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A note on the Koethe conjecture

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Abstract

In this note we provide some results related to the Koethe conjecture and report some significant observations on some results related to the Koethe conjecture appeared in [2–3].

1 Introduction

The well-known Koethe conjecture was introduced in 1930. It is a longstanding problem in ring theory, with a long and complicated history. It can be stated as follows. Every one-sided nil ideal of a ring R is contained in a two sided nil ideal of R [4]. Let $N(R)$ is the set of all nilpotent elements of a ring R. As per Theorem 2.6 [2–3], the following statements are equivalent for a ring R.

(a) $N(R)$ is closed under addition.

(b) $N(R)$ is closed under multiplication and R satisfies Koethe conjecture.

(c) $N(R)$ is a subring of R.

Thus it has been shown in [2–3] that R satisfies the Koethe conjecture provided $N(R)$ is a subring of R. Also, it has been shown in [2–3] that if $N(R)$ is multiplicatively closed and R satisfies the Koethe conjecture then $N(R)$ is a subring of R.

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In this note we exhibit that the condition 'R satisfies the Koethe conjecture' given above (Theorem 2.6, [2–3]) is superfluous at least under certain conditions described here.

In addition we provide some results when $N(R)$ does not form a subring of R and these are seen to be related to an open question appeared in [2–3] which asks that if $N(R)$ is multiplicatively closed, then does it imply that R satisfies the Koethe conjecture?

2 Some Results

We provide the following results.

Proposition 2.1. *Let R is a non-commutative ring and $N(R)$ is the set of all nilpotent elements of R. If R fails to satisfy the Koethe conjecture, then $N(R)$ is not a subring of R.*

Proof. Let R is a non-commutative ring and $N(R)$ is the set of all nilpotent elements of R. If $N(R)$ is a subring of R, then R satisfies the Koethe conjecture [2–3, Theorem 2.6].

Therefore, it follows that if R fails to satisfy the Koethe conjecture, then $N(R)$ is not a subring of R. □

Proposition 2.2. *The converse of the proposition* 2.1 *is not true.*

Proof. Let R is the ring of all 2×2 matrices over the field of order two. Then $N(R)$ is not a subring of R. However, R satisfies the Koethe conjecture. □

Proposition 2.3. *Let R is a non-commutative ring and $N(R)$ is the set of all nilpotent elements of R. Then the following are equivalent.*

(i) $N(R)$ is not a subring of R.

(ii) $N(R)$ is not additively closed.

Proof. Let R is a non-commutative ring and $N(R)$ is the set of all nilpotent elements of R. It is trivial that (ii) $\Rightarrow$ (i). Let $N(R)$ is additively closed. Then $N(R)$ is a subring of R [2–3. Theorem 2.6]. This is a contradiction. Hence, if $N(R)$ is not a subring of R, then $N(R)$ is not additively closed. Thus, (i) $\Rightarrow$ (ii). □

Proposition 2.4. *Let R is a non-commutative ring and $N(R)$ is the set of all nilpotent elements of R. Then $N(R)$ is not multiplicatively closed is equivalent to $N(R)$ is not additively closed.*

Proof. Let $N(R)$ is not multiplicatively closed. This implies that $N(R)$ is not a subring of R. Now from Proposition 2.3, it follows that $N(R)$ is not additively closed. □

Remark 2.1. *It may be noted that there is an open question given in [2–3]. The question is as follows. Let R is a ring such that $N(R)$ is multiplicatively closed. Does R satisfy the Koethe conjecture? It should be emphasized that this open question posed in [1–2] remains open if the converse of the Proposition 2.4 is not true. This is due to the fact that if $N(R)$ is multiplicatively as well as additively closed, then R satisfies the Koethe conjecture.*

In the next few results we exhibit that the condition 'R satisfies the Koethe conjecture' given in [2–3, Theorem 2.6] is superfluous at least under certain conditions.

Proposition 2.5. *Let R is a ring such that $N(R)$ is multiplicatively closed. Then $N(R)$ is additively closed and it is superfluous to assume that R satisfies the Koethe conjecture provided each element of $N(R)$ is of index two.*

Proof. Let R be a ring such that $N(R)$ is multiplicatively closed. Let $a, b \in N(R)$ are any two elements of $N(R)$. Then it easily follows that $a+b \in N(R)$ since $(a+b)^4 = 0$. Therefore, $N(R)$ a subring of R and hence R satisfies the Koethe conjecture [2–3, Theorem 2.6]. Clearly, in this case, in order to prove that $N(R)$ is a subring of R it suffices that $N(R)$ is multiplicatively closed and unlike [2–3] one does not require additional condition, namely, R satisfies the Koethe conjecture. □

Theorem 2.1. *Let R is a ring such that $N(R)$ is multiplicatively closed. Then $N(R)$ is additively closed and it is superfluous to assume that R satisfies the Koethe conjecture provided any one of the following hold.*

(i) $S(R)[x] \subseteq N(R[x])$.

(ii) $S(R)[x] = N(R[x])$.

Here $R[x]$ stands for polynomial ring defined over R and $S(R)[x]$ is the set of polynomials defined over $N(R)$.

Proof. We refer [1, Theorem 2.11]. □

Theorem 2.2. *Let R is a ring such that $N(R)$ is multiplicatively closed. Then $N(R)$ is additively closed and it is superfluous to assume that R satisfies the Koethe conjecture provided $S(R)[[x]] = N(R[[x]])$. Here $R[[x]]$ stands for power series ring defined over R and $S(R)[[x]]$ is the set of power series defined over $N(R)$.*

Proof. We refer [1, Theorem 2.11]. □

Proposition 2.6. *If R is a non-commutative ring, then $N(R)$ is an ideal of R provided $N(R) = E(R)$. Here $E(R)$ is the set of all even elements of R.*

Proof. Let R is a ring and $N(R) = E(R)$. Let a, b are arbitrary elements of $N(R)$. Then we have $a = 2c$ for some $c \in R$ and $b = 2d$ for some $d \in R$ (since $N(R) = E(R)$). Therefore, $a - b \in N(R)$. Similarly $ra \in N(R)$ and $ar \in N(R)$ for each $a \in N(R)$ and for each $r \in R$. Hence $N(R)$ is an ideal of R. □

Corollary 2.1. *If R is a ring such that $N(R) = E(R)$, then R satisfies the Koethe conjecture.*

Remark 2.2. *The notion of even elements of a ring has been introduced and studied in [5].*

Below we shall generalize Proposition 2.6 to get the next Proposition.

Proposition 2.7. *If R is a non-commutative ring, then $N(R)$ is an ideal of R provided $N(R) = E_k(R)$. Here $E_k(R)$ is the set of all elements of R satisfying $a = kb$ for some fixed positive integer k and some $b \in R$.*

Proof. Similar to the proof of Proposition 2.6. □

Corollary 2.2. *If R is a ring such that $N(R) = E_k(R)$, then R satisfies the Koethe conjecture.*

Statements and Declaration

The author declares that there is no competing interest and this is an original work of this author. Also no funds, grants were available for this research.

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