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Some sufficient conditions of clean comodules

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Abstract

Let R be a commutative ring with the multiplicative identity. In this paper, we apply the notions of cleanness on modules and rings to comodules and coalgebras. Based on the cleanness concept in modules theory, a C-comodule M is a clean comodule provided the endomorphism ring of C-comodule M is clean. A clean coalgebra is defined by considering every R-coalgebra C as a comodule over itself. In the trivial case, every clean R-module is a clean R-comodule. Here, we obtained some sufficient conditions of clean comodules by generalizing the cleanness condition of comodules over $R[G]$.

1 Introduction and preliminaries

Throughout R is a commutative ring with the multiplicative identity and (C, Δ, ε) is a coassociative and counital coalgebra over R. A ring R is called a clean ring if every element of R can be expressed as a sum of a unit and

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an idempotent element (see [1]). In [2] and [3], the authors showed that a clean ring is one of the subclasses of exchange rings. It is well known that fields, local rings and von Neumann regular rings are clean. Furthermore, a direct product of the family of clean rings is a clean ring if and only if each factor is clean. Also, if R is a clean ring then the ring matrix $M_n(R)$ is clean (see [4] and [5]). Furthermore, [6] gives the property of the tensor product of two clean algebras. Some necessary and sufficient conditions of the clean group ring $R[G]$ can refer to [4] and [7].

By considering R as an R-module, the ring of endomorphism R-module R (denoted by $End_R(R)$) is isomorphic to the ring R. It implies R is a clean ring if and only if $End_R(R)$ is clean. As a special case, in [8], the ring of linear transformations on a countable vector space V is clean. In the general case, the ring of linear transformations of an arbitrary vector space over a field is clean (see [9]). Moreover, in Lemma 1, [10] have modified the results in [8, 9] for any vector spaces over a division ring. Based on the result of [8], [9] and [10], a clean module is defined as an R-module M which the endomorphism of R-module M (denoted by $End_R(M)$) is a clean ring [11]. We recall the important result in [11], i.e., necessary and sufficient conditions of clean elements of an endomorphism R-module (see Proposition 2.2 and 2.3). One example of clean modules is a continuous module (see [11]). Furthermore, in [12] this property has been proved more shortly, i.e., by proving that every non M-singular self-injective module M is clean (see Lemma 4).

In a clean module, its submodules are not necessarily clean. For example, $\mathbb{Z}$ is not clean in $\mathbb{Z}$-module $\mathbb{Q}$. Based on this fact, in [13], Ismarwati, et.al., have introduced the notion of nice modules. An R-module M is called a nice module if every submodule of M is a clean module. For examples, the duo modules, finite-length modules and semi-simple modules are nice.

In [14], the author has introduced coalgebras over a field as the dualization of algebras over a field. This ground field has been generalized to any commutative ring with multiplicative identity (see [15]). A comodule over a coalgebra is well-known as a dualization of a module over a ring. For any R-coalgebra C, we can construct $C^* = Hom_R(C, R)$ (the set of all homomorphism of R-module C to R-module R). Here, C^* is an algebra (ring) over the convolution product. We call C^* as the dual algebra of C. Hence, if M is a C-comodule, then M is a module over the dual algebra C^* [15].

In this research, the cleanness of rings and modules are transferred on comodules and coalgebras, by introducing the notion of clean comodules and clean coalgebras. Since any R-module M can be considering as the trivial comodule over the trivial coalgebra R, we have a relation between the cleanness of the endomorphism ring of R-module M ($End_R(M)$) and the endomorphism of R-comodule M ($End^R(M)$). Then, the ring of $End_R(M)$ is clean if and only if $End^R(M)$ is also a clean ring. We generalize it for any C-comodules. A C-comodule M is called a clean C-comodule if the endomorphism ring of C-comodule (denoted by $End^C(M)$) is clean. Moreover, by using the α-condition of C, we have that the ring of $End^C(M) \simeq {}_{C^*}End(M)$ ([15]). It implies a C-comodule M is clean if M is clean as a C^*-module. An R-coalgebra is a comodule over itself, an R-coalgebra C is said to be a clean coalgebra if C is a clean C-comodule.

To recall some notions of comodules and coalgebras we refer to [15]. An R-module C is called an R-coalgebra if there exist (R,R)-bilinear maps

$$\Delta : C \to C \otimes_R C \text{ and } \varepsilon : C \to R$$

where

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta \text{ and } (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta.$$

The maps are called coassociative and counital comultiplication, respectively. For any R-coalgebra C, [15] has been defined the convolution product $(*)$ of $C^* = Hom_R(C,R)$ such that $(C^*,+,*)$ is an R-algebra and we call it as the dual algebra of C.

