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Some common fixed point theorems on partial metric spaces involving auxiliary function

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Abstract

In this paper, we prove some common fixed point theorems in the framework of partial metric spaces by using auxiliary function and give some consequences of the main result. Also we give some examples in support of the result. The presented results in this paper extend and generalize several results from the existing literature.

1 Introduction and Preliminaries

Metric fixed point theory has been the centre of extensive research for several researchers. Fixed point theory has become an important tool for solving many nonlinear problems related to science and engineering because of its applications. The Banach contraction mappings principle is the opening and vital result in the direction of fixed point theory. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $\mathcal{T}x = x$ (where $\mathcal{T}$ is a self mapping defined on a nonempty set $\mathcal{X}$), is the most widely used fixed point theorem in all of analysis. In a metric space setting it can be briefly stated as follows.

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Theorem 1.1. *([7]) Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{S}\colon \mathcal{X} \to \mathcal{X}$ be a map satisfying*

$$d(\mathcal{S}(p), \mathcal{S}(q)) \leq m\, d(p, q), \text{ for all } p, q \in \mathcal{X}, \tag{1.1}$$

where $0 < m < 1$ is a constant. Then

(1) *$\mathcal{S}$ has a unique fixed point z in $\mathcal{X}$;*

(2) *The Picard iteration $\{y_n\}_{n=0}^{\infty}$ defined by*

$$y_{n+1} = \mathcal{S} y_n, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

converges to z, for any $y_0 \in \mathcal{X}$.

Remark 1.1. *(i) A self-map satisfying (1) and (2) is said to be a Picard operator (see, [28, 29]).*

(ii) Inequality (1.1) also implies the continuity of $\mathcal{S}$.

In literature, there are many generalizations of Banach contraction principle in metric and generalized metric spaces. These generalizations are made either by using different contractive conditions or by imposing some additional condition on the ambient spaces. On the other hand, a number of generalizations of metric spaces have been done and one of such generalization is partial metric space introduced in 1992 by Matthews [22, 23]. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews proved the partial metric version of Banach fixed point theorem ([7]). Then, many authors gave some generalizations of the result of Matthews and proved some fixed point theorems in this space (see, i.e., [1], [2], [3], [16], [17], [18], [19], [25], [27], [30], [36]-[39], [40] and many others).

Recently, many authors proved fixed point and common fixed point results via contractive type conditions in various ambient spaces (see, e.g., [4, 5, 8, 9, 10, 11, 14, 15, 19, 20, 26, 31, 32, 33, 34, 35] and many others).

The purpose of this work is to prove some common fixed point theorems for contractive type condition involving auxiliary function in the setting of partial metric spaces.

Now, we recall some basic concepts on partial metric spaces defined as follows.

Definition 1.1. *([23]) Let $\mathcal{X}$ be a nonempty set and $p\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ be a self mapping of $\mathcal{X}$ such that for all $u,v,w\in\mathcal{X}$ the followings are satisfied:*

$(P1)$ $u=v\Leftrightarrow p(u,u)=p(u,v)=p(v,v)$,

$(P2)$ $p(u,u)\leq p(u,v)$,

$(P3)$ $p(u,v)=p(v,u)$,

$(P4)$ $p(u,v)\leq p(u,w)+p(w,v)-p(w,w)$.

Then p is called partial metric on $\mathcal{X}$ and the pair $(\mathcal{X},p)$ is called partial metric space (in short PMS).

Remark 1.2. *It is clear that if $p(u,v)=0$, then $u=v$. But, on the contrary $p(u,u)$ need not be zero.*

Example 1.1. *([6]) Let $\mathcal{X}=\mathbb{R}^+$ and $p\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ be given by $p(u,v)=\max\{u,v\}$ for all $u,v\in\mathbb{R}^+$. Then $(\mathbb{R}^+,p)$ is a partial metric space.*

Example 1.2. *([6]) Let I denote the set of all intervals $[a,b]$ for any real numbers $a\leq b$. Let $p\colon I\times I\to[0,\infty)$ be a function such that*

$$p\Big([a,b],[c,d]\Big)=\max\{b,d\}-\min\{a,c\}.$$

Then (I,p) is a partial metric space.

Example 1.3. *([12]) Let $\mathcal{X}=\mathbb{R}$ and $p\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ be given by $p(u,v)=e^{\max\{u,v\}}$ for all $u,v\in\mathbb{R}$. Then $(\mathbb{X},p)$ is a partial metric space.*

Various applications of this space has been extensively investigated by many authors (see, K$\ddot{u}$nzi [21] and Valero [40] for details).

Remark 1.3. *([17]) Let $(\mathcal{X},p)$ be a partial metric space.*

(1) *The function $d_p\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ defined as $d_p(u,v)=2p(u,v)-p(u,u)-p(v,v)$ is a metric on $\mathcal{X}$ and $(\mathcal{X},d_p)$ is a metric space.*

(2) *The function $d_s\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ defined as $d_s(u,v)=\max\{p(u,v)-p(u,u),p(u,v)-p(v,v)\}$ is a metric on $\mathcal{X}$ and $(\mathcal{X},d_s)$ is a metric space.*

Note also that each partial metric p on $\mathcal{X}$ generates a T_0 topology τ_p on $\mathcal{X}$, whose base is a family of open p-balls $\{B_p(u,\varepsilon):u\in\mathcal{X},\varepsilon>0\}$ where,

$$B_p(u,\varepsilon)=\{v\in\mathcal{X}:p(u,v)<p(u,u)+\varepsilon\},$$

for all $u \in \mathcal{X}$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as

$$B_p[u, \varepsilon] = \{v \in \mathcal{X} : p(u, v) \leq p(u, u) + \varepsilon\},$$

for all $u \in \mathcal{X}$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [22].

Definition 1.2. *([22]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is said to be convergent to a point $r \in \mathcal{X}$ if and only if $p(r, r) = \lim_{n\to\infty} p(r_n, r)$;*

(2) *a sequence $\{r_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(r_m, r_n)$ exists and finite;*

(3) *$(\mathcal{X}, p)$ is said to be complete if every Cauchy sequence $\{r_n\}$ in $\mathcal{X}$ converges to a point $r \in \mathcal{X}$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(r_m, r_n) = \lim_{n\to\infty} p(r_n, r) = p(r, r).$$

(4) *A mapping $f\colon \mathcal{X} \to \mathcal{X}$ is said to be continuous at $r_0 \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f\Big(B_p(r_0, \delta)\Big) \subset B_p\Big(f(r_0), \varepsilon\Big)$.*

Definition 1.3. *([24]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is called 0-Cauchy if $\lim_{m,n\to\infty} p(r_m, r_n) = 0$;*

(2) *$(\mathcal{X}, p)$ is said to be 0-complete if every 0-Cauchy sequence $\{r_n\}$ in $\mathcal{X}$ converges to a point $r \in \mathcal{X}$, such that $p(r, r) = 0$.*

Lemma 1.1. *([22, 23]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(1) a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{X}, d_p)$,

(2) $(\mathcal{X}, p)$ is complete if and only if the metric space $(\mathcal{X}, d_p)$ is complete,

(3) a subset E of a partial metric space $(\mathcal{X}, p)$ is closed if a sequence $\{r_n\}$ in E such that $\{r_n\}$ converges to some $r \in \mathcal{X}$, then $r \in E$.

Lemma 1.2. *([2]) Assume that $r_n \to r$ as $n \to \infty$ in a partial metric space $(\mathcal{X}, p)$ such that $p(r, r) = 0$. Then $\lim_{n\to\infty} p(r_n, u) = p(r, u)$ for every $u \in \mathcal{X}$.*

Lemma 1.3. *(see [19]) Let $(\mathcal{X}, p)$ be a partial metric space.*
(i) If $p(u, v) = 0$, then $u = v$;
(ii) If $u \neq v$, then $p(u, v) > 0$.

2 Main Results

In this section, we shall prove some unique common fixed point theorems in the framework of partial metric spaces by using auxiliary function.

We shall denote Ψ the set of functions $\psi\colon [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(Ψ_1) ψ is continuous; (Ψ_2) $\psi(t) < t$ for all $t > 0$.
Obviously, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) \leq t$ for all $t \geq 0$.

Theorem 2.1. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two self-maps on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{R}_1 y, \mathcal{R}_2 z) \leq \alpha_1 \Lambda_1^p(y, z) + \alpha_2 \Lambda_2^p(y, z), \tag{2.1}$$

for all $y, z \in \mathcal{X}$, where

$$\Lambda_1^p(y, z) = \psi\Big(p(y, \mathcal{R}_1 y)\frac{1 + p(z, \mathcal{R}_2 z)}{1 + p(y, z)}\Big), \tag{2.2}$$

and

$$\begin{aligned}\Lambda_2^p(y, z) = \max\Big\{&\psi(p(y, z)), \psi(p(y, \mathcal{R}_1 y)), \psi\Big(\frac{1}{2}[p(z, \mathcal{R}_1 y) + p(y, \mathcal{R}_2 z)]\Big),\\ &\psi\Big(\frac{p(y, \mathcal{R}_1 y)[1 + p(z, \mathcal{R}_2 z)]}{1 + p(y, z)}\Big)\Big\},\end{aligned} \tag{2.3}$$

for all $\psi \in \Psi$, where $\alpha_1, \alpha_2 \in [0,1)$ with $\alpha_1 + \alpha_2 < 1$. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ have a unique common fixed point in $\mathcal{X}$.

Proof. For each $u_0 \in \mathcal{X}$. Let $u_{2n+1} = \mathcal{R}_1 u_{2n}$ and $u_{2n+2} = \mathcal{R}_2 u_{2n+1}$ for $n = 0, 1, 2, \ldots$, we prove that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$. It follows from (2.1) for $y = u_{2n}$ and $z = u_{2n-1}$ that

$$\begin{aligned} p(u_{2n+1}, u_{2n}) &= p(\mathcal{R}_1 u_{2n}, \mathcal{R}_2 u_{2n-1}) \\ &\leq \alpha_1 \, \Lambda_1^p(u_{2n}, u_{2n-1}) + \alpha_2 \, \Lambda_2^p(u_{2n}, u_{2n-1}), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Lambda_1^p(u_{2n}, u_{2n-1}) &= \psi\Big(p(u_{2n}, \mathcal{R}_1 u_{2n}) \frac{1 + p(u_{2n-1}, \mathcal{R}_2 u_{2n-1})}{1 + p(u_{2n}, u_{2n-1})}\Big) \\ &= \psi\Big(p(u_{2n}, u_{2n+1}) \frac{1 + p(u_{2n-1}, u_{2n})}{1 + p(u_{2n}, u_{2n-1})}\Big) \\ &= \psi\Big(p(u_{2n+1}, u_{2n}) \frac{1 + p(u_{2n-1}, u_{2n})}{1 + p(u_{2n-1}, u_{2n})}\Big) \text{ (by (P3))} \\ &= \psi\Big(p(u_{2n+1}, u_{2n})\Big), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \Lambda_2^p(u_{2n}, u_{2n-1}) &= \max\Big\{\psi(p(u_{2n}, u_{2n-1})), \psi(p(u_{2n}, \mathcal{R}_1 u_{2n})), \\ &\qquad \psi\Big(\frac{1}{2}[p(u_{2n-1}, \mathcal{R}_1 u_{2n}) + p(u_{2n}, \mathcal{R}_2 u_{2n-1})]\Big), \\ &\qquad \psi\Big(\frac{p(u_{2n}, \mathcal{R}_1 u_{2n})[1 + p(u_{2n-1}, \mathcal{R}_2 u_{2n-1})]}{1 + p(u_{2n}, u_{2n-1})}\Big)\Big\} \\ &= \max\Big\{\psi(p(u_{2n}, u_{2n-1})), \psi(p(u_{2n}, u_{2n+1})), \end{aligned} \tag{2.6}$$

$$\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n+1})+p(u_{2n},u_{2n})]\Big), \tag{2.7}$$

$$\begin{aligned}
&\quad\psi\Big(\frac{p(u_{2n},u_{2n+1})[1+p(u_{2n-1},u_{2n})]}{1+p(u_{2n},u_{2n-1})}\Big)\Big\} \\
&\le \max\Big\{\psi(p(u_{2n-1},u_{2n})),\psi(p(u_{2n+1},u_{2n})), \\
&\quad\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n})+p(u_{2n+1},u_{2n})]\Big), \\
&\quad\psi\Big(\frac{p(u_{2n+1},u_{2n})[1+p(u_{2n-1},u_{2n})]}{1+p(u_{2n-1},u_{2n})}\Big)\Big\} \text{ (by (P3) and (P4))}
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&= \max\Big\{\psi(p(u_{2n-1},u_{2n})),\psi(p(u_{2n+1},u_{2n})), \\
&\quad\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n})+p(u_{2n+1},u_{2n})]\Big)\Big\}.
\end{aligned} \tag{2.9}$$

The following cases arise.

Case (i) If $\Lambda_2^p(u_{2n},u_{2n-1})=\psi(p(u_{2n+1},u_{2n}))$, then from (2.4), (2.5), (2.9) and using the property of ψ that

$$\begin{aligned}
p(u_{2n+1},u_{2n}) &\le \alpha_1\,\psi\Big(p(u_{2n+1},u_{2n})\Big)+\alpha_2\,\psi\Big(p(u_{2n+1},u_{2n})\Big) \\
&= (\alpha_1+\alpha_2)\psi\Big(p(u_{2n+1},u_{2n})\Big) \\
&< (\alpha_1+\alpha_2)p(u_{2n+1},u_{2n}) \\
&< p(u_{2n+1},u_{2n}),\ \text{(since, } (\alpha_1+\alpha_2)<1)
\end{aligned} \tag{2.10}$$

a contradiction.

