \psi(\eta)_{\tau}[\mathfrak{v}, \mathfrak{u}] + (\eta^{\tau} - \eta)\mathfrak{v} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.31}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Now we proceed by considering the following cases:
Case I. When $\psi(\eta) = 0$. From (3.31), we have

$$\mathfrak{u}\Psi(\mathfrak{v}) + \mathfrak{v} \circ \mathfrak{u} \in \mathcal{C} \tag{3.32}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Taking $\mathfrak{u} = \eta$ in the previous relation, we see that $\Psi(\mathfrak{v})+2\mathfrak{v} \in \mathcal{C}$ for all $\mathfrak{v} \in \mathfrak{A}$. Applying [[11], Lemma 3], it follows that $\Psi(\mathfrak{v}) = -2\mathfrak{v}$ for all $\mathfrak{v} \in \mathfrak{A}$. Therefore from (3.32), we have $[\mathfrak{v}, \mathfrak{u}] \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{A}$. Hence $\mathfrak{A}$ is commutative.
Case II. When $\psi(\eta) \neq 0$. Replacing $\mathfrak{u}$ by $\eta\mathfrak{u}$ in (3.27), we have

$$\eta\eta^{\tau}\mathfrak{u}^{\tau}\Psi(\mathfrak{u}) + \eta^{\tau}\psi(\eta)\mathfrak{u}^{\tau}\mathfrak{u} - \eta^{2}\Psi(\mathfrak{u})\mathfrak{u} - \eta\psi(\eta)\mathfrak{u}^{2} + \eta\eta^{\tau}\mathfrak{u} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.33}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Also from (3.27), we have

$$\eta\eta^{\tau}\mathfrak{u}^{\tau}\Psi(\mathfrak{u}) - \eta\eta^{\tau}\Psi(\mathfrak{u})\mathfrak{u} + \eta\eta^{\tau}\mathfrak{u} \circ \mathfrak{u}^{\tau} \in \mathcal{C} \tag{3.34}$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.33) and (3.34), we find that

$$\eta(\eta^{\tau} - \eta)\Psi(\mathfrak{u})\mathfrak{u} + \eta^{\tau}\psi(\eta)\mathfrak{u}^{\tau}\mathfrak{u} - \eta\psi(\eta)\mathfrak{u}^{2} \in \mathcal{C} \tag{3.35}$$

for all $\mathfrak{u} \in \mathfrak{A}$. Using $\eta\mathfrak{u}$ in place of $\mathfrak{u}$ in (3.35), we have

$$\eta^3(\eta^\tau - \eta)\Psi(\mathfrak{u})\mathfrak{u} + \eta^2\psi(\eta)(\eta^\tau - \eta)\mathfrak{u}^2 + \eta(\eta^\tau)^2\psi(\eta)\mathfrak{u}^\tau\mathfrak{u} - \eta^3\psi(\eta)\mathfrak{u}^2 \in \mathcal{C} \quad (3.36)$$

for all $\mathfrak{u} \in \mathfrak{A}$. Also from (3.35), we have

$$\eta^3(\eta^\tau - \eta)\Psi(\mathfrak{u})\mathfrak{u} + \eta^\tau\eta^2\psi(\eta)\mathfrak{u}^\tau\mathfrak{u} - \eta^3\psi(\eta)\mathfrak{u}^2 \in \mathcal{C} \quad (3.37)$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.36) and (3.37), we find that

$$\eta^2\psi(\eta)(\eta^\tau - \eta)\mathfrak{u}^2 + \eta\eta^\tau(\eta^\tau - \eta)\psi(\eta)\mathfrak{u}^\tau\mathfrak{u} \in \mathcal{C} \quad (3.38)$$

for all $\mathfrak{u} \in \mathfrak{A}$. Consequently,

$$\eta\mathfrak{u}^2 + \eta^\tau\mathfrak{u}^\tau\mathfrak{u} \in \mathcal{Z}(\mathfrak{A}) \quad (3.39)$$

for all $\mathfrak{u} \in \mathfrak{A}$. Replacing $\mathfrak{u}$ by $\eta\mathfrak{u}$ in (3.39), we have

$$\eta^3\mathfrak{u}^2 + \eta(\eta^\tau)^2\mathfrak{u}^\tau\mathfrak{u} \in \mathcal{Z}(\mathfrak{A}) \quad (3.40)$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.39), we have