An R-coalgebra (C,Δ,ε) satisfies the α-condition if the map

$$\alpha_N : N \otimes_R C \to Hom_R(C^*,N), n \otimes c \mapsto [f \mapsto f(c)n]$$

is injective, for every $N \in \mathbf{M}_R$. Moreover, for any R-coalgebra (C,Δ,ε) the following statements are equivalent:

1. C satisfies the α-condition;
2. for any $N \in \mathbf{M}_R$ and $u \in N \otimes_R C, (I_N \otimes f)(u) = 0$ for all $f \in C^*$, implies $u = 0$;
3. C is locally projective as an R-module.

A right C-comodule M is a right R-module together with R-linear map $\varrho^M : M \to M \otimes_R C$ called a right C-coaction, with properties coassociative and counital:

$$(I_M \otimes \Delta) \circ \varrho^M = (\varrho^M \otimes I_C) \circ \varrho^M \text{ and } (I_M \circ \varepsilon) \circ \varrho^M = I_M.$$

Throughout, comodules on this paper mean a right comodule. A comodule morphism $f : M \to N$ between C-comodule M and N is an R-linear map f which satisfies

$$\varrho^N \circ f = (f \otimes I_C) \circ \varrho^M.$$

The set of all comodule morphisms from M to N is denoted by $Hom^C(M, N)$.

We have already known from [15] that any right C-comodule M can be considered as a left module over the dual algebra C^* by the following action.

$$\begin{gathered} \rightharpoonup: C^* \otimes_R M \to M, \\ f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m) = \Sigma m_{\underline{0}} f(m_{\underline{1}}) \end{gathered}$$

and any comodule morphisms $h : M \to N$ is a left C^*-module homomorphism. Consequently, the category of $\mathbf{M}^C$ is a full subcategory of the category of left ${}_{C^*}\mathbf{M}$. The category $\mathbf{M}^C$ become a full subcategory of ${}_{C^*}\mathbf{M}$ if and only if C satisfies the α-condition, i.e., if and only if C is locally projective R-module (see [15]).

For a group G, in comodule theory we have special condition that any G-graded module over a ring R is an $R[G]$-comodule. Moreover, in [15], an R-module M is a G-graded if and only if it is an $R[G]$-comodule. It motivated us to start investigation from the clean $R[G]$-comodule. Here, we give some properties of cleanness notion on comodules and obtained some sufficient conditions of clean comodules by generalizing clean $R[G]$-comodules.

2 Clean Comodules and Clean Coalgebras

Assume that C is an R-coalgebra with the α condition. Here, we combine two concepts, i.e., the clean property of modules and the structure of comodules to obtain the notions of clean comodules and clean coalgebras.

Definition 2.1. *Let R be a ring and (C, Δ, ε) an R-coalgebra. A right (left) C-comodule M is called a clean comodule if the ring of endomorphism of C-comodule M ($End^C(M)$) is a clean ring.*

Based on the α-condition of C, since the ring $End^C(M) \simeq {}_{C^*}End(M)$, Definition 2.1 means that a right C-comodule M is a clean if M is a clean C^*-module. In this paper we also extend to clean coalgebras. Since every R-coalgebra C is a right and left comodule over itself, based on Definition 2.1 we introduce a clean coalgebra as follows.

Definition 2.2. *Let R be a ring. An R-coalgebra C is called a clean coalgebra if C is a clean comodule over itself.*

Definition 2.2 means that C is a clean coalgebra if C is clean as a C^*-module. We start our example with a trivial clean coalgebra. It is obtained based on the fact that every ring R is a trivial R-coalgebra with the trivial comultiplicaion $\Delta_T : R \to R \otimes_R R, r \mapsto r \otimes r$ and counit $\varepsilon_T : R \to R, r \mapsto 1$ for any $r \in R$. Hence, as an R-coalgebra, the dual algebra of R, i.e., $(R^*, +, *)$ is isomorphic to the ring R by mapping $f \mapsto f(1)$ for all $f \in R^*$. Then we have the relationship between clean rings and clean coalgebras.

Proposition 2.1. *Let R be a ring. A trivial R-coalgebra $(R, \Delta_T, \varepsilon_T)$ is clean if and only if R is a clean ring.*

Proof. The proof is obvious since the ring $End_R(R) \simeq R$. □

The R-coalgebra R with comultiplication Δ_T and counit ε_T is called the trivial R-coalgebra. The cleanness of R-coalgebra R depends on its comultiplication and counit. Even tough R is a clean ring, it does not imply that the ring R is clean as an R-coalgebra.