Case (ii) If $\Lambda_2^p(u_{2n},u_{2n-1})=\psi(p(u_{2n-1},u_{2n}))$, then from (2.4), (2.5), (2.9) and using the property of ψ that

$$\begin{aligned}
p(u_{2n+1},u_{2n}) &\le \alpha_1\,\psi\Big(p(u_{2n+1},u_{2n})\Big)+\alpha_2\,\psi\Big(p(u_{2n-1},u_{2n})\Big) \\
&\le \alpha_1\,p(u_{2n+1},u_{2n})+\alpha_2\,p(u_{2n-1},u_{2n}),
\end{aligned}$$

or

$$(1-\alpha_1)p(u_{2n+1},u_{2n}) \le \alpha_2\, p(u_{2n-1},u_{2n}),$$

or

$$p(u_{2n+1},u_{2n}) \le \Big(\frac{\alpha_2}{1-\alpha_1}\Big)p(u_{2n-1},u_{2n}). \tag{2.11}$$

Case (iii) If $\Lambda_2^p(u_{2n},u_{2n-1}) = \psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n+1},u_{2n})]\Big)$, then from (2.4), (2.5), (2.9) and using the property of ψ that

$$\begin{aligned} p(u_{2n+1},u_{2n}) &\le \alpha_1\,\psi\Big(p(u_{2n+1},u_{2n})\Big) + \alpha_2\,\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n+1},u_{2n})]\Big) \\ &\le \alpha_1\, p(u_{2n+1},u_{2n}) + \frac{\alpha_2}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n+1},u_{2n})], \end{aligned}$$

or

$$\begin{aligned} 2p(u_{2n+1},u_{2n}) &\le 2\alpha_1\, p(u_{2n+1},u_{2n}) + \alpha_2\, p(u_{2n-1},u_{2n}) + \alpha_2\, p(u_{2n+1},u_{2n}) \\ &= (2\alpha_1+\alpha_2)\, p(u_{2n+1},u_{2n}) + \alpha_2\, p(u_{2n-1},u_{2n}), \end{aligned}$$

or

$$(2-2\alpha_1-\alpha_2)p(u_{2n+1},u_{2n}) \le \alpha_2\, p(u_{2n-1},u_{2n}),$$

or

$$p(u_{2n+1},u_{2n}) \le \Big(\frac{\alpha_2}{2-2\alpha_1-\alpha_2}\Big)p(u_{2n-1},u_{2n}). \tag{2.12}$$

Put $\theta = \max\Big\{\frac{\alpha_2}{1-\alpha_1}, \frac{\alpha_2}{2-2\alpha_1-\alpha_2}\Big\} < 1$, since $(\alpha_1+\alpha_2) < 1$. Then from (2.12), we obtain

$$p(u_{2n+1},u_{2n}) \le \theta\, p(u_{2n-1},u_{2n}), \tag{2.13}$$

which implies

$$p(u_{n+1},u_n) \le \theta\, p(u_n,u_{n-1}). \tag{2.14}$$

Let $\mathcal{D}_n = p(u_{n+1}, u_n)$ and $\mathcal{D}_{n-1} = p(u_n, u_{n-1})$. Then from (2.14), it can be concluded that

$$\mathcal{D}_n \leq \theta\, \mathcal{D}_{n-1} \leq \theta^2\, \mathcal{D}_{n-2} \leq \cdots \leq \theta^n\, \mathcal{D}_0. \tag{2.15}$$

Therefore, since $0 \leq \theta < 1$, taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} p(u_{n+1}, u_n) = 0. \tag{2.16}$$

Now, we shall show that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$.

Thus for any $n, m \in \mathbb{N}$ with $m > n$, then we have

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{n+m-1}, u_m) \\ &\quad - p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \cdots - p(u_{n+m-1}, u_{n+m-1}) \\ &\leq \theta^n p(u_0, u_1) + \theta^{n+1} p(u_0, u_1) + \cdots + \theta^{n+m-1} p(u_0, u_1) \\ &= \theta^n [p(u_0, u_1) + \theta p(u_0, u_1) + \cdots + \theta^{m-1} p(u_0, u_1)] \\ &= \theta^n [1 + \theta + \cdots + \theta^{m-1}] \mathcal{D}_0 \\ &\leq \theta^n \Big(\frac{1 - \theta^{m-1}}{1 - \theta}\Big) \mathcal{D}_0. \end{aligned}$$

Taking the limit as $n, m \to \infty$ in the above inequality, we get $p(u_n, u_m) \to 0$, since $0 < \theta < 1$, hence $\{u_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$. Hence, by Lemma 1.1, this sequence will also Cauchy in $(\mathcal{X}, d_p)$. In addition, since $(\mathcal{X}, p)$ is complete, $(\mathcal{X}, d_p)$ is also complete. Thus there exists $v \in \mathcal{X}$ such that $u_n \to v$ as $n \to \infty$. Moreover, by Lemma 1.1,

$$p(v, v) = \lim_{n\to\infty} p(v, u_n) = \lim_{n,m\to\infty} p(u_n, u_m) = 0, \tag{2.17}$$

implies

$$\lim_{n\to\infty} d_p(v, u_n) = 0. \tag{2.18}$$

Now, we shall show that v is a common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$. Notice that due to (2.17), we have $p(v, v) = 0$. By (2.1) with $y = u_{2n}$ and $z = v$ and using (2.17), we have

$$\begin{aligned} p(u_{2n+1}, \mathcal{R}_2 v) &= p(\mathcal{R}_1 u_{2n}, \mathcal{R}_2 v) \\ &\le \alpha_1\, \Lambda_1^p(u_{2n}, v) + \alpha_2\, \Lambda_2^p(u_{2n}, v), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \Lambda_1^p(u_{2n}, v) &= \psi\Big(p(u_{2n}, \mathcal{R}_1 u_{2n}) \frac{1 + p(v, \mathcal{R}_2 v)}{1 + p(u_{2n}, v)}\Big) \\ &= \psi\Big(p(u_{2n}, u_{2n+1}) \frac{1 + p(v, \mathcal{R}_2 v)}{1 + p(u_{2n}, v)}\Big). \end{aligned} \tag{2.20}$$

Passing to limit as $n \to \infty$ and using the properties of ψ and (2.17), we obtain

$$\Lambda_1^p(u_{2n}, v) \to 0, \tag{2.21}$$

and

$$\begin{aligned} \Lambda_2^p(u_{2n}, v) = \max\Big\{&\psi(p(u_{2n}, v)), \psi(p(u_{2n}, \mathcal{R}_1 u_{2n})), \\ &\psi\Big(\frac{1}{2}[p(z, \mathcal{R}_1 u_{2n}) + p(u_{2n}, \mathcal{R}_2 v)]\Big), \\ &\psi\Big(\frac{p(u_{2n}, \mathcal{R}_1 u_{2n})[1 + p(v, \mathcal{R}_2 v)]}{1 + p(u_{2n}, v)}\Big)\Big\} \\ = \max\Big\{&\psi(p(u_{2n}, v)), \psi(p(u_{2n}, u_{2n+1})), \\ &\psi\Big(\frac{1}{2}[p(v, u_{2n+1}) + p(u_{2n}, \mathcal{R}_2 v)]\Big), \\ &\psi\Big(\frac{p(u_{2n}, u_{2n+1})[1 + p(v, \mathcal{R}_2 v)]}{1 + p(u_{2n}, v)}\Big)\Big\}. \end{aligned} \tag{2.22}$$

Passing to limit as $n \to \infty$ and using the properties of ψ and (2.17) in equation

(2.22), we obtain

$$\begin{aligned}\Lambda_2^p(u_{2n}, v) &= \max\Big\{0, 0, \psi\Big(\frac{p(v, \mathcal{R}_2 v)}{2}\Big), 0\Big\}\\ &= \psi\Big(\frac{p(v, \mathcal{R}_2 v)}{2}\Big) < \frac{p(v, \mathcal{R}_2 v)}{2} < p(v, \mathcal{R}_2 v).\end{aligned} \tag{2.23}$$

Now from equations (2.19), (2.21) and (2.23), we obtain

$$\begin{aligned}p(u_{2n+1}, \mathcal{R}_2 v) &\leq \alpha_1 .0 + \alpha_2\, p(v, \mathcal{R}_2 v)\\ &= \alpha_2\, p(v, \mathcal{R}_2 v).\end{aligned} \tag{2.24}$$

Passing to limit as $n \to \infty$, we obtain

$$\begin{aligned}p(v, \mathcal{R}_2 v) &\leq \alpha_2\, p(v, \mathcal{R}_2 v)\\ &< p(v, \mathcal{R}_2 v), \text{ since } \alpha_2 < 1,\end{aligned} \tag{2.25}$$

which is a contradiction. Hence $p(v, \mathcal{R}_2 v) = 0$, that is, $v = \mathcal{R}_2 v$. This shows that v is a fixed point of $\mathcal{R}_2$. By similar fashion, we can show that $v = \mathcal{R}_1 v$. Thus v is a common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Now, we shall show the uniqueness of common fixed point. Assume that v' is another common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R}_1 v' = v' = \mathcal{R}_2 v'$ with $v \neq v'$. Using (2.1) for $y = v$, $z = v'$ and using the properties of ψ and (2.17), we have

$$\begin{aligned}p(v, v') &= p(\mathcal{R}_1 v, \mathcal{R}_2 v')\\ &\leq \alpha_1\, \Lambda_1^p(v, v') + \alpha_2\, \Lambda_2^p(v, v'),\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}\Lambda_1^p(v, v') &= \psi\Big(p(v, \mathcal{R}_1 v)\frac{1 + p(v', \mathcal{R}_2 v')}{1 + p(v, v')}\Big)\\ &= \psi\Big(p(v, v)\frac{1 + p(v', v')}{1 + p(v, v')}\Big) = 0,\end{aligned} \tag{2.27}$$

and

$$
\begin{aligned}
\Lambda_2^p(v,v') = \max\Big\{&\psi(p(v,v')), \psi(p(v,\mathcal{R}_1 v)), \psi\Big(\frac{1}{2}[p(v',\mathcal{R}_1 v) + p(v,\mathcal{R}_2 v')]\Big), \\
&\psi\Big(\frac{p(v,\mathcal{R}_1 v)[1+p(v',\mathcal{R}_2 v')]}{1+p(v,v')}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \max\Big\{\psi(p(v,v')), \psi(p(v,v)), \psi\Big(\frac{1}{2}[p(v',v) + p(v,v')]\Big), \\
&\qquad \psi\Big(\frac{p(v,v)[1+p(v',v')]}{1+p(v,v')}\Big)\Big\} \\
&= \max\Big\{\psi(p(v,v')), 0, \psi(p(v,v')), 0\Big\} = \psi(p(v,v')). \qquad (2.28)
\end{aligned}
$$

From equations (2.26), (2.27), (2.28) and using the property of ψ, we obtain

$$
\begin{aligned}
p(v,v') &\leq \alpha_1 .0 + \alpha_2\, \psi(p(v,v')) = \alpha_2\, \psi(p(v,v')) \\
&< \alpha_2\, p(v,v') < p(v,v'), \text{ since } \alpha_2 < 1,
\end{aligned}
$$

which is a contradiction. Hence, $p(v,v') = 0$, that is, $v = v'$. Thus the common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$ is unique. This completes the proof. □

Theorem 2.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous self-maps on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$
p(\mathcal{F}_1^m y, \mathcal{F}_2^n z) \leq L_1\, \mathcal{H}_1^p(y,z) + L_2\, \mathcal{H}_2^p(y,z), \qquad (2.29)
$$

for all $y, z \in \mathcal{X}$, where m and n are some positive integers,

$$
\mathcal{H}_1^p(y,z) = \psi\Big(p(y,\mathcal{F}_1^m y)\frac{1+p(z,\mathcal{F}_2^n z)}{1+p(y,z)}\Big), \qquad (2.30)
$$

and

$$\begin{aligned}\mathcal{H}_2^p(y,z) = \max\Big\{&\psi(p(y,z)),\\ &\psi(p(y,\mathcal{F}_1^m y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{F}_1^m y) + p(y,\mathcal{F}_2^n z)]\Big),\\ &\psi\Big(\frac{p(y,\mathcal{F}_1^m y)[1+p(z,\mathcal{F}_2^n z)]}{1+p(y,z)}\Big)\Big\},\end{aligned} \tag{2.31}$$

for all $\psi \in \Psi$, and $L_1, L_2 \in [0,1)$ with $L_1 + L_2 < 1$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{X}$.