$$\eta^3\mathfrak{u}^2 + \eta^2\eta^\tau\mathfrak{u}^\tau\mathfrak{u} \in \mathcal{Z}(\mathfrak{A}) \quad (3.41)$$

for all $\mathfrak{u} \in \mathfrak{A}$. From (3.40) and (3.41), we have $\eta\eta^\tau(\eta^\tau - \eta)\mathfrak{u}^\tau\mathfrak{u} \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Hence $\mathfrak{u}^\tau\mathfrak{u} \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Thus $\mathfrak{u}\mathfrak{u}^{\tau^{-1}} \in \mathcal{Z}(\mathfrak{A})$ for all $\mathfrak{u} \in \mathfrak{A}$. Since τ^{-1} is also of the second kind. Applying Lemma 2.1, we conclude that $\mathfrak{A}$ is commutative.

By using similar arguments, we can prove that $\mathfrak{A}$ is commutative if ${}_\tau[\mathfrak{u}, \Psi(\mathfrak{u})] - \mathfrak{u} \circ \mathfrak{u}^\tau \in \mathcal{C}$ holds for all $\mathfrak{u} \in \mathfrak{A}$. □

Corollary 3.4 ([2], Theorem 7)**.** *Let $\mathfrak{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind and let $(\Psi, \psi) : \mathfrak{A} \to \mathfrak{A}$ be a generalized derivation such that $[\mathfrak{u}, \Psi(\mathfrak{u}^*)]_* + \mathfrak{u} \circ \mathfrak{u}^* \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Then $\mathfrak{A}$ is a commutative integral domain.*

Proof. Suppose $[\mathfrak{u}, \Psi(\mathfrak{u}^*)]_* + \mathfrak{u} \circ \mathfrak{u}^* \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Using $\mathfrak{u}^*$ in place of $\mathfrak{u}$ in the previous relation, we find that ${}_*[\mathfrak{u}, \Psi(\mathfrak{u})] + \mathfrak{u} \circ \mathfrak{u}^* \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Applying Theorem 3.3, we conclude that $\mathfrak{A}$ is commutative. □

Similarly, we have the following corollary.

Corollary 3.5 ([1], Theorem 4)**.** *Let $\mathfrak{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind and let $\Psi : \mathfrak{A} \to \mathfrak{A}$ be a left multiplier such that $[\mathfrak{u}, \Psi(\mathfrak{u}^*)]_* + \mathfrak{u} \circ \mathfrak{u}^* \in \mathcal{C}$ for all $\mathfrak{u} \in \mathfrak{A}$. Then either Ψ is a multiplier or $\mathfrak{A}$ is a commutative integral domain.*

Finally, we provide an example to show that Theorems 3.1-3.3 do not hold for semiprime rings and hence the condition of primeness is not superflous.

Example 3.1. *Consider the noncommutative ring $\mathfrak{A} = \mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ is the ring of real quaternions and $\mathbb{C}$ is the field of complex numbers. Define the maps $\tau, \Psi : \mathfrak{A} \to \mathcal{Q}_{mr}(\mathfrak{A})$ by $\Psi(A, \zeta) = (0, \zeta)$ and $(A, \zeta)^\tau = (\bar{A}, \bar{\zeta})$, where $\bar{\lambda}$ denotes the conjugate of λ. Then it can be easily verified that Ψ is a generalized derivation and τ is an anti-automorphism of the second kind. Moreover,*

(i) $\Psi([\mathfrak{u}, \mathfrak{u}]_\tau) \in \mathcal{C}$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

(ii) $\Psi({}_\tau[\mathfrak{u}, \mathfrak{u}]) \in \mathcal{C}$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

(iii) $[\mathfrak{u}, \Psi(\mathfrak{u})]_\tau \pm [\mathfrak{u}, \mathfrak{u}^\tau] \in \mathcal{C}$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

(iv) ${}_\tau[\mathfrak{u}, \Psi(\mathfrak{u})] \pm \mathfrak{u} \circ \mathfrak{u}^\tau \in \mathcal{C}$ *for all* $\mathfrak{u} \in \mathfrak{A}$.

Note that $\mathfrak{A}$ is not a prime ring.

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