Example 2.1. *Let $(\mathbb{Z}_4, +, \cdot)$ be a clean ring and define a comultiplication in $\mathbb{Z}_4$ as below:*

$$\begin{gathered}\Delta : \mathbb{Z}_4 \to \mathbb{Z}_4 \otimes_{\mathbb{Z}_4} \mathbb{Z}_4 \\ a \to a \otimes 1 + 1 \otimes a.\end{gathered}$$

Since $\mathbb{Z}_4$ is a projective $\mathbb{Z}_4$-module, $\mathbb{Z}_4$ satisfies the α-condition and the ring $End_{\mathbb{Z}_4}(\mathbb{Z}_4) \simeq \mathbb{Z}_4^*$. It means, the cleanness of $\mathbb{Z}_4$-coalgebra $\mathbb{Z}_4$ depends on the dual ring $\mathbb{Z}_4^*$. The elements of $\mathbb{Z}_4$-endomorphism $\mathbb{Z}_4$ are

1. $f_0(x) = \bar{0}$ *for all* $x \in \mathbb{Z}_4$;
2. $f_1(x) = x$ *for all* $x \in \mathbb{Z}_4$;
3.

$$f_2(x) = \begin{cases} \bar{0} & \textit{if } x = \bar{0} \\ \bar{2} & \textit{if } x = \bar{1} \\ \bar{0} & \textit{if } x = \bar{2} \\ \bar{2} & \textit{if } x = \bar{3} \end{cases} \tag{2.1}$$

Table 1: Convolution Product for $\mathbb{Z}_4^*$

$*$	f_0	f_1	f_2	f_3
f_0	f_0	f_0	f_0	f_0
f_1	f_0	f_1	f_0	f_2
f_2	f_0	f_0	f_0	f_0
f_3	f_0	f_2	f_0	f_2

4.

$$f_3(x) = \begin{cases} \bar{0} & \textit{if } x = \bar{0} \\ \bar{3} & \textit{if } x = \bar{1} \\ \bar{2} & \textit{if } x = \bar{2} \\ \bar{1} & \textit{if } x = \bar{3}. \end{cases} \tag{2.2}$$

Hence, we will investigate the structure of the ring $\mathbb{Z}_4^* = End_{\mathbb{Z}_4}(\mathbb{Z}_4, +, *)$, where $\mathbb{Z}_4^* = \{f_1, f_2, f_3, f_4\}$. In general, for any $f_i, f_j \in \mathbb{Z}_4^*$, the convolution product of any element of $\mathbb{Z}_4^*$ are $f_i * f_j = (f_i \otimes f_j) \circ \Delta$ such that we have the Table 1. Based on Table 1, $\mathbb{Z}_4^*$ does not contain the multiplicative identity. It means, the ring $\mathbb{Z}_4^*$ does not contain the unit element. Hence, $\mathbb{Z}_4^*$ is not a clean ring. Consequently, although the ring $\mathbb{Z}_4$ is a clean ring, the $\mathbb{Z}_4$-coalgebra $\mathbb{Z}_4$ with comultiplication Δ as above is not clean.

Therefore, we must be precise in determining whether an R-coalgebra is clean or not, since it depends on the comultiplication. For the trivial coalgebra $(R, \Delta_T, \varepsilon_T)$, any R-module M can be considered as a trivial right and left R-comodule with an R-coaction, i.e., $m \mapsto m \otimes 1$. Here, we give a necessary and sufficient condition of the trivial clean R-comodules.

Proposition 2.2. *Let $(R, \Delta_T, \varepsilon_T)$ be a trivial R-coalgebra. An R-module M is a clean module if and only if M is a right and left clean R-comodule*

Proof. $\Rightarrow$ Let M be a clean R-module and R is the trivial R-coalgebra with a comultiplication Δ_T and counit ε_T. For any R-module M, we can define a right coaction ϱ on M

$$\varrho^M : M \to M \otimes_R R, m \mapsto m \otimes 1.$$

Hence, (M, ϱ^M) is a right C-comodule. Thus M is an R^*-module. Hence, the dual algebra of R-coalgebra R, (i.e., $R^* = Hom_R(R, R)$) is isomorphic to R. Thus, if M is a clean R-module, then we have $End_{R^*}(M) \simeq End_R(M)$ is a clean ring and it implies M is a clean R-comodule.
$\Leftarrow$ Suppose M is a clean R-comodule. Then $End_{R^*}(M)$ is a clean ring. Since $R^* = Hom_R(R, R) \simeq R$ and $End_{R^*}(M)$ is a clean ring, we have $End_{R^*}(M) \simeq End_R(M)$ is a clean ring, or M is a clean R-module.

Based on Proposition 2.2, a clean comodule is a generalization of the cleanness concept of modules. The cleanness property of trivial coalgebras or comodules is not delightful to be discussed, because it is an evident from cleanness as a ring and as a module. However, from an R-module M, it is possible that M could be a comodule over an R-coalgebra C where C is not always the same as R. In this paper, we are going to discuss it further and give more examples of clean comodule. □

3 Some Sufficient Conditions of Clean Comodules and Clean Coalgebras

In this section, we investigate some conditions of R-coalgebra C to be a clean coalgebra. As a comodule over itself, we have $End^C(C) \simeq End_{C^*}(C) \simeq C^*$. Thus, the cleanness of ring $End^C(C)$ can be determined from the structure of C^*. Hence, we get a sufficient condition of clean coalgebras as a direct consequence of the α-condition of C.