Proof. Since $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ satisfy the conditions of the Theorem 2.1. So $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ have a unique common fixed point. Let w be the common fixed point. Then we have

$$\begin{aligned}\mathcal{F}_1^m w = w &\Rightarrow \mathcal{F}_1(\mathcal{F}_1^m w) = \mathcal{F}_1 w\\ &\Rightarrow \mathcal{F}_1^m(\mathcal{F}_1 w) = \mathcal{F}_1 w.\end{aligned}$$

If $\mathcal{F}_1 w = w_0$, then $\mathcal{F}_1^m w_0 = w_0$. So $\mathcal{F}_1 w$ is a fixed point of $\mathcal{F}_1^m$. Similarly, $\mathcal{F}_2^n(\mathcal{F}_2 w) = \mathcal{F}_2 w$, that is, $\mathcal{F}_2 w$ is a fixed point of $\mathcal{F}_2^n$.
Now, using equations (2.29) and (2.17) and using the properties of ψ, we have

$$\begin{aligned}p(w,\mathcal{F}_1 w) &= p(\mathcal{F}_1^m w, \mathcal{F}_1^m(\mathcal{F}_1 w))\\ &\leq L_1\, \mathcal{H}_1^p(w,\mathcal{F}_1 w) + L_2\, \mathcal{H}_2^p(w,\mathcal{F}_1 w),\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}\mathcal{H}_1^p(w,\mathcal{F}_1 w) &= \psi\Big(p(w,\mathcal{F}_1^m w)\frac{1+p(\mathcal{F}_1 w, \mathcal{F}_2^n(\mathcal{F}_1 w))}{1+p(w,\mathcal{F}_1 w)}\Big)\\ &= \psi\Big(p(w,w)\frac{1+p(\mathcal{F}_1 w,\mathcal{F}_1 w)}{1+p(w,\mathcal{F}_1 w)}\Big)\\ &= \psi(0) = 0,\end{aligned} \tag{2.33}$$

and

$$\mathcal{H}_2^p(w, \mathcal{F}_1 w) = \max\Big\{\psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1^m(\mathcal{F}_1 w))),$$

$$\psi\Big(\frac{1}{2}[p(\mathcal{F}_1 w, \mathcal{F}_1^m w) + p(w, \mathcal{F}_2^n(\mathcal{F}_1 w))]\Big),$$

$$\begin{aligned}
&\psi\Big(\frac{p(w, \mathcal{F}_1^m w)[1 + p(\mathcal{F}_1 w, \mathcal{F}_2^n(\mathcal{F}_1 w))]}{1 + p(w, \mathcal{F}_1 w)}\Big)\Big\} \\
&= \max\Big\{\psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1 w)), \\
&\quad \psi\Big(\frac{1}{2}[p(\mathcal{F}_1 w, w) + p(w, \mathcal{F}_1 w)]\Big), \\
&\quad \psi\Big(\frac{p(w, w)[1 + p(\mathcal{F}_1 w, \mathcal{F}_1 w)]}{1 + p(w, \mathcal{F}_1 w)}\Big)\Big\} \\
&= \max\Big\{\psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1 w), 0\Big\} \\
&= \psi(p(w, \mathcal{F}_1 w)). \qquad (2.34)
\end{aligned}$$

From equations (2.32)-(2.34) and using the property of ψ, we obtain

$$\begin{aligned}
p(w, \mathcal{F}_1 w) &\leq L_1 .0 + L_2\, \psi(p(w, \mathcal{F}_1 w)) = L_2\, \psi(p(w, \mathcal{F}_1 w)) \\
&< L_2\, p(w, \mathcal{F}_1 w) < p(w, \mathcal{F}_1 w), \text{ since } L_2 < 1,
\end{aligned}$$

which is a contradiction. Hence, we deduce that $p(w, \mathcal{F}_1 w) = 0$, that is, $w = \mathcal{F}_1 w$ for all $w \in \mathcal{X}$. Similarly, we can show that $w = \mathcal{F}_2 w$. This shows that w is a common fixed point of $\mathcal{F}_1$ and $\mathcal{F}_2$. For the uniqueness of w, let $w' \neq w$ be another common fixed point of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then clearly w' is also a common fixed point of $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ which implies $w = w'$. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{X}$. This completes the proof. □

If we take $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{S}$ in Theorem 2.1, then we have the following result as corollary.

Corollary 2.1. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$*

satisfying the condition:

$$p(\mathcal{S}y,\mathcal{S}z) \leq \beta_1\, \mathcal{Q}_1^p(y,z) + \beta_2\, \mathcal{Q}_2^p(y,z), \tag{2.35}$$

for all $y,z \in \mathcal{X}$*, where*

$$\mathcal{Q}_1^p(y,z) = \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big),$$

and

$$\mathcal{Q}_2^p(y,z) = \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big), \psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\},$$

for all $\psi \in \Psi$*, where* $\beta_1, \beta_2 \in [0,1)$ *with* $\beta_1+\beta_2<1$*. Then* $\mathcal{S}$ *has a unique fixed point in* $\mathcal{X}$*.*

If we take $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{G}$ in Theorem 2.2, then we have the following result as corollary.

Corollary 2.2. *Let* $\mathcal{G}$ *be a self-map on a complete partial metric space* $(\mathcal{X},p)$ *satisfying the inequality for some positive integer* n*:*

$$p(\mathcal{G}^n y,\mathcal{G}^n z) \leq s_1\, \mathcal{M}_1^p(y,z) + s_2\, \mathcal{M}_2^p(y,z), \tag{2.36}$$

for all $y,z \in \mathcal{X}$*, where*

$$\mathcal{M}_1^p(y,z) = \psi\Big(p(y,\mathcal{G}^n y)\frac{1+p(z,\mathcal{G}^n z)}{1+p(y,z)}\Big),$$

and

$$\mathcal{M}_2^p(y,z) = \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{G}^n y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{G}^n y)+p(y,\mathcal{G}^n z)]\Big), \psi\Big(\frac{p(y,\mathcal{G}^n y)[1+p(z,\mathcal{G}^n z)]}{1+p(y,z)}\Big)\Big\},$$

for all $\psi \in \Psi$*, and* $s_1, s_2 \in [0,1)$ *with* $s_1+s_2<1$*. Then* $\mathcal{G}$ *has a unique fixed point in* $\mathcal{X}$*.*

Proof. Let $\mathcal{U} = \mathcal{G}^n$, then from (2.36), we have

$$p(\mathcal{U}y,\mathcal{U}z) \leq s_1\,\mathcal{M}_1^p(y,z) + s_2\,\mathcal{M}_2^p(y,z),$$

for all $y, z \in \mathcal{X}$, where

$$\mathcal{M}_1^p(y,z) = \psi\Big(p(y,\mathcal{U}y)\frac{1+p(z,\mathcal{U}z)}{1+p(y,z)}\Big),$$

and

$$\mathcal{M}_2^p(y,z) = \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{U}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{U}y)+p(y,\mathcal{U}z)]\Big), \psi\Big(\frac{p(y,\mathcal{U}y)[1+p(z,\mathcal{U}z)]}{1+p(y,z)}\Big)\Big\},$$

So by Corollary 2.1, $\mathcal{U}$, that is, $\mathcal{G}^n$ has a unique fixed point u_0. But $\mathcal{G}^n(\mathcal{G}u_0) = \mathcal{G}(\mathcal{G}^n u_0) = \mathcal{G}u_0$. So $\mathcal{G}u_0$ is also a fixed point of $\mathcal{G}^n$. Hence $\mathcal{G}u_0 = u_0$, i.e., u_0 is a fixed point of $\mathcal{G}$. Since the fixed point of $\mathcal{G}$ is also a fixed point of $\mathcal{G}^n$, so the fixed point of $\mathcal{G}$ is unique. This completes the proof. □

Corollary 2.3. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$. Suppose that there exists a nondecreasing function $\psi \in \Psi$ satisfying the condition:*

$$p(\mathcal{S}y,\mathcal{S}z) \leq \beta_1\,\psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big) + \beta_2\,\psi\Big(\max\Big\{p(y,z), p(y,\mathcal{S}y), \frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)], \frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big\}\Big), \tag{2.37}$$

for all $y, z \in \mathcal{X}$, where $\beta_1, \beta_2 \in [0, 1)$ with $\beta_1 + \beta_2 < 1$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Proof. It follows from Corollary 2.1 by taking that if $\psi \in \Psi$ is a nondecreasing function, we have

$$\mathcal{Q}_2^p(y,z) = \psi\Big(\max\Big\{p(y,z), p(y,\mathcal{S}y), \frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)], \frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big\}\Big).$$

□

Remark 2.1. *It is clear that the conclusions of the Corollary 2.3 remain valid if in condition (2.37), the second term of the right-hand side is replaced by one of the following terms:*

$$\beta_2\,\psi(p(y,z));\quad \beta_2\,\psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big);$$

$$\beta_2\,\max\Big\{\psi(p(y,z)),\psi(p(y,\mathcal{S}y))\Big\};$$

$$\text{or}\quad \beta_2\,\max\Big\{\psi(p(y,z)),\psi(p(y,\mathcal{S}y)),\psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big)\Big\}.$$

Corollary 2.4. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X},p)$. Suppose that there exist five positive constants a_j, $j=1,2,3,4,5$ with $\sum_{j=1}^{5} a_j < 1$ satisfying the inequality:*

$$\begin{aligned} p(\mathcal{S}y,\mathcal{S}z) \le\, & a_1\left(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\right) + a_2\,p(y,z) \\ & + a_3\,p(y,\mathcal{S}y) + a_4\,\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)] \\ & + a_5\,\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}, \end{aligned} \tag{2.38}$$

for all $y,z\in\mathcal{X}$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Proof. It follows from Corollary 2.1 with $\psi(t)=(a_1+a_2+a_3+a_4+a_5)t$. □

As a special case, we obtain partial metric space versions of Banach ([7]) and Chatterjae ([13]) fixed point results from Corollary 2.4.

Corollary 2.5. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$. Suppose that there exists $\mu \in [0,1)$ such that one of the following conditions hold:*

$$p(\mathcal{S}y, \mathcal{S}z) \leq \mu\, p(y,z),$$

$$p(\mathcal{S}y, \mathcal{S}z) \leq \frac{\mu}{2}\,[p(z, \mathcal{S}y) + p(y, \mathcal{S}z)],$$

for all $y, z \in \mathcal{X}$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Proof. It follows from Corollary 2.4 by taking (1) $a_2 = \mu$ and $a_1 = a_3 = a_4 = a_5 = 0$ and (2) $a_4 = \mu$ and $a_1 = a_2 = a_3 = a_5 = 0$. □

If we take $\beta_1 = 0$, $\beta_2 = 1$ and

$$\max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y) + p(y,\mathcal{S}z)]\Big), \psi\Big(\frac{p(y,\mathcal{S}y)[1 + p(z,\mathcal{S}z)]}{1 + p(y,z)}\Big)\Big\} = \psi(p(y,z)),$$

in Corollary 2.1, then we obtain the following result.

Corollary 2.6. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{S}y, \mathcal{S}z) \leq \psi(p(y,z)),$$

for all $y, z \in \mathcal{X}$ and $\psi \in \Psi$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

If we take $\psi(t) = k\,t$, where $0 < k < 1$ is a constant in Corollary 2.6, then we obtain the following result.

Corollary 2.7. *(see [23]) Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{S}y, \mathcal{S}z) \leq k\, p(y,z),$$

for all $y, z \in \mathcal{X}$ and $k \in [0,1)$ is a constant. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Remark 2.2. *Corollary 2.7 generalizes Banach contraction mapping principle ([7]) from complete metric space to the setting of complete partial metric space.*

Now, we give some examples in support of the result.

Example 2.1. *Let* $\mathcal{X} = \{1, 2, 3, 4\}$ *and* $p\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *be defined by*

$$p(y,z) = \begin{cases} |y-z| + \max\{y,z\}, & \text{if } y \neq z, \\ y, & \text{if } y = z \neq 1, \\ 0, & \text{if } y = z = 1, \end{cases}$$

for all $y, z \in \mathcal{X}$. *Then* $(\mathcal{X}, p)$ *is a complete partial metric space.*

Define the mapping $\mathcal{S}\colon \mathcal{X} \to \mathcal{X}$ *by*

$$\mathcal{S}(1) = 1,\ \mathcal{S}(2) = 1,\ \mathcal{S}(3) = 2,\ \mathcal{S}(4) = 2.$$

Now, we have

$$p(\mathcal{S}(1), \mathcal{S}(2)) = p(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p(1,2),$$

$$p(\mathcal{S}(1), \mathcal{S}(3)) = p(1,2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4}p(1,3),$$

$$p(\mathcal{S}(1), \mathcal{S}(4)) = p(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p(1,4),$$

$$p(\mathcal{S}(2), \mathcal{S}(3)) = p(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p(2,3),$$

$$p(\mathcal{S}(2), \mathcal{S}(4)) = p(1,2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4}p(2,4),$$

$$p(\mathcal{S}(3), \mathcal{S}(4)) = p(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p(3,4).$$

Thus, $\mathcal{S}$ *satisfies all the conditions of Corollary 2.7 with* $k = \frac{3}{4} < 1$. *Now by applying Corollary 2.7,* $\mathcal{S}$ *has a unique fixed point. Indeed* 1 *is the required unique fixed point in this case.*

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Example 2.2. *Let* $\mathcal{X} = [0, \infty)$ *and* $p\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *be defined by* $p(y,z) = \max\{y,z\}$ *for all* $y, z \in \mathcal{X}$. *Then* $(\mathcal{X}, p)$ *is a complete partial metric space. Con-*

sider the mappings $\mathcal{S}\colon \mathcal{X} \to \mathcal{X}$ *defined by*

$$\mathcal{S}(y) = \begin{cases} 0, & \textit{if } 0 \le y < 1, \\ \frac{y^2}{1+y}, & \textit{if } y \ge 1, \end{cases}$$

and $\psi\colon [0,\infty) \to [0,\infty)$ *is defined by* $\psi(t) = \frac{3t}{4}$.