Proposition 3.1. *Let (C, Δ, ε) be an R-coalgebra.*

1. *If C^* is a clean ring, then C is a clean R-coalgebra.*
2. *If C^* is a division ring, then C is a clean R-coalgebra.*

Proof. Let C be an R-coalgebra. Since C satisfies the α-condition and C is a C-comodule, we have :

1. C^* is a clean ring, then the ring $End_{C^*}C \simeq C^*$ is clean. Then C is a clean C^*-module and C is a clean C-comodule.

2. Since C^* is a division ring by considering C as a comodule over itself, then C is a C^*-module. Thus, C is a clean module over C^*. Consequently, C is a clean R-coalgebra.

Recall definition of injective C-comodule [15]. Let N be a right C-comodule. A C-comodule Q is said to be N-injective if

$$Hom^C(f,Q) : Hom^C(N,U) \to Hom^C(K,U)$$

is a surjective map for every comodule monomorphism $f : K \to N$. A C-comodule Q is said to be an injective comodule, if Q is N-injective for any $N \in \mathbf{M}^C$. □

In [15], if C satisfies the α-condition then $\mathbf{M}^C$ is a full subcategory of ${}_{C^*}\mathbf{M}$. Moreover, if C is a finitely generated projective R-module, then ${}_{C^*}\mathbf{M} = \mathbf{M}^C$. Consequently, if M is an injective C-comodule, then M is also injective as a C^*-module. From the result of [11] every (quasi)-injective module is a clean module, thus we have this proposition.

Proposition 3.2. *If C is an R-coalgebra and a finitely generated projective R-module, then every injective C-comodule is a clean comodule.*

Proof. Suppose that M is an injective comodule over C. Since C satisfies the α-condition and the finitely generated projective R-module,

$${}_{C^*}\mathbf{M} = \mathbf{M}^C.$$

Consequently, M is also injective as a C^*-module. Based on [11] every injective module is a clean module; thus M is a clean C^*-module. That is, M is a clean C-comodule.

In [15], if C is a coalgebra over a quasi-Frobenius ring A, then C has satisfied the α-condition. It implies $\mathbf{M}^C$ is a full subcategory ${}_{C^*}\mathbf{M}$. Here, we give a sufficient condition of the cleanness of C-comodules when C is module over A. □

Furthermore, for any C-comodule M we have a condition as below:

Proposition 3.3. *Let A be a QF ring and M, C be a finitely generated A-module. If M is an injective C-comodule, then M is a clean C-comodule.*

Proof. Suppose that C is a finitely generated A-module and A is a QF ring. It implies C is a locally projective A-module. Since M is finitely generated as an A-module and C satisfies the α-condition, then $\mathbf{M}^C =_{C^*} \mathbf{M}$. Hence, if M is an injective C-comodule, then M is an injective C^*-module. By [11] M is a clean C^*-module, thus M is a C-clean comodule. □

3.1 The Cleanness of $R[G]$-Comodules

Let R be a ring and G a group. It is well-known that the group ring $R[G]$ is an R-coalgebra by the following comultiplication

$$\Delta : R[G] \to R[G] \otimes_R R[G], g \mapsto g \otimes g.$$

Since the group ring $R[G]$ is a free R-module, $R[G]$ is projective as an R-module. It implies that the coalgebra $R[G]$ satisfies the α-condition. On the following proposition, we give the cleanness properties of comodules over coalgebra $R[G]$ and obtain a sufficient condition of clean $R[G]$-comodules.

Theorem 3.1. *Let G be a finite group and R be a field. If M is a G-graded module over R, then M is a clean $R[G]$-comodule.*

Proof. From [15] it is clear that every G-graded R-module is an $R[G]$-comodule. We want to prove that M is a clean $R[G]$-comodule by proving $R[G]^*$ is a semisimple ring, i.e., $R[G]^*$ is a finite product of simple rings. Since R is a field, $R[G]$ and $R[G]^*$ can be considered as vector spaces over R. We have some facts of $R[G]$ and $R[G]^*$ as below

1. Since G is finite with $|G| = n$, we have $dim(R[G]^*) = dim(R[G]) = n$;

2. Since both of $R[G]^*$ and R^n are vector spaces with dimension n, we have $R[G]^* \simeq R^n$ as a vector space;

3. Suppose that basis of $R[G]^*$ is $\{\delta_{g_i} | g_i \in G\}$ where

$$\delta_{g_i}(g_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Here, we need to know that $R[G]^*$ and R^n are isomorphic as a ring. We define the following map

$$\psi : R[G]^* \to R^n, \delta_{g_i} \mapsto (0, 0, ..., 1, 0, ..., 0)$$

where 1 lies at the i-th entry and $n = |G|$. $\square$

1. It has been known that $R[G]^* \simeq R^n$ as a vector spaces by ψ. We only need to prove that ψ preserves convolution product on $R[G]^*$ to multiplication on R^n (multiplication on R^n is a point-wise product). For any $\delta_{g_i}, \delta_{g_j} \in R[G]^*$, based on definition ψ we obtain:

$$\psi(\delta_{g_i}) \cdot \psi(\delta_{g_j}) = \begin{cases} (0, ..., 0) & \text{if } i \neq j \\ (0, 0, ..., 0, 1_i, 0, ..., 0) & \text{if } i = j. \end{cases}$$

By definition of convolution product on $R[G]^*$, for any $g \in R[G]$ we have

$$\begin{aligned} (\delta_{g_i} * \delta_{g_j})(g) &= (\mu \circ (\delta_{g_i} \otimes \delta_{g_j}) \circ \Delta)(g) \\ &= (\mu \circ (\delta_{g_i} \otimes \delta_{g_j}))(g \otimes g) \\ &= (\delta_{g_i}(g))(\delta_{g_j}(g)) \end{aligned}$$

where

$$\delta_{g_i}(g)\delta_{g_j}(g) = \begin{cases} 0 & \text{if } g_i \neq g \text{ or } g_j \neq g \\ 1 & \text{if } g_i = g \text{ and } g_j = g. \end{cases}$$

It means

$$\delta_{g_i}(g)\delta_{g_j}(g) = 1 \text{ if } g_i = g_j = g.$$

Thus $\delta_g^2 = \delta_g * \delta_g = \delta_g$ and $\delta_{g_i} * \delta_{g_j} = 0$ if $g_i \neq g_j$. Then

$$\psi(\delta_{g_i} * \delta_{g_j}) = \begin{cases} \psi(0) = (0, 0, ..., 0) & \text{if } i \neq j \\ \psi(\delta_{g_j}) = (0, 0, ..., 0, 1_i, 0, ..., 0) & \text{if } i = j. \end{cases}$$

Consequently, $\psi(\delta_{g_i}) \cdot \psi(\delta_{g_j}) = \psi(\delta_{g_i} * \delta_{g_j})$.

2. Suppose $\delta_{g_i} \neq \delta_{g_j} \in R[G]^*$. It implies that $\psi(\delta_{g_i}) \neq \psi(\delta_{g_i})$, then ψ is injective;

3. Finally, we want to prove ψ is surjective. It is clear that

$$\{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 0, 1)\}$$

is a basis of R^n. For every $(0, 0, ..., 0, 1_i, 0, ..., 0) \in R^n$ where 1 lies at the i-th entry, there is $\delta_{g_i} \in R[G]^*$ such as $\psi(\delta_{g_i}) = (0, ..., 0, 1, 0, ..., 0)$. Then ψ is surjective.

From point 1-3, ψ is an isomorphism ring. Hence, $R[G]^*$ is isomorphic to finite product of fields (simple rings) R. Consequently, $R[G]^*$ is a semisimple ring. Since $R[G]^*$ is a semisimple ring, M is an injective $R[G]^*$-module. It implies M is a clean $R[G]^*$-module, and hence M is a clean $R[G]$-comodule.

Since every R-module M can be considered as a G-graded module over R with the trivial grading, it implies that every R-module M is an $R[G]$-comodule. Theorem 3.1 has the following direct consequence.

Corollary 3.1. *If R is a field and G is a finite group, then every R-module M is a clean $R[G]$-comodule.*

Proof. Consider M a G-graded module over R with the trivial grading and the proof is clear. $\square$

Furthermore, we generalize the Theorem 3.1 for any simple ring as follow.

Theorem 3.2. *Let G be a finite group and R be a simple ring. If M is a G-graded module over R, then M is a clean $R[G]$-comodule.*

Proof. Suppose G be a finite group, then $R[G]$ is a free R-module with basis G. Since M is a G-graded module over R, M is an $R[G]$-comodule. Considering M as an $R[G]^*$-module, we need to prove $R[G]^*$ is a semisimple ring. Hence, ring $R[G]^*$ is a free R-module with basis $\{p_{g_i} | g_i \in G\}$ where $p_{g_i}(g_j) = \delta_{i,j}$. For any $f \in R[G]^*$ with $f = \sum r_i p_{g_i}$ we define

$$\beta : R[G]^* \to R^n,$$

$\beta(f) = \beta(\sum r_i p_{g_i}) = \sum r_i (0, 0, ..., 0, 1_i, 0, ..., 0)$.
We see β is a ring isomorphism. Given $f, h \in R[G]^*$, where $f = \sum r_i p_{g_i}$ and $h = \sum s_i p_{g_i}$.