We have the following cases:

Case (i) *If* $y, z \in [0,1)$ *and assume that* $y \ge z$*, then we have*

$$p(\mathcal{S}(y), \mathcal{S}(z)) = 0,$$

$$\begin{aligned} \mathcal{Q}_1^p(y,z) &= \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big) \\ &= \psi\Big(\frac{y(1+z)}{(1+y)}\Big) = \frac{3y(1+z)}{4(1+y)}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_2^p(y,z) &= \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big), \\ &\qquad \psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\} \\ &= \max\Big\{\psi(y), \psi(y), \psi(\frac{y+z}{2}), \psi\Big(\frac{y(1+z)}{1+y}\Big)\Big\} = \psi(y) = \frac{3y}{4}. \end{aligned}$$

Hence from above inequalities, we see that

$$p(\mathcal{S}(y), \mathcal{S}(z)) = 0 \le \beta_1\, \mathcal{Q}_1^p(y,z) + \beta_2\, \mathcal{Q}_2^p(y,z).$$

Thus the inequality holds.

Case (ii) *If* $z \in [0,1)$, $y \ge 1$ *and assume that* $y \ge z$*, then we have*

$$p(\mathcal{S}(y), \mathcal{S}(z)) = \max\Big\{\frac{y^2}{1+y}, 0\Big\} = \frac{y^2}{1+y},$$

and

$$\begin{aligned}\mathcal{Q}_1^p(y,z) &= \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big)\\ &= \psi\Big(\frac{y(1+z)}{(1+y)}\Big) = \frac{3y(1+z)}{4(1+y)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}_2^p(y,z) = \max\Big\{&\psi(p(y,z)),\psi(p(y,\mathcal{S}y)),\psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big),\\ &\psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\}\\ = \max\Big\{&\psi(y),\psi(y),\psi\Big(\frac{1}{2}\Big[\frac{y^2}{1+y}+y\Big]\Big),\psi\Big(\frac{y(1+z)}{1+y}\Big)\Big\} = \psi(y) = \frac{3y}{4}.\end{aligned}$$

Using contractive condition (2.35), we have

$$\frac{y^2}{1+y} \le \beta_1\left(\frac{3y(1+z)}{4(1+y)}\right) + \beta_2\left(\frac{3y}{4}\right).$$

If we take $y=1$ *and* $z=0$*, then the above inequality reduces to*

$$\frac{1}{2} \le \Big(\frac{3\beta_1}{8}\Big) + \Big(\frac{3\beta_2}{4}\Big),$$

or

$$4 \le 3\beta_1 + 6\beta_2.$$

The above inequality is satisfied for (i) $\beta_1 = \frac{1}{5}$ *and* $\beta_2 = \frac{3}{5}$, (ii) $\beta_1 = \frac{1}{5}$ *and* $\beta_2 = \frac{2}{3}$ *with* $\beta_1 + \beta_2 < 1$.

Case (iii) *If* $y \ge z \ge 1$ *and assume that* $y \ge z$*, then we have*

$$p(\mathcal{S}(y),\mathcal{S}(z)) = \max\left\{\frac{y^2}{1+y},\frac{z^2}{1+z}\right\} = \frac{y^2}{1+y},$$

and

$$\mathcal{Q}_1^p(y,z) = \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big)$$
$$= \psi\Big(\frac{y(1+z)}{(1+y)}\Big) = \frac{3y(1+z)}{4(1+y)},$$

and

$$\mathcal{Q}_2^p(y,z) = \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big), \psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\}$$
$$= \max\Big\{\psi(y), \psi(y), \psi\Big(\frac{1}{2}[z+y]\Big), \psi\Big(\frac{y(1+z)}{1+y}\Big)\Big\} = \psi(y) = \frac{3y}{4}.$$

Using contractive condition (2.35), we have

$$\frac{y^2}{1+y} \leq \beta_1\Big(\frac{3y(1+z)}{4(1+y)}\Big) + \beta_2\Big(\frac{3y}{4}\Big).$$

If we take $y = z = 1$, then the above inequality reduces to

$$\frac{1}{2} \leq \frac{3\beta_1}{4} + \frac{3\beta_2}{4},$$

or

$$2 \leq 3\beta_1 + 3\beta_2.$$

The above inequality is satisfied for (i) $\beta_1 = \frac{1}{5}$ and $\beta_2 = \frac{1}{2}$, (ii) $\beta_1 = \frac{1}{3}$ and $\beta_2 = \frac{2}{5}$ and (iii) $\beta_1 = \frac{1}{4}$ and $\beta_2 = \frac{4}{7}$ with $\beta_1 + \beta_2 < 1$. Thus, in all the above cases $\mathcal{S}$ satisfies all the conditions of Corollary 2.1. Hence, $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$, indeed, $y = 0$ is the required point.

Conclusion

In this paper, we establish some unique common fixed point theorems in the framework of complete partial metric spaces involving auxiliary function and give some consequences of the established results as corollaries. We also give some examples in support of the results. The results of findings in this paper extend and generalize several results from the existing literature regarding partial metric spaces.

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Strong commutativity preserving endomorphisms in prime rings with involution

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Abstract

Let $\mathscr{R}$ be a noncommutative prime ring with involution of the second kind and $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ be the set of symmetric and skew symmetric elements of $\mathscr{R}$. The aim of the present paper is to show that every strong commutativity preserving endomorphism on $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ is strong commutativity preserving on $\mathscr{R}$.

1. Introduction

Let $\mathscr{R}$ be a ring with centre $\mathcal{Z}(\mathscr{R})$. The symbol $[x, y] = xy - yx$ denotes the commutator of $x, y \in \mathscr{R}$. A mapping $\phi : \mathscr{R} \to \mathscr{R}$ preserves commutativity

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if $[\phi(x), \phi(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in \mathscr{R}$. The commutativity preserving maps has been studied intensively in matrix theory, operator theory and ring theory (see [5, 11]). Following [4], let $\mathcal{S}$ be a subset of $\mathscr{R}$, a map $\phi : \mathscr{R} \to \mathscr{R}$ is said to be strong commutativity preserving (SCP) on $\mathcal{S}$ if $[\phi(x), \phi(y)] = [x, y]$ for all $x, y \in \mathcal{S}$. In the course of time several techniques have been developed to investigate the behaviour of strong commutativity preserving maps using restrictions on polynomials invoking derivations, generalized derivations etcetera.
In [3], Bell and Daif investigated the commutativity in rings admitting a derivation which is strong commutativity preserving on a nonzero right ideal. More precisely, they proved that if a semiprime ring $\mathscr{R}$ admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of $\mathscr{R}$, then $I \subseteq \mathcal{Z}(\mathscr{R})$. In particular, $\mathscr{R}$ is commutative if $I = \mathscr{R}$. Later, Deng and Ashraf [8] proved that if there exists a derivation d of a semiprime ring $\mathscr{R}$ and a map $f : I \to \mathscr{R}$ defined on a nonzero ideal I of $\mathscr{R}$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then $\mathscr{R}$ contains a nonzero central ideal. Thus, $\mathscr{R}$ is commutative in the special case when $I = \mathscr{R}$. Further Al and Huang [2] showed that if $\mathscr{R}$ is a 2-torsion free semi prime ring and d is a derivation of $\mathscr{R}$ satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of $\mathscr{R}$, then $\mathscr{R}$ contains a nonzero central ideal. Many other results in this direction can be found in [1, 5–7, 9] and references therein.
Recall that a ring $\mathscr{R}$ is called $*$-ring or ring with involution if there is an additive map $* : \mathscr{R} \to \mathscr{R}$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in \mathscr{R}$. Let $\mathcal{H}(\mathscr{R}) = \{x \in \mathscr{R} | x^* = x\}$ and $\mathcal{S}(\mathscr{R}) = \{x \in \mathscr{R} | x^* = x\}$ denote the set of symmetric and skew symmetric elements of $\mathscr{R}$. The involution is said to be of the first kind if $\mathcal{Z}(\mathscr{R}) \subseteq \mathcal{H}(\mathscr{R})$, otherwise it is said to be of the second kind. In the later case, $\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$ (e.g. involution in the case of ring of quaternions).

One can observe that every strong commutativity preserving endomorphism on $\mathscr{R}$ is strong commutativity preserving on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$ but the converse is not true in general (see Example 3.1). Now if we take the statement that an endomorphism θ is strong commutativity preserving on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$, does it follow that θ is strong commutativity preserving on $\mathscr{R}$. The answer is obviously affirmative in case $\mathscr{R}$ is commutative or θ is the identity map. However some restrictions must certainly be imposed here for the answer is negative in case of noncommutative rings, if θ is not the identity map. Thus we will assume that $\mathscr{R}$ is a 2-torsion free noncommutative prime ring with involution of the second kind and $\theta \neq I$, the identity map in order that the said question makes sense.

2. Preliminary Results

In the present section, we present following facts which are very crucial for developing the proofs of our main results.

Fact 2.1. *If the involution is of the second kind, then $\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$. which indeed implies that $\mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$.*

Fact 2.2. *Let $\mathscr{R}$ be a 2-torsion free prime ring with involution of the second kind. Then every $x \in \mathscr{R}$ can uniquely be represented as $2x = h + k$, where $h \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$.*

Fact 2.3. *Let $\mathscr{R}$ be a 2-torsion free prime ring with involution of the second kind such that*
(1) If $[h, h^{'}] = 0$ for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$, then $\mathscr{R}$ is commutative.
(2) If $[k, k^{'}] = 0$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$, then $\mathscr{R}$ is commutative.

Proof. (1) Suppose that $[h, h^{'}] = 0$. Replacing h by kk_0, with $k \in \mathcal{S}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we obtain $[k, h^{'}]k_0 = 0$ for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$, which because of primeness yields that $[k, h^{'}] = 0$ for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2, we obtain $2[x, h^{'}] = [2x, h^{'}] = [h+k, h^{'}] = [h, h^{'}] + [k, h^{'}] = 0$. Hence $[x, h^{'}] = 0$ for all $x \in \mathscr{R}$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Again replacing $h^{'}$ by $k^{'}k_0$, where $k^{'} \in \mathcal{S}(\mathscr{R})$, $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we find that $[x, k^{'}] = 0$ for all $x \in R$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again making use of Fact 2.2, we finally arrive at $[x, y] = 0$ for all $x, y \in \mathscr{R}$. Hence $\mathscr{R}$ is commutative.
(2) Assume that $[k, k^{'}] = 0$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we obtain $[h, k^{'}]k_0 = 0$ for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$, which because of primeness yields that $[h, k^{'}] = 0$ for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2, we obtain $2[x, k^{'}] = [2x, k^{'}] = [h + k, k^{'}] = [h, k^{'}] + [k, k^{'}] = 0$. Hence $[x, k^{'}] = 0$ for all $x \in \mathscr{R}$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again replacing $k^{'}$ by $h^{'}k_0$, where $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we find that $[x, h^{'}] = 0$ for all $x \in R$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Again making use of Fact 2.2, we finally arrive at $[x, y] = 0$ for all $x, y \in \mathscr{R}$. Hence $\mathscr{R}$ is commutative. □

Fact 2.4. *Let $\mathscr{R}$ be a 2-torsion free noncommutative prime ring with involution of the second kind.*
(1) If $a[h,h^{'}]b = 0$ for all $h,h^{'} \in \mathcal{H}(\mathscr{R})$, $a,b \in \mathscr{R}$, then $a = 0$ or $b = 0$.
(2)If $a[k,k^{'}]b = 0$ for all $k,k^{'} \in \mathcal{S}(\mathscr{R})$, $a,b \in \mathscr{R}$, then $a = 0$ or $b = 0$.

Proof. (1) Suppose that $a[h,h^{'}]b = 0$ for all $h,h^{'} \in \mathcal{H}(\mathscr{R})$. Arguing on similar lines as in the proof of Fact 2.3, we obtain $a[x,y]b = 0$ for all $x,y \in \mathscr{R}$. Substituting yb for y, we get $ay[x,b]b = 0$ so that $a = 0$ or $b \in \mathcal{Z}(\mathscr{R})$. In the later case, our hypothesis leads to $a = 0$ or $b = 0$.
(2) Suppose that $a[k,k^{'}]b = 0$ for all $k,k^{'} \in \mathcal{S}(\mathscr{R})$. Again arguing on similar lines as in the proof of Fact 2.3, we obtain $a[x,y]b = 0$ for all $x,y \in \mathscr{R}$. Hence $a = 0$ or $b = 0$ as shown above. □

Fact 2.5. *Let $\mathscr{R}$ be a 2-torsion free prime ring. If $[[a,y],a] = 0$ for all $y \in \mathscr{R}$, then $a \in \mathcal{Z}(\mathscr{R})$.*

Proof. Let $a \in \mathscr{R}$ is such that $[a,[a,y]] = 0$ for all $y \in \mathscr{R}$. First applying $2[a,y][a,x] = [a,[a,yx]] - y[a,[a,x]] - [a,[a,y]]x$, we conclude that $[a,y][a,x] = 0$ for all $y,x \in \mathscr{R}$. Replacing x by xy in this identity and using $[a,xy] = [a,x]y + x[a,y]$, we get $[a,y]\mathscr{R}[a,y] = 0$ for all $y \in \mathscr{R}$. Thus $[a,y] = 0$ by the primeness of $\mathscr{R}$. □

3. When θ is SCP on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$

We begin this section with the following examples which show that a strong commutativity preserving endomorphism on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ need not be strong commutativity preserving on $\mathscr{R}$.