1. The map β preserve addition operation as below

$$\begin{aligned}
\beta(f+g) &= \beta((\sum r_i p_{g_i}) + (\sum s_i p_{g_i}))\\
&= \beta(\sum (r_i + s_i) p_{g_i})\\
&= \sum ((r_i + s_i)(0, 0, ..., 0, 1_i, 0, ..., 0))\\
&= \sum ((r_i)(0, 0, ..., 0, 1_i, 0, ..., 0))\\
&+ ((s_i)(0, 0, ..., 0, 1_i, 0, ..., 0))\\
&= \sum ((r_i)(0, 0, ..., 0, 1_i, 0, ..., 0))\\
&+ \sum ((s_i)(0, 0, ..., 0, 1_i, 0, ..., 0))\\
&= \beta(\sum r_i p_{g_i}) + \beta(\sum s_i p_{g_i})\\
&= \beta(f) + \beta(h).
\end{aligned}$$

2. We want to see that $\beta(fg) = \beta(f)\beta(g)$. Analogue with the Theorem 3.1 for any i, j, then $p_i * p_j = 0$ if $j \neq i$ and $p_i * p_j = p_i$ if $i = j$. Therefore for $f, g \in R[G]^*$,

$$f * g = (\sum r_i p_{g_i}) * (\sum s_i p_{g_i}) = \sum r_i s_i p_{g_i},$$

such that

$$\begin{aligned}
\beta(f * g) &= \beta(\sum r_i s_i p_{g_i})\\
&= \sum r_i s_i (0, 0, ..., 0, 1_i, 0, ..., 0)\\
&= (r_1 s_1, ..., r_n s_n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\beta(f)\beta(g)\\
&= (\sum r_i (0, 0, ..., 0, 1_i, 0, ..., 0))(\sum s_i (0, 0, ..., 0, 1_i, 0, ..., 0))\\
&= (r_1, ..., r_n)(s_1, ..., s_n) = (r_1 s_1, ..., r_n s_n).
\end{aligned}$$

It means $\beta(f * g) = \beta(f)\beta(g)$. Then β is a homomorphism ring.

3. Moreover, by mapping every basis on $R[G]^*$ to basis of R^n i.e.,

$$p_{g_i} \mapsto (0, ..., 1_i, 0, ..., 0),$$

β is a bijective map, then β is an isomorphism ring.

Hence, $R^n \simeq R[G]^*$ for any simple ring R and finite group G. Consequently, $R[G]^*$ is a semisimple ring and M is an injective $R[G]^*$-module. Thus, M is a clean $R[G]$-comodules. □

Every module over commutative Artinian principal ideal ring is clean [11]. On the other hand, if R is a finite ring, then R is Artinian. We prove the following proposition as a generalization the situation of Proposition 3.2.

Proposition 3.4. *Let R be an Artinian principal ideal ring and G be a finite group. Every $R[G]$-comodule M is a clean comodule.*

Proof. Suppose that R is an Artinian principal ideal ring and G is a finite group with order n. Since M is a comodule over $R[G]$, M is an $R[G]^*$-module. Furthermore, $R[G]^* \simeq R^n$ as rings. Since R is a commutative Artinian principal ideal ring, by using point-wise operation we have that R^n is also a commutative Artinian principal ideal ring. As an $R[G]^*$-module, since $R[G]^* \simeq R^n$ and R^n is a commutative Artinian principal ideal ring, M is a clean $R[G]^*$-module (by [11]). Therefore, M is a clean $R[G]$-comodule. □

3.2 The generalization of the Clean $R[G]$-Comodules

In the previous result, if G is finite with order n, then the dual algebra $R[G]^*$ is isomorphic to the ring R^n. It motivates us to make this concepts more general for any R-coalgebra C. Based on module theory, we know that if C is a free R-module with dimension n, then $Hom_R(C, R) \simeq R^n$ as an R-module. Moreover, if C is an R-coalgebra we have a particular condition as below.

Theorem 3.3. *Let C be a free R-module with basis $\{x_i\}_{i=1}^n$. If C is an R-coalgebra with comultiplication*

$$\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i,$$

then the dual algebra C^* is isomorphic to the ring R^n.

Proof. Suppose that C is a free R-module with basis $\{x_i\}_{i=1}^n$ and the comultiplication on R-coalgebra C is

$$\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i.$$

The dual algebra C^* is also a finitely generated R-module with basis $\{x_i^*\}_{i=1}^n$ where $x_i^*(x_j) = \delta_{i,j}$ for any i, j. Define a map

$$\sigma : C^* \mapsto R^n$$

i.e., for any $f \in C^*$

$$\begin{aligned}
\sigma(f) &= \sigma(\sum_{i=1}^{n} r_i x_i^*) \\
&= \sum_{i=1}^{n} r_i(0, 0, ..., 0, 1_i, 0, ..., 0) \\
&= (r_1, r_2, ..., r_n).
\end{aligned}$$

We want to prove σ is a ring isomorphism.

1. For $f = \sum_{i=1}^n r_i x_i^*, g = \sum_{i=1}^n s_i x_i^* \in C^*$, the map σ preserves the addition operation as below

$$\begin{aligned}
\sigma(f+g) &= \sigma((\sum_{i=1}^{n} r_i x_i^*) + (\sum s_i x_i^*)) \\
&= \sigma(\sum_{i=1}^{n} (r_i + s_i) x_i^*) \\
&= (r_1 + s_1, r_2 + s_2, ..., r_n + s_n) \\
&= (r_1, r_2, ..., r_n) + (s_1, s_2, ..., s_n) \\
&= \sigma(\sum_{i=1}^{n} r_i x_i^*) + \beta(\sum_{i=1}^{n} s_i x_i^*) \\
&= \sigma(f) + \sigma(g).
\end{aligned}$$

2. Suppose that $f = \sum_{i=1}^n r_i x_i^*, g = \sum_{i=1}^n s_i x_i^* \in C^*$. Using the definition of convolution product of C^*, we see that $f * g = \sum_{i=1}^n r_i s_i x_i^*$. Here,

we put any $c \in C$ where $c = \sum_{i=1}^{n} a_i x_i$ and see the result of $f * g(c)$ as below

$$
\begin{aligned}
& f * g(\sum_{i=1}^{n} a_i x_i) = \mu \circ (f \otimes g) \circ \Delta(\sum_{i=1}^{n} a_i x_i) \\
& = \mu \circ (f \otimes g)(\sum_{i=1}^{n} a_i (x_i \otimes x_i)) \\
& = \mu(\sum_{i=1}^{n} a_i f(x_i) \otimes g(x_i)) \\
& = \mu(\sum_{i=1}^{n} a_i ((\sum_{i=1}^{n} r_i x_i^*)(x_i) \otimes (\sum_{i=1}^{n} r_i x_i^*)(x_i))) \\
& = \mu(\sum_{i=1}^{n} a_i (r_i \otimes s_i) \\
& = \sum_{i=1}^{n} a_i r_i s_i.
\end{aligned}
$$

On the other hand, the result $\sum_{i=1}^{n} r_i s_i x_i^*(c)$ is

$$
\begin{aligned}
& \sum_{i=1}^{n} r_i s_i x_i^* (\sum_{i=1}^{n} a_i x_i) \\
& = r_1 s_1 x_1^* (\sum_{i=1}^{n} a_i x_i) + ... + r_n s_n x_n^* (\sum_{i=1}^{n} a_i x_i) \\
& = a_1 r_1 s_1 + ... + a_n r_n s_n \\
& = \sum_{i=1}^{n} a_i r_i s_i.
\end{aligned}
$$

Hence we can see that for any $c \in C, f * g(c) = \sum_{i=1}^{n} r_i s_i x_i^*(c)$. Consequently, the function $f * g = \sum_{i=1}^{n} r_i s_i x_i^*$ on C^*. It implies that

$$
\sigma(f * g) = \sigma(\sum_{i=1}^{n} r_i s i x_i^*) = (r_1 s_1, r_2 s_2, ..., r_n s_n),
$$

and for other side we have this condition,

$$\begin{aligned}\sigma(f)\sigma(g) &= \sigma(\sum_{i=1}^{n} r_i x_i^*)\sigma(\sum_{i=1}^{n} s_i x_i^*)\\ &= (r_1, r_2, ..., r_n)(s_1, s_2, ..., s_n)\\ &= \sigma(f)\sigma(g) = (r_1 s_1, r_2 s_2, ..., r_n s_n).\end{aligned}$$

Since the left and right side give the equivalent result, i.e., $\sigma(f * g) = \sigma(f)\sigma(g)$, σ is a ring homomorphism.

3. For $f = \sum_{i=1}^{n} r_i x_i^*, g = \sum_{i=1}^{n} s_i x_i^* \in C^*$ with $\sigma(f) = \sigma(g)$, we have $(r_1, r_2, ..., r_n) = (s_1, s_2, ..., s_n)$. It means $r_i = s_i$ for any i. Therefore, $\sum_{i=1}^{n} r_i x_i^* = \sum_{i=1}^{n} s_i x_i^*$ if and only if $f = g$. Then σ is injective.

4. Let $(a_1, a_2, ..., a_n)$ is an element of R^n. Since C^* is an R-module, $a_i x_i^* \in C^*$ for every $i = 1, 2, ..., n$. Furthermore, $\sum_{i=1}^{n} a_i x_i^* \in C^*$. Putting $h = \sum_{i=1}^{n} a_i x_i^* \in C^*$, then we have $\sigma(h) = \sigma(\sum_{i=1}^{n} a_i x_i^*) = (a_1, a_2, ..., a_n)$. It is proved that σ is surjective.

□

From this explanation we have a conclusion that σ is an isomorphism ring. In particular, $C^* \simeq R^n$.

In general condition, the isomorphism between C^* and R^n as a ring really depends on the comultiplication Δ. We give the following example as an illustration that there is Δ that make the ring C^* is not isomorphic with the ring R^n.

Example 3.1. *Let C be a free R-module with basis $\{x, y\}$. We consider C as an R-coalgebra with comultiplication*

$$\begin{aligned}\Delta : C &\to C \otimes_R C\\ x &\mapsto x \otimes y + y \otimes x\\ y &\mapsto y \otimes y - x \otimes x.\end{aligned}$$

Let C^* be the dual algebra of C with basis $\{x^*, y^*\}$. We prove that for any $f = a_1 x^* + a_2 y^*, g = b_1 x^* + b_2 y^* \in C^*$, we have $f * g \neq a_1 b_1 x^* + a_2 b_2 y^*$.