Example 3.1. *Let $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{Q} \right\}$. Of course, $\mathscr{R}$ with matrix addition and matrix multiplication is a prime ring. Define $* : \mathscr{R} \longrightarrow \mathscr{R}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. Let $\mathcal{S}(\mathscr{R})$ be the set of skew symmetric elements of $\mathscr{R}$. If $\theta : \mathscr{R} \to \mathscr{R}$ is an inner automorphism of $\mathscr{R}$ defined by $\theta(X) = PXP^{-1}$, where $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly $\theta(K) = K$ for all $K \in \mathcal{S}(\mathscr{R})$. Thus one can easily see that $[\theta(K),\theta(K^{'})] = [K,K^{'}]$*

for all $K, K^{'} \in \mathcal{S}(\mathscr{R})$. *But* $[\theta(X), \theta(Y)] \neq [X, Y]$ *for all* $X, Y \in \mathscr{R}$. *For instance if* $X = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, *It is easy to verify that* $[\theta(X), \theta(Y)] \neq [X, Y]$.

Further if we define the involution $* : \mathscr{R} \longrightarrow \mathscr{R}$ *such that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. *Then clearly* $\theta(H) = H$ *for all* $H \in \mathcal{H}(\mathscr{R})$, *symmetric elements of* $\mathscr{R}$. *Hence* $[\theta(H), \theta(H^{'})] = [H, H^{'}]$ *for all* $H, H^{'} \in \mathcal{H}(\mathscr{R})$. *But again* $[\theta(X), \theta(Y)] \neq [X, Y]$ *for all* $X, Y \in \mathscr{R}$.

Regarding the converse part, we prove the following result.

Theorem 3.1. *Let* $\mathscr{R}$ *be a 2-torsion free noncommutative prime ring with involution of the second kind. If* θ *is a nontrivial endomorphism of* $\mathscr{R}$, *then the following assertions are equivalent;*
(1) $[\theta(h), \theta(h^{'})] = [h, h^{'}]$ *for all* $h, h^{'} \in \mathcal{H}(\mathscr{R})$;
(2) $[\theta(k), \theta(k^{'})] = [k, k^{'}]$ *for all* $k, k^{'} \in \mathcal{S}(\mathscr{R})$;
(3) $[\theta(x), \theta(y)] = [x, y]$ *for all* $x, y \in \mathscr{R}$.

Proof. It is obvious that (3) implies both (1) and (2). Hence we need to prove that (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3).
(1) $\Longrightarrow$ (3) Suppose that

$$[\theta(h), \theta(h^{'})] - [h, h^{'}] = 0 \tag{3.1}$$

for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. Replacing h by hh_0, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, we obtain

$$[\theta(h), \theta(h^{'})]\theta(h_0) - [h, h^{'}]h_0 = 0 \tag{3.2}$$

for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. Right multiplying (3.1) by $\theta(h_0)$, we have

$$[\theta(h), \theta(h^{'})]\theta(h_0) - [h, h^{'}]\theta(h_0) = 0. \tag{3.3}$$

On comparing equations (3.2) and (3.3) one can easily see that

$$[h,h'](\theta(h_0)-h_0)=0 \tag{3.4}$$

for all $h,h'\in\mathcal{H}(\mathscr{R})$. Since $\theta(h_0)\in\mathcal{Z}(\theta(\mathscr{R}))$, the above equation implies that

$$[h,h']\theta(\mathscr{R})(\theta(h_0)-h_0)=0 \tag{3.5}$$

In particular

$$[h,h'][\theta(u),\theta(v)](\theta(h_0)-h_0)=0 \tag{3.6}$$

for all $h,h',u,v\in\mathcal{H}(\mathscr{R})$. Thus our hypothesis forces that

$$[h,h'][u,v](\theta(h_0)-h_0)=0 \tag{3.7}$$

for all $h,h',u,v\in\mathcal{H}(\mathscr{R})$. Applying Fact 2.4, we get either $[h,h']=0$ for all $h,h'\in\mathcal{H}(\mathscr{R})$ or $\theta(h_0)=h_0$ for all $h_0\in\mathcal{H}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$. Now $[h,h']=0$ forces $\mathscr{R}$ to be commutative in view of Fact 2.3, which leads us to contradiction.
So $\theta(h_0)=h_0$ for all $h_0\in\mathcal{H}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$; hence $\theta(k_0^2)=k_0^2$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$, therefore $(\theta(k_0)+k_0)(\theta(k_0)-k_0)=0$. This implies that $(\theta(k_0)+k_0)\theta(\mathscr{R})(\theta(k_0)-k_0)=0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$. In particular $(\theta(k_0)+k_0)[\theta(u),\theta(v)](\theta(k_0)-k_0)=0$ for all $u,v\in\mathcal{H}(\mathscr{R})$. Making use of our hypothesis, we obtain $(\theta(k_0)+k_0)[u,v](\theta(k_0)-k_0)=0$ for all $u,v\in\mathcal{H}(\mathscr{R})$. Thus invoking Fact 2.4, it follows that either $\theta(k_0)=k_0$ or $\theta(k_0)=-k_0$. Using Brauer's trick, we conclude that $\theta(k_0)=k_0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$ or $\theta(k_0)=-k_0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$.
Suppose $\theta(k_0)=-k_0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$. Replacing h by kk_0, where $k\in\mathcal{S}(\mathscr{R})$ and $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})/\{0\}$ in (3.1), we obtain $([\theta(k),\theta(h')]+[k,h'])k_0=0$. Thus primeness of $\mathscr{R}$ forces that

$$[\theta(k),\theta(h')]+[k,h']=0 \tag{3.8}$$

for all $h'\in\mathcal{H}(\mathscr{R})$ and $k\in\mathcal{S}(\mathscr{R})$. Now since for $x\in\mathscr{R}$, $x+x^*\in\mathcal{H}(\mathscr{R})$ and $x-x^*\in\mathcal{S}(\mathscr{R})$, one can easily derive from equation (3.8) that

$$[\theta(x),\theta(x^*)]+[x,x^*]=0 \tag{3.9}$$

for all $x \in \mathscr{R}$. Linearizing equation (3.9), one can find that

$$[\theta(x), \theta(y^*)] + [\theta(y), \theta(x^*)] + [x, y^*] + [y, x^*] = 0 \tag{3.10}$$

for all $x, y \in \mathscr{R}$. Substituting yk_0 for y in (3.10), where $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we have $([\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] - [x, y^*] + [y, x^*])k_0 = 0$ which leads us to

$$[\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] - [x, y^*] + [y, x^*] = 0 \tag{3.11}$$

for all $x, y \in \mathscr{R}$. Combining equation (3.10) with (3.11), we get $[\theta(x), \theta(y^*)] = [x^*, y]$, which further implies that $\theta[x, y] = [y, x]^*$ for all $x, y \in \mathscr{R}$. Replacing y by yx, we obtain $[y, x]^*\theta(x) = x^*[y, x^*]$ for all $x, y \in \mathscr{R}$. Taking $x = [r, s]$, where $r, s \in \mathscr{R}$, one can verify that $[s, r][y, [r, s]] = [y, [r, s]][r, s]$ for all $r, s, y \in \mathscr{R}$. Thus obtaining $[[r, s], y] \circ [r, s] = 0$ for all $r, s, y \in \mathscr{R}$. This further implies that $[[r, s]^2, y] = 0$ for all $r, s, y \in \mathscr{R}$ and thus $[r, s]^2 \in \mathcal{Z}(\mathscr{R})$ for all $r, s \in \mathscr{R}$. On linearizing one can see that

$$[r, s][r, t] + [r, t][r, s] \in \mathcal{Z}(\mathscr{R}) \tag{3.12}$$

for all $r, s, t \in \mathscr{R}$. If $d_r(x) = [r, x]$, then d_r is an inner derivation and $d_r(s) \circ d_r(t) \in \mathcal{Z}(\mathscr{R})$. Thus in view of [[10], Corollary 3.6], either $\mathscr{R}$ is commutative or $d_r = 0$, which again implies commutativity of $\mathscr{R}$, a contradiction.
Therefore we have $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Substituting kk_0, where $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ for h in equation (3.1), we get $[\theta(k), \theta(h')]\theta(k_0) - [k, h']k_0 = 0$. This gives $([\theta(k), \theta(h')] - [k, h'])k_0 = 0$. Using the primeness of $\mathscr{R}$, we obtain

$$[\theta(k), \theta(h')] - [k, h'] = 0 \tag{3.13}$$

for all $h' \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2 and using equations (3.1) and (3.13), we find that

$$\begin{aligned} 2([\theta(x), \theta(h')] - [x, h']) &= [\theta(2x), \theta(h')] - [2x, h'] \\ &= [\theta(h + k), \theta(h')] - [h + k, h'] \\ &= [\theta(h), \theta(h')] + [\theta(k), \theta(h')] - [h, h'] - [k, h'] \\ &= 0. \end{aligned} \tag{3.14}$$

Using 2-torsion freeness of R, we have

$$[\theta(x), \theta(h^{'})] - [x, h^{'}] = 0 \tag{3.15}$$

for all $x \in \mathscr{R}$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Again replacing $h^{'}$ by $k^{'}k_0$, where $k^{'} \in \mathcal{S}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we get

$$[\theta(x), \theta(k^{'})] - [x, k^{'}] = 0 \tag{3.16}$$

for all $x \in \mathscr{R}$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Thus invoking Fact 2.2 and using (3.15) and (3.16), one can easily derive that $[\theta(x), \theta(y)] = [x, y]$ for all $x, y \in \mathscr{R}$, as desired.

(2) $\implies$ (3) Assume that

$$[\theta(k), \theta(k^{'})] - [k, k^{'}] = 0 \tag{3.17}$$

for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$. Replacing k by kh_0, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ and proceeding on similar lines as in the first case, one can easily find that $[k, k^{'}](\theta(h_0) - h_0) = 0$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$ and $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Since $\theta(h_0) \in \mathcal{Z}(\theta(\mathscr{R}))$, the above equation implies that $[k, k^{'}]\theta(\mathscr{R})(\theta(h_0) - h_0) = 0$. In particular $[k, k^{'}][\theta(u), \theta(v)](\theta(h_0) - h_0) = 0$ for all $k, k^{'}, u, v \in \mathcal{S}(\mathscr{R})$. Thus our hypothesis forces that

$$[k, k^{'}][u, v](\theta(h_0) - h_0) = 0 \tag{3.18}$$

for all $k, k^{'}, u, v \in \mathcal{S}(\mathscr{R})$. Applying Fact 2.4, we get either $[k, k^{'}] = 0$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$ or $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Now $[k, k^{'}] = 0$ implies $\mathscr{R}$ is commutative in view of Fact 2.3, a contradiction.
So $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. This gives $\theta(k_0^2) = k_0^2$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, therefore $(\theta(k_0) + k_0)(\theta(k_0) - k_0) = 0$. This implies that $(\theta(k_0) + k_0)\theta(\mathscr{R})(\theta(k_0) - k_0) = 0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. In particular $(\theta(k_0)+k_0)[\theta(u), \theta(v)](\theta(k_0)-k_0) = 0$ for all $u, v \in \mathcal{S}(\mathscr{R})$. Making use of our hypothesis, we obtain $(\theta(k_0) + k_0)[u, v](\theta(k_0) - k_0) = 0$ for all $u, v \in \mathcal{S}(\mathscr{R})$. Thus invoking Fact 2.4 again, it follows that either $\theta(k_0) = k_0$ or $\theta(k_0) = -k_0$. Using Brauer's trick, we conclude that $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ or $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$.

If $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ in (3.17), we obtain

$$[\theta(h), \theta(k^{'})] + [h, k^{'}] = 0 \tag{3.19}$$

for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. For $x \in \mathscr{R}$, $x + x^* \in \mathcal{H}(\mathscr{R})$ and $x - x^* \in \mathcal{S}(\mathscr{R})$, therefore one can easily derive from equation (3.19) that

$$[\theta(x), \theta(x^*)] + [x, x^*] = 0 \tag{3.20}$$

for all $x \in \mathscr{R}$ which is same as equation (3.9), thus on similar lines one can get $\mathscr{R}$ is commutative, a contradiction.
Therefore, we have $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ in equation (3.17), we obtain

$$[\theta(h), \theta(k^{'})] - [h, k^{'}] = 0 \tag{3.21}$$

for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2 and making use of the equations (3.17) and (3.21), we find that

$$[\theta(x), \theta(k^{'})] - [x, k^{'}] = 0 \tag{3.22}$$

for all $x \in \mathscr{R}$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again replacing $k^{'}$ by $h^{'}k_0$, where $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ and arguing as above, one can find that

$$[\theta(x), \theta(h^{'})] - [x, h^{'}] = 0 \tag{3.23}$$

for all $x \in \mathscr{R}$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Thus in view of Fact 2.2 and equations (3.22) and (3.23), one can obtain $[\theta(x), \theta(y)] = [x, y]$ for all $x, y \in \mathscr{R}$. Thus the proof is complete. □

In view of the the above result and Theorem 1 [6] ,we have the following corollary:

Corollary 3.1. *Let $\mathscr{R}$ be a 2-torsion free noncommutative prime ring with involution of the second kind. Let $\mathcal{S}$ be the set of symmetric elements of $\mathscr{R}$. Suppose $\theta : \mathcal{S} \to \mathscr{R}$ is a nontrivial endomorphism such that $[\theta(x), \theta(y)] = [x, y]$ for all $x, y \in \mathcal{S}$, then $\theta(x) = \lambda x + \mu(x)$ where $\lambda \in \mathscr{C}$, $\lambda^2 = 1$ and μ is an additive map of $\mathscr{R}$ into $\mathscr{C}$.*

4. When θ is SSCP on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$

In [2], Ali and Huang established that if $\mathscr{R}$ is a 2-torsion free semiprime ring and d is a derivation of $\mathscr{R}$ such that $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of $\mathscr{R}$, then $\mathscr{R}$ contains a nonzero central ideal. To be more general in the class of such mappings. We call a mapping $f : \mathscr{R} \to \mathscr{R}$ strong skew-commutativity preserving (SSCP) if $[f(x), f(y)] = -[x, y]$ for all $x, y \in \mathscr{R}$. Here again one can observe that every SSCP endomorphism θ on $\mathscr{R}$ is SSCP on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$. But the converse is not true in general.

Example 4.1. *Let $\mathscr{R}$ be the ring of real quaternions. If we define $* : R \longrightarrow R$ by $(\alpha+\beta i+\gamma j+\delta k)^* = \alpha-\beta i+\gamma j+\delta k$. Let $\mathcal{S}(\mathscr{R})$ be the set of skew symmetric elements of $\mathscr{R}$. Clearly one can see that all skew symmetric elements commute with one another. Therefore if θ is any non trivial endomorphism of $\mathscr{R}$, the condition $[\theta(k), \theta(k')] = -[k, k']$ for all $k, k' \in \mathcal{S}(\mathscr{R})$ holds. However $[\theta(x), \theta(y)] \neq -[x, y]$ for all $x, y \in \mathscr{R}$, because $\mathscr{R}$ is non commutative.*

However if $*$ is the usual conjugation $(\alpha+\beta i+\gamma j+\delta k)^* = \alpha-\beta i-\gamma j-\delta k$, all symmetric elements are central and hence the property $[\theta(h), \theta(h')] = -[h, h']$ for all symmetric elements h, h' holds. However $[\theta(x), \theta(y)] \neq -[x, y]$ for all $x, y \in \mathscr{R}$,

Again one can observe that if $\mathscr{R}$ is commutative, then the converse is also true. Moreover, if $\theta = I$, the identity map, then in our case $[\theta(h), \theta(h')] = -[h, h']$ implies that $[h, h'] = 0$ for all $h, h' \in \mathcal{H}(\mathscr{R})$. That is, $\mathscr{R}$ is commutative in view of Lemma 2.3. Hence we will again assume $\mathscr{R}$ is a 2-torsion free noncommutative prime ring with involution of the second kind and $\theta \neq I$, the identity map.

Theorem 4.1. *Let $\mathscr{R}$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If θ is a nontrivial endomorphism of $\mathscr{R}$, then the following assertions are equivalent;*
(1) $[\theta(h), \theta(h')] = -[h, h']$ for all $h, h' \in \mathcal{H}(\mathscr{R})$;
(2) $[\theta(k), \theta(k')] = -[k, k']$ for all $k, k' \in \mathcal{S}(\mathscr{R})$;
(3) $[\theta(x), \theta(y)] = -[x, y]$ for all $x, y \in \mathscr{R}$.

Proof. Clearly (3) implies both (1) and (2). Hence we need to prove that (1) $\implies$ (3) and (2) $\implies$ (3).

(1) $\implies$ (3) Suppose that

$$[\theta(h), \theta(h^{'})] + [h, h^{'}] = 0 \tag{4.1}$$

for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. Replacing h by hh_0, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ and reasoning as in the case of Theorem 3.1, we obtain $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. This further implies that $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ or $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Suppose that $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing h by kk_0, where $k \in \mathcal{S}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ in (4.1), we obtain

$$[\theta(k), \theta(h^{'})] - [k, h^{'}] = 0 \tag{4.2}$$

for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Now for $x \in \mathscr{R}$, $x + x^* \in \mathcal{H}(\mathscr{R})$ and $x - x^* \in \mathcal{S}(\mathscr{R})$, therefore equation (4.2) leads us to

$$[\theta(x), \theta(x^*)] - [x, x^*] = 0 \tag{4.3}$$

for all $x \in \mathscr{R}$. Linearizing (4.3), we find that

$$[\theta(x), \theta(y^*)] + [\theta(y), \theta(x^*)] - [x, y^*] - [y, x^*] = 0 \tag{4.4}$$

for all $x, y \in \mathscr{R}$. Substituting yk_0 for y in (4.4), where $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ and using $\theta(k_0) = -k_0$, we have $([\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] + [x, y^*] - [y, x^*])k_0 = 0$ for all $x, y \in \mathscr{R}$ which proves that

$$[\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] + [x, y^*] - [y, x^*] = 0 \tag{4.5}$$

for all $x, y \in \mathscr{R}$. Comparing equations (4.4) and (4.5), it follows that $[\theta(x), \theta(y)] - [y^*, x^*] = 0$. Hence $\theta[x, y] = [x, y]^*$ for all $x, y \in \mathscr{R}$. Replacing y by yx in the last expression and using it, one can find that $[x, y]^*\theta(x) = x^*[x, y]^*$ for all $x, y \in \mathscr{R}$. Taking $x = [r, s]$, $r, s \in \mathscr{R}$, we get $[r, s][[r, s], y] = [[r, s], y][r, s]$ for all $r, s, y \in \mathscr{R}$. This implies that $[[[r, s], y], [r, s]] = 0$ for all $r, s, y \in \mathscr{R}$. Applying Fact 2.5, we get $[r, s] \in \mathcal{Z}(\mathscr{R})$ for all $r, s \in \mathscr{R}$, therefore $\mathscr{R}$ is commutative, a contradiction.
So $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Following the similar steps as in the proof Theorem 3.1, One finally obtains

$$[\theta(x), \theta(y)] = -[x, y] \tag{4.6}$$

for all $x, y \in \mathscr{R}$, as desired.
(2) $\implies$ (3) Suppose that

$$[\theta(k), \theta(k^{'})] + [k, k^{'}] = 0 \tag{4.7}$$

for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$. Taking $k = kh_0$, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ and arguing as in Theorem 3.1, one obtains $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, since $\mathscr{R}$ is noncommutative.
Again following the proof of Theorem 3.1, one can easily show that $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ or $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Assume that $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ in (4.7), we obtain

$$[\theta(h), \theta(k^{'})] - [h, k] = 0 \tag{4.8}$$

for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again taking $h = x + x^{'}$ and $k^{'} = x - x^{'}$, where $x \in \mathscr{R}$, we get

$$[\theta(x), \theta(x^*)] - [x, x^*] = 0 \tag{4.9}$$

for all $x \in \mathscr{R}$. which is same as equation (4.3). Thus arguing on similar lines, one obtain $\mathscr{R}$ is commutative, a contradiction. Now assume $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Thus following the same steps as in Theorem 3.1, one can easily derive that

$$[\theta(x), \theta(y)] = -[x, y] \tag{4.10}$$

for all $x, y \in \mathscr{R}$, thereby completing the proof of the theorem.

□

We end our paper by providing an example which shows that the said question does not hold in case θ is simply an additive map. Hence we conclude that for the said question to hold, θ needs to be of some special type such as endomorphism in our case.

Example 4.2. *Let* $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{C} \right\}$. *Define* $* : R \longrightarrow R$ *such that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$. *Clearly* $*$ *is of the second kind. Let*

$\mathcal{H}(\mathscr{R})$ be the set of symmetric elements of $\mathscr{R}$. If $\theta : \mathscr{R} \to \mathscr{R}$ is an additive map of $\mathscr{R}$ defined by $\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \bar{b} \\ c & d \end{pmatrix}$. Then one can see that $\theta(H) = H$ for all $H \in \mathcal{H}(\mathscr{R})$ and hence the condition $[\theta(H), \theta(H^{'})] = [H, H^{'}]$ holds for all $H, H^{'} \in \mathcal{H}(\mathscr{R})$. However $[\theta(X), \theta(Y)] \neq [X, Y]$ for all $X, Y \in \mathscr{R}$.

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POLOIDS AND MATRICES

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Abstract

This study is about the poloids that are obtained by the formation of a new algebraic structure obtained by adding the condition (G4) obtained from the solution of the equation $XA = B$ and $B|A$ to the definition of monoid. The (G4) property is based on the factorial property of a noncommutative matrix. The (G4) property is based on the factorial property of a noncommutative matrix. Divisibility in matrices contributes to the existence of common factors of a matrix. This necessitates the distinguishing feature in data in theoretical and applied computer sciences. For example, it paves the way for detecting the truth of lying in the syntax of the person who is lying.

1 Introduction

Here is a brief history of the monoid. The name "monoid" was first used in mathematics by Arthur Cayley for a surface of order n which has a multiple point of order $n-1$.

In the context of semigroups the name is due to Bourbaki.

It is also worth commenting on the related term monoid, meaning an associative magma with identity. This term is a little more recent than semigroup, and seems to originate with Bourbaki. Before this, Birkhoff (1934) was using the term groupoid

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for an associative magma with identity. More precisely Bourbaki (1942, p. 7): A set endowed with the structure determined by an associative law every where defined takes the name of monoid. Perhaps this was motivated by Eilenberg & Mac Lanes upcoming A monoid is a category with one object? (They started categories around 1942.) [2, 3].

The monoid, which briefly forms the algebraic structures of mathematics defined by a binary operation, is the basis for the study of monoids, automata theory (Krohn-Rhodes theory) and formal language theory (star height problem) in theoretical computer science.
The definition of poloid, defined by this binary operation, which includes a monoid, was discovered during work on factoring a matrix. It is defined by us, considering that it will make a wider contribution to theoretical computer science and formal language theory. The (G4) condition added to a monoid definition preserves the algebraic structure being applied in computer science, and also offers new unobservable paths and alternative options.

Let us start with the row co-divisor definition that I gave in the study in 2022.

Here F is a field and $M_n(F) = \left\{ [a_{ij}]_n \middle| a_{ij} \in F, n \in \mathbb{Z}^+ \right\}$ is the set of regular matrices. The transpose of $A \in M_n(F)$ is denoted by A^T.

Let A and B be two regular square matrices of order n. The determinant of the new matrix obtained by writing the i^{th} row of the matrix A on the j^{th} row of the matrix B is called the *co-divisor by row* of the matrix A by the row on the matrix B. It denoted by $\underset{ij}{AB}$. Their number is n^2. The matrix co-divisor by row is $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$ [9].

Example 1.1. *Let* $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$ *be regular matrices. Matrix of co-divisors by row of matrix* A *on matrix* B *is* $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$.

$$\underset{11}{AB} = \begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} = -5, \underset{12}{AB} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5. \underset{21}{AB} = \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} = -6, \underset{22}{AB} = \begin{vmatrix} 2 & 1 \\ 2 & 5 \end{vmatrix} = 8.$$

$$\left[\left(\underset{ij}{BA} \right)_{ij} \right] = \begin{bmatrix} -5 & 5 \\ -6 & 8 \end{bmatrix}$$

Likewise, the matrix of rows co-dividing matrix B over matrix A is a matrix $\left[\left(\underset{ij}{BA} \right)_{ij} \right]$.

$$\underset{11}{BA} = \begin{vmatrix} 2 & 1 \\ 2 & 5 \end{vmatrix} = 8, \underset{12}{BA} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5. \underset{21}{BA} = \begin{vmatrix} 4 & 7 \\ 2 & 5 \end{vmatrix} = 6, \ \underset{22}{BA} \begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} = -5.$$

$$\left[\left(\underset{ij}{AB} \right)_{IJ} \right] = \begin{bmatrix} 8 & -5 \\ 6 & -5 \end{bmatrix}.$$

For the two matrices satisfying the above conditions, the matrix division is also given by $\frac{A}{B} := \frac{1}{|B|} \left[\left(\frac{A_i}{B_j} \right)_{ji} \right]$ and at the same time, the solution of the equation $AX = B$ is $X = \frac{B}{A}[4, 7, 6, 5, 8]$.

Volodymyr P. Shchedryk gave the following proved theorem in [12]. It is been determined that this theorem has to do with column division. The proof is given in my study called " Different Approaches on the Matrix Division and Generalization of Cramers Rule" in 2017 [5].

Lemma 1.1. *Let $A, B \in M_n(F)$. If $B|A$, then it is $A|B$.*

Proof. For all $A, B \in M_n(F)$, If $B|A$ then,

$$B|A \Leftrightarrow \exists T \in M_n(F) : A = BT$$
$$A = BT \Leftrightarrow A \left(\frac{I_n}{T} \right) = B \Leftrightarrow A|B.$$

□

Theorem 1.1. *Let R be a commutative elementary divisor domain. If $BX = A$ is a solvable matrix equation over R, where $A, B, X \in M_n(F)$ then a left g.c.d. and* *a left l.c.m. of solutions of this equation are also solutions of $BX = A$ [10].*

The solution of the equation $BX = A$ is $X = [x_{ij}] = \left[\frac{\binom{A_i}{B_j}_{ji}}{|B|}\right]$ in terms of column co-divisors and $X = \frac{A}{B}$ according to the division operation [5,8,9].