Put any $c \in C$ where $c = c_1x + c_2y$, then we have

$$\begin{aligned}
f * g(c) &= f * g(c_1x + c_2y) \\
&= \mu \circ (f \otimes g) \circ \Delta(c_1x + c_2y) \\
&= \mu \circ (f \otimes g)(\Delta(c_1x) + \Delta(c_2y)) \\
&= \mu \circ (f \otimes g)((c_1(x \otimes y \\
&+ y \otimes x)) + (c_2(y \otimes y - x \otimes x))) \\
&= \mu((c_1(f(x) \otimes g(y) + f(y) \otimes g(x))) \\
&+ (c_2(f(y) \otimes g(y) - f(x) \otimes g(x))))
\end{aligned}$$

by subtituting $f = a_1x^* + a_2y^*, g = b_1x^* + b_2y^*$ we get

$$f * g(c) = c_1a_1b_2 + c_1a_2b_1 + c_2a_2b_2 - c_2a_1b_1,$$

and $a_1b_1x_1^* + a_2b_2x_2^*(c) = a_1b_1x_1^* + a_2b_2x_2^*(c_1x + c_2y) = a_1b_1c_1 + a_2b_2c_2$. Hence, $f * g \neq a_1b_1x_1^* + a_2b_2x_2^*$ in C^*. Thus, the map σ on Theorem 3.3 does not preserve the multiplication operation. In particular, C^* and R^2 is not isomorphic as a ring.

As a direct consequence of Theorem 3.3, we give some sufficient conditions for clean C-comodules. The proof of the following proposition is obviously obtained by using the fact that $C^* \simeq R^n$ as a ring.

Proposition 3.5. *Let R be a field and C be a vector space over R with basis $\{x_i\}_{i=1}^n$. If C is an R-coalgebra with comultiplication $\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i$, then every C-comodule M is clean.*

Proof. By using Theorem 3.3, since C is a vector space over R, we have $C^* \simeq R^n$ is a simple ring. If we consider C-comodule M as a module over simple ring C^*, then M is an injective C^*-module. Hence, M is a clean C^*-module [11]. It means M is a clean C-comodule. □

Proposition 3.6. *Let R be a simple ring and C a free R-module with basis $\{x_i\}_{i=1}^n$. If C is an R-coalgebra with comultiplication $\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i$, then every C-comodule M is clean.*

Proof. Analogue to Proposition 3.5, since the dual algebra $C^* \simeq R^n$ is a semisimple ring, M is an injective C^*-module. It means M is a clean C-coalgebra.

Recall that every R-coalgebra C is a comodule over itself. We have the structure of clean coalgebra. Here, we give the conditions that make the coalgebra C is clean. □

Proposition 3.7. *Let R be a clean ring and C a free R-module with basis $\{x_i\}_{i=1}^n$. If C is an R-coalgebra with comultiplication $\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i$, then C is a clean R-coalgebra.*

Proof. Suppose that C is an R-coalgebra. Since C is a free R-module, C is a projective R-module. Thus the colagebra C satisfies the α-condition. It means ${}_{C^*}End(C) \simeq C^*$. Moreover, from Theorem 3.3 $C^* \simeq R^n$ where R is a clean ring. From [5] direct product of clean ring is also clean, then C^* is a clean ring. It implies C is a clean C^*-module. Then C is a clean R-coalgebra. □

In [11], a continuous module as a generalization of (quasi)-injective module is clean. Most of the proof of the clean C-comodule M in this paper is shown by proving that M is an injective module over C^*. On the other hand, generally, module over a commutative Artinian principal ideal domain is not always be an injective module, but it is a clean module [11]. By this fact, we have a generalization of the Proposition 3.4.

Proposition 3.8. *Let R be a Artinian principal ideal ring and C is a free R-module with basis $\{x_i\}_{i=1}^n$. If C is an R-coalgebra with comultiplication $\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i$, then every C-comodule M is a clean comodule.*

Proof. Suppose that R is an Artinian principal ideal ring and C is a free R-module with dimension n. Consider C-comodule M as a C^*-module. Based on Theorem 3.3 $C^* \simeq R^n$ as a ring. Since R is a commutative Artinian principal ideal ring, R^n is a commutative principal ideal ring (by pointwise multiplication). By Theorem 3.3 $C^* \simeq R^n$, then M is a module over a commutative Artinian principal ideal ring C^*. It implies M is a clean C^*-module. Consequently, M is a clean C-comodule. □

4 Conclusions

The sufficient conditions of clean comodules have been obtained by observing the relationship between the category of C-comodules and C^*-modules. Moreover, we also get the sufficient conditions of $R[G]$-comodule and its generalization for any C-comodules. The clean comodules (coalgebras) on this paper is intensely depending on the structure of R-module and the comultiplication of C, since we assumed that the α-condition of C.

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