Lemma 1.2. *Let $A \in M_n(F)$.Then,*

$$\left(\frac{I_n}{A}\right)^T = \frac{I_n}{A^T}.$$

Proof. Let a regular matrix $A = [a_{ij}]_n$ be given.

$$\frac{I_n}{A^T}.A^T = I_n \wedge A^T.\frac{I_n}{A^T} = I_n$$

$$\frac{I_n}{A^T} = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \left(\frac{I_n}{A}\right)^T$$

$$\frac{I_n}{A^T} = \left(\frac{I}{A}\right)^T.$$

The following lemma is given which simply explains the relationship between the row co-divisors matrix and the transpose. □

Lemma 1.3. *Let $A, B \in M_n(F)$. Then,*

$$\frac{1}{|A|}\left[\left(\underset{ij}{BA}\right)_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T.$$

Proof. For all $A, B \in M_n(F)$ then $\underset{ij}{BA} = \frac{B^T}{A^T}ij$. Because, the row co-divisors of matrix B on matrix A are the same as the column co-divisors of matrix B^T on matrix A^T.

$$\left[\left(\underset{ij}{BA}\right)\right] = \left[\left(\tfrac{B^T}{A^T}ij\right)_{ij}\right]$$

$$\frac{1}{|A|}\left[\left(\underset{ij}{BA}\right)_{ij}\right] = \frac{1}{|A^T|}\left[\left(\tfrac{B^T}{A^T}ij\right)\right]^T = \left(\frac{B^T}{A^T}\right)^T.$$

□

Proposition 1.1. *Let* $A, B \in M_n(F)$. *Then, the solution of the linear matrix equation* $XA = B$

$$X = \left(\frac{B^T}{A^T}\right)^T .$$

Proof. The solution of the equation $AX = B$ is $X = \frac{B}{A}$, for all $A, B \in M_n(F)$. Then

$$XA = B \Leftrightarrow (XA)^T = B^T \Leftrightarrow A^T X^T = B^T$$

$$X^T = \frac{1}{|A^T|}\left[\left(B^T \underset{ij}{} A^T\right) ij\right] \Rightarrow X = \frac{1}{|A_T|}\left[\left(B^T \underset{ij}{} A^T\right)_{ji}\right]^T$$

$$X = \frac{1}{|A^T|}\left[\left(B^T \underset{ij}{} A^T\right)_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T .$$

□

Due to the properties as given is [8], the following Proposition regarding the solution of this equation is obtained.

Proposition 1.2. *Let* $A, B \in M_n(F)$. *If the factors of matrix* A *is* BA_1 *and the factors of matrix* B *is* AB_1 *then*

(i) The rational matrix $\frac{A}{B}$ *is equal to matrix* A_1.

(ii) The rational matrix $\frac{A}{B}$ *is equal to matrix* $\frac{I_n}{B_1}$.

Proof. (i) The matrix A is written in terms of B as $A = BA_1$.

$$\frac{A}{B} = \frac{BA_1}{B} = A_1.$$

(ii) The matrix B is written in terms of A as $B = AB_1$.

$$\frac{A}{B} = \frac{A}{AB_1} = \frac{I_n}{B_1}.$$

□

Theorem 1.2. *Let* $A, B, X \in M_n(F)$ *and* X *unknowns matrix. Then, in the solution of the equation* $AX = B$, *there are regular matrices* $A = B_2A_3$, $B =$

B_2B_3, such as B_2, A_3 and B_3, and the rational matrix $\frac{B_3}{A_3}$ is the solution of the equation $AX = B$. This solution is equal to the rational matrix $\frac{B}{A}$.

Proof. Since the solution of $Ax = B$ is the rational matrix $\frac{B}{A}$, where any factor of matrix B is matrix B_2

$$B = B_2B_3,$$

Likewise, matrix A in terms of this B_2 matrix multiplier.
It can be written as

$$A = B_2A_3$$

Therefore

$$X = \frac{B}{A} = \frac{B_3}{A_3}.$$

□

Example 1.2. *Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$ be two matrices in $M_2(F)$, the solution of the equation $AX = B$ is $X = \frac{B}{A}$. If $B_2 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ is selected*

$$B = B_2 \underbrace{\begin{bmatrix} -\frac{8}{3} & -\frac{19}{3} \\ \frac{10}{3} & \frac{20}{3} \end{bmatrix}}_{B_3}$$

Likewise,

$$A = B_2 \underbrace{\begin{bmatrix} -\frac{4}{3} & -3 \\ \frac{5}{3} & 4 \end{bmatrix}}_{A_3}$$

Then we obtain,

$$X = \frac{B}{A} = \frac{B_2B_3}{B_2A_3} = I_3\frac{B_3}{A_3} = \frac{B_3}{A_3} = \begin{bmatrix} 2 & 16 \\ 0 & -5 \end{bmatrix}$$

2 Matrix Poloids

Let's start this section with the following definition.

Definition 2.1. *A group is a set G equipped with a binary operation $\cdot : G \times G \to G$ that associates an element $a.b \in G$ to every pair of elements $a, b \in G$, and having the following properties: is associative, has an identity element $e \in G$, and every element in G is invertible (w.r.t.). More explicitly, this means that the following equations hold for all $a, b, c \in G$:*

(G1) $a.(b.c) = (a.b).c$. *(associativity);*

(G2) $a.e = e.a = a$ *(identity);*

(G3) For every $a \in G$, there is some $a^{-1} \in G$ such that $a.a^{-1} = a^{-1}.a = e$ (inverse)[11].

A set M together with an operation $: \cdot M \times M \to M$ and an element e satisfying only Conditions (G1) and (G2) is called a monoid [1].

Noticed that if the conditions for G1 and G2 are met on the multiplication operation in matrices. So let's briefly examine whether it is a monoid or not. The set of $M_n(F)$-square matrices satisfies the conditions (G1) and (G2), However, the following example has $A = BA_1$, and $A = A_1C$ whereas $B \neq C$.

Example 2.1. *We have matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ written as follows:*

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A,$$

And it is also,

$$\underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix}}_{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A.$$

Here, neither $B = C$ nor $A_1 = I_n$ nor $A_1 = A$.

To further explore this expression, the following new definition is given.

Definition 2.2. *A group is a set G equipped with a binary operation $* : G \times G \to G$ that associates an element $a.b \in G$ to every pair of elements $a, b \in G$, and having* *the following properties: is associative, has an identity element* $e \in G$ *, and every element in G is invertible (w.r.t.* $*$ *). More explicitly, this means that the following equations hold for all* $a, b, c, d, e \in G$*:*

(G1) $a * (b * c) = (a * b) * c$*. (associativity)*

(G2) $a * e = e * a$*. (identity);*

(G3) For every $a \in G$*, there is some* $a^{-1} \in G$ *such that* $a * a^{-1} = a^{-1} * a = e$*. (inverse)*

(G4) For every $a \in G$*, there some* $d, f \in G$ *such that* $b * f = f * d = a$ *with* $b \neq d$*. (escort).*

A set M *together with an operation* $* : G \times G \to G$ *and an element* e *satisfying only Conditions (G1), (G2), (G3) and (G4) is called a poloid. It is denoted by* $(G, *)$*.*

Example 2.2. *For example, the set* $M_n(\mathbb{R})$ *of square matrices is poloid under multiplication. But, the set of real numbers* $\mathbb{R}$ *is not poloid by multiplication. Let us take the real number 2.*

$$2 = \frac{1}{3}.6 \wedge 2 = \frac{12}{2}.\frac{1}{2}$$

Here although

$$6 = \frac{12}{2}$$

The condition (G4) is not satisfied. Therefore, every poloid is also a monoid. The converse of the statement is not always true.

The set $M_n(F)$ *is poloid when the multiplication operation in the matrices is considered.*

Lemma 2.1. *Let* $M_n(F)$ *be a poloid. For all* $A_1 \in M_n(F)$ *then,*

(i) There are $A, C \in M_n(F)$ *regular matrices such that* $A_1 = \frac{A}{C}$*.*

(ii) There are $A, C \in M_n(F)$ *regular matrices such that* $A_1 = \frac{1}{|C|} \left[\left(\underset{ij}{AC} \right)_{ij} \right]$*.*

Proof. The proofs of (i) and (ii) are easily obtained from Lemma 2. □

Theorem 2.1. *Let $M_n(F)$ be a poloid. Then, there are matrices $A, C \in M_n(F)$ such that $A_1 = \frac{1}{|C|}\left[\begin{pmatrix} AC \\ ij \end{pmatrix}_{ij}\right] = \left(\frac{A^T}{C^T}\right)$, for all $A_1 \in M_n(F)$.*

Proof. The proof of the theorem 5 is easily obtained from Lemma 2, Lemma 3 and (G4). □

Theorem 2.2. *Let $M_n(F)$ be a poloid. Then, there are matrices $B, C \in M_n(F)$ such that satisfying the equation $A = BAC$, for all $A \in M_n(F)$.*

Proof. For all $A \in M_n(F)$, there are $S, R, A_1 \in M_n(F)$ such that $A = SA_1 = A_1R$ from (G4)

$$C|S \Rightarrow S = CS_1, \text{ where } S_1 \in M_n(F).$$

$$A = SA_1 = CS_1A_1,$$

$$A|S_1 \Rightarrow S_1 = AS_2, \text{ where } S_2 \in M_n(F)$$

$$A = SA_1 = CAS_2A_1, B := S_2A_1$$

$$A = SA_1 = CAB.$$

$$C|A \Leftrightarrow A_1 = CC_1, \text{ where } C_1 \in M_n(F)$$

$$A = A_1R = CC_1R \wedge A|C_1 \Leftrightarrow C_1 = AC_2, \text{ where } C_2 \in M_n(F)$$

$$A = A_1R = CAC_2R, B' = C_2R.$$

□

We want to prove that $B = B'$. Assume that B is not equal to B'. Then the fact that the B matrix is different in the B' matrix contradicts the (G4) condition.

Theorem 2.3. *Let $M_n(F)$ be a poloid. Then, there are matrices $K, L \in M_n(F)$ such that satisfying the equations $A = AKC = BLA$, for all $A \in M_n(F)$.*

Proof. It is clear if A is the unit matrix and the zero matrix. Since (G4) is provided

$$A_1 = AK \Rightarrow KC = I_n \Rightarrow A_1C = AKC = A,$$

and because of Lemma 4, $A_1 = LA \Rightarrow L = \frac{1}{|A|}\left[\left(\underset{ij}{A_1 A}\right)_{ij}\right]$ [6].

$$BL = I_n \Rightarrow BA_1 = BLA = A.$$

□

Example 2.3. *We have the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *written as follows:*

$$\underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix}}_{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix} = I_2$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -\frac{7}{3} & -4 \\ \frac{11}{6} & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 9 & 12 \\ -\frac{11}{2} & -7 \end{bmatrix}}_{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

And it is also,

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \frac{4}{3} & 2 \\ \frac{1}{3} & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$A_1 = L \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$L = \frac{1}{|A|}\left[\left(\underset{ij}{A_1 A}\right)_{ij}\right] = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = I_2$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A.$$

Conclusions and Discussions

It is obvious that the concept of "poloid", which has just been defined as our knowledge, will find many application areas. The existence of an algebraic structure that manifests itself when any element is processed from the right and left is still an open problem.

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On the graded $^*-$rings

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Abstract

A prime ring A is called a $*-$ring if $\beta(A/I) = A/I$ for every nonzero ideal proper I of A, where β is the prime radical. Gardner in 1988 asked whether the upper radical $\mathcal{U}(*_k)$ of the essential closure $*_k$ of the class of all $*-$rings coincide with the prime radical β. Until now, this problem remains open. In this paper, we construct a graded $*-$ring that motivates a further research to bring the Gardner problem into a graded version.

1 Introduction

In this paper, R (respectively, A) will denote a ring which has identity (respectively, a ring which is not necessary to have identity). A nonzero ideal of J of a ring R

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is essential if $I \cap J \neq 0$ for every nonzero ideal I of R and it will be denoted by $I \lhd \circ R$.

A class γ of rings is called a radical class in the sense of Amitsur and Kurosh if γ satisfies the following property [7]:

1. $A \in \gamma \Rightarrow \forall A \to B \neq 0$, there exists an ideal C of B such that $0 \neq C \in \gamma$.
2. A is arbitrary ring and $\forall A \to B \neq 0$, there exists an ideal C of B such that $0 \neq C \in \gamma \Rightarrow A \in \gamma$.

Example 1.1. *The following examples are radical class of rings:*

1. *The class $\mathcal{N} = \{A \mid \forall a \in A$, then there exists a positive integer n such that $a^n = 0\}$ of all nil rings is a radical class.*
2. *The class $\mathcal{J} = \{A$ is a ring $|(A, \circ)$ forms a group, where $a \circ b = a + b - ab, \forall\, a, b \in A\}$ forms a radical class.*
3. *Let π be the class of all prime rings. The class $\mathcal{U}(\pi) = \{A|A$ has no nonzero homomorphic image is $\pi\}$ is a radical class. It is famously named the prime radical class and it is denoted by β.*

For any radical class γ. The symbol $\gamma(A)$ represents the largest ideal of a ring A which belongs to γ. A prime ring A is called a $*-$ring if $\beta(A/I) = A/I$ for every nonzero ideal proper I of A, where β is the prime radical. The class of all $*-$ring is denoted by $*$ [2]. The definition of $*-$ring was introduced in [11] and some further properties of $*-$rings and their implementation in radical theory can be seen in [3, 4, 5, 6].

Example 1.2. *Consider the following examples of $*-$rings.*

1. *Every field is a $*-$ring.*
2. *The set $J = \{j \in \mathbb{Q}|j$ has even numerator and odd denumerator$\}$ forms a $*-$ring.*
3. *Let W be a simple idempotent ring of characteristic 0, but with no unity. The ring W is also a $*-$ring.*
4. *Every simple ring with unity is a $*-$ring.*

On the other hand, we have also the following conditions (S1 and S2) which motivate the existence of the definition of semisimple class of rings

1. (S1) If $A \in \sigma$, then for every nonzero $B \lhd A$ there exists a nonzero homomorphic image C of B such that $C \in \sigma$.

2. (S2) If A is a ring of the universal class $\mathbb{A}$ and for every nonzero $B \lhd A$ there exists a nonzero homomorphic image C of B such that $C \in \sigma$, then A is in σ.

A class σ of rings which satisfies the condition S1 is called a regular class of rings. Furthermore, if the class σ of rings satisfies S1 and S2, σ is called a semisimple class of rings [7].

The relationship between radical class and semisimple class is described in the Proposition 1.1 below.

Proposition 1.1. *[7] Let γ be a radical class of rings. The class $S\gamma = \{A$ is a ring $|\gamma(A) = 0\}$ is the semisimple class of γ.*

Proof. Please see the Proof of Proposition 2.3.2 in [7]. □

Furthermore, for any regular class ϱ of rings. We have the following property:

Theorem 1.1. *[7] If ϱ is a regular class of rings, then the class $\mathcal{U}(\varrho) = \{A|A$ has no nonzero homomorphic image in $\varrho\}$ is a radical class, $\mathcal{U}(\varrho) \cap \varrho = \{0\}$ and $\mathcal{U}(\varrho)$ is the largest radical having zero intersection with ϱ.*

Proof. Please see the proof of Theorem 2.2.3 in [7]. □

A class μ of prime rings is called a special class of rings if μ satisfies the following properties [7]:

1. $A \in \mu \Rightarrow I \in \mu$ for every ideal I of A.

2. for every essential ideal J of R such that $I \in \mu$ implies $R \in \mu$.

Example 1.3. *The class π of all prime rings is a special class. Moreover, the upper radical $\mathcal{U}(\pi)$ is precisely the prime radical β.*

In 1988, Gardner in his paper [8] asked whether the prime radical β coincides with the upper radical $\mathcal{U}(*_k)$, where $\mathcal{U}(*_k)$ will be explained more detail in the Section 2. However, this question remains open in general. Hence, it is important to investigate this problem at least on a specific condition. In this research, we give an insight of the problem specifically on the restricted graded version for radical class of rings.

2 The structure of $'*'$

We start this section revisiting the properties and the structure of $*$.

Remark 2.1. *[2] The class $*$ is neither a radical class nor a semisimple class.*

Consider the following counter examples.

Example 2.1. *Let W be a simple idempotent ring of characteristic 0, but with no unity. It is clear that the ring J in the Example 1.2 is a $*-$ring. The ring W is also a $*-$ring. Define $R = \{(j, w)|j \in J, w \in W\}$ with $(j, w)+(k, x) = (j+k, w+x)$ and $(j, w)(k, x) = (jk, jx + kw + wx)$ for all $j, k \in J$ and $w, x \in W$. Then, R is a subdirectly irreducible ring with the heart $H(R) \cong W$ such that $R/H(R) \cong J$ and any Noetherian homomorphic image of R (other then the identical one and $R/H(R)$ is nilpotent. Therefore R is not a $*-$ring in spite of the fact that both $R/H(R) \cong J$ and $H(R)$ are a $*-$rings.*

Remark 2.2. *The existence of a simple idempotent ring of characteristic 0, but with no unity can be seen in [10] and [14].*

Example 2.2. *Let $M_\infty(R)$ be the ring of all infinite matrices which has infinite row over a ring R. In the other words, every matrix in $M_\infty(R)$ has countably infinite number of rows but almost all entries in each row are equal to 0. If R is a simple ring, then so $M_\infty(R)$ and, clearly, the center of $M_\infty(R)$ is $\{0\}$. Therefore, $M_\infty(R)$ does not contain the identity element. So, in particular, if R is any simple ring with characteristic 0, then so is $M_\infty(R)$.*

Remark 2.3. *[2] The class $*$ is not essentially closed. Therefore, the class $*$ is not a special class.*

Example 2.3. *Example 2.1 shows that the ring $R = \{(j, w)|j \in J, w \in W\}$, where $J = \{j \in \mathbb{Q}|j$ has even numerator and odd dinominator$\}$ and W is a simple idempotent ring of characteristic 0 without unity, is a subdirectly irreducible. The heart $H(R)$ of R is $W' \cong W$. So, we may deduce that the ideal W' is an essential ideal of R and R is an essential extension of W'. In fact, $W' \cong W$ is a $*-$ring. However, the ring R is not a $*-$ring by Example 2.1. Therefore, the class $*$ is not a special class.*

Let δ be any class of rings which is not essentially closed. The class $\varepsilon(\delta) = \{A|\exists B \in \delta$ such that $B \lhd \circ A\}$ is the essential cover of δ, where $B \lhd \circ A$ expresses

B is the essential ideal of A. The definition of an essential ideal is previously described in the beginning Section 1. But, the class $\varepsilon(\delta)$ is not generally closed under essential extensions. This motivates the existence of essential closure of δ.

Definition 2.1. *[9] Let $\delta^{(0)} = \delta$ and $\delta^{(i+1)} = \varepsilon(\delta^{(i)})$. The class $\delta_k = \bigcup_{i=0}^{\infty} \delta^{(i)}$ is the essential closure of δ.*

The essential closure of class of rings is always essentially closed.

Proposition 2.1. *[9] Let δ be any class of rings which is not essentially closed. The essential closure $\delta_k = \bigcup_{i=0}^{\infty} \delta^{(i)}$ of δ is essentially closed*

Proposition 2.2. *[2] Let $*$ be the class of all $*-$rings. The class $*_k = \bigcup_{i=0}^{\infty} *^{(i)}$ is the essential closure of $*$ and $*_k$ is a special class of rings.*

Definition 2.2. *[7] The upper radical $\mathcal{U}(\mu)$ of a special class μ of rings is called a special radical.*

In fact, we have both the prime radical β and the upper radical $\mathcal{U}(*_k)$ are special radical classes, where $\mathcal{U}(*_k)$ is the upper radical of the essential closure $*_k$ of the class of all $*-$rings. Hence, it is important to scrutinize the properties of $\mathcal{U}(*_k)$ and the structure of $*-$ring especially for restricted graded version.

3 Graded $*$-rings

Since the question which asked whether $\beta = \mathcal{U}(*_k)$ is still open, we will be trying to bring this open problem to the graded version. We start from the following definition.

Let $\mathbb{Z}$ be the set of all integers. A ring R is called a $\mathbb{Z}-$graded if there is a family of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R which satisfies

1. $R = \oplus_n R_n$, and
2. $R_n R_m \subseteq R_{n+m}, \forall n, m \in \mathbb{Z}$.

In general, we have the following definition.

Definition 3.1. *[1] Let G be an additive group. A ring R is called a $G-$graded ring if $R = \oplus_{g\in G} R_g$ where the set $\{R_g | g \in G\}$ is the additive subgroups of R such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.*

Example 3.1. *For every ring A. The polynomial ring $A(x)$ over A is a graded ring by its degree.*

Further implementation of graded rings in radical theory motivates the restricted graded radical class. In specific case, restricted graded Jacobson radical class can be explored in [13].

Definition 3.2. *A ring A is called a graded $*-$ring if $A = \oplus_{g\in G} A_g$, where the set $\{A_g | g \in G\}$ is the additive subgroups of A such that $A_g R_h \subseteq A_{gh}$ for all $g, h \in G$ and it is a $*-$ring or simply A is a graded ring which is a $*-$ring.*

Remark 3.1. *It is clear that every polynomial ring $R(x)$ over a ring R can be seen as a graded ring by its degree. Furthermore, if R is a semiprime ring, then so is $R(x)$. Now, a natural question asks whether the polynomial ring $R(x)$ over $*-$ring R is also a $*-$ring?.*

The answer of the question described in the Remark 3.1 will be explained in the following proposition.

Proposition 3.1. *Let R be a $*-$ring with unity. The polynomial ring $R(x)$ is not a $*-$ring.*

Proof. Let R be a ring with unity which is $*-$ring. Then it follows from the definition of $*-$ring that R is a prime ring and $\beta(R/I) = R/I$ for every nonzero ideal proper I of R. Now, suppose $R(x)$ is the polynomial ring over R. It follows from [12] that $R(x)/\langle x \rangle \cong R$ which implies that $\beta(R(x)/\langle x \rangle) = 0$ since, R is a prime ring. Thus, $R(x)$ is not a $*-$ring. □

So, we do not have any natural graded $*-$rings constructed by polynomial rings.

In this part, we provide a graded $*-$ring.

Theorem 3.1. *Let $\mathbb{R}$ be set of all real numbers. Then the ring $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices over $\mathbb{R}$ is a graded $*-$ring.*

Proof. Let $\mathbb{R}$ be set of all real numbers and let δ be the class of all simple rings with unity. Then $R \in \delta$. It follows from Example 3.8.14 in [7] that δ is a special class of rings which implies δ consists of prime rings. Thus, the ring R is a prime ring. It is clear that R is a $*-$ring since R is simple. Furthermore, it follows from Example 11.2.15 in [12] that the ring $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices over $\mathbb{R}$ is a simple ring with unity. Thus $M_{2\times 2}(\mathbb{R}) \in \delta$. Hence, $M_{2\times 2}(\mathbb{R})$ is a $*-$ring. □

In fact, the speciality of the class π of all prime rings implies that the essential closure π_k of π is the class π itself since π is closed under essential extension. A radical class γ is said to have the Amitsur property if $\gamma(A(x)) = (\gamma(A))(x)$ for every ring A [5]. Furthermore, it follows from the Amitsur property of β that if A is a prime ring, then the polynomial $A(x) \in S\beta$, where $S\beta$ is the semisimple class of the prime radical β. We continue to compare with property of the class $*$ of all $*-$rings.

Proposition 3.2. *The class $*$ of all $*-$rings is properly contained in the semisimple class $S\mathcal{U}(*_k)$ of $\mathcal{U}(*_k)$.*

Proof. It is clear that $*$ is contained in $*_k$. Now let R be any $*-$ring. This means that $R \in *$. It follows from Proposition 3.1 that the polynomial ring $R(x)$ is not a $*-$ring. Thus, $R(x) \notin *$. We will show that $R(x) \in S\mathcal{U}(*_k)$.

Since $R \in *, R \in *_k$. It follows from Corollary 13 in [5] that $R(x) \in S\mathcal{U}(*_k)$. This implies $*$ is properly contained in $S\mathcal{U}(*_k)$ which ends the proof. □

Now, we shall follow the construction of the graded radical in $G-$category introduced by [1]. Let G be any arbitrary group and let A be a $G-$graded ring. The graded prime radical of A is denoted by $\beta_G(A)$ and it is defined as the intersection of all the $G-$graded prime ideals of A. Furthermore, the class of rings

$$\beta_G = \{A | A \text{ is a } G-\text{graded and } \beta_G(A) = A\}$$

is the graded prime radical class determined by the class of $G-$graded prime rings [1].

Definition 3.3. *[1] A nonempty class μ_G of $G-$graded rings is called a $G-$graded special class if satisfies the following conditions:*

1. *γ_G consists of $G-$graded prime rings.*
2. *If $A \in \mu_G$, then every nonzero graded ideal I of A is in μ_G.*
3. *If A is any $G-$graded ring such that there exists a graded two-sided essential ideal I of A and $I \in \mu_G$ implies $A \in \mu$.*

Let G be any group. Examples of $G-$graded special class of rings can be seen in the Example 3.2 and the Example 3.3 below.

Example 3.2. *Let G be any group. It is clear that the class π_G of all $G-$graded prime rings is a $G-$graded special class of rings.*

It is clear that the class π_G of all $G-$graded prime rings is the largest $G-$graded special class of rings. The Example 3.3 explains a specific example of $G-$graded special class of rings which is properly contained in the class π_G of all $G-$graded prime rings.

Example 3.3. *Define $(*_k)_G = \{A|A$ is a $G-$graded ring such that $A^u \in *_k\}$, where $*_k$ is the essential closure of the class of all $*-$rings which is previously described in the Proposition 2.2. Then class $(*_k)_G$ is graded special.*

Remark 3.2. *In specific case, when $G = \mathbb{Z}_4$, the class $(*_k)_{\mathbb{Z}_4}$ is not an empty set since $M_{2\times 2}(\mathbb{R})$ is contained in $(*_k)_{\mathbb{Z}_4}$. In general, we can follows the construction of graded special radical [1] which is the graded upper radical of graded special class. Hence, for any group G. The graded radical $\mathcal{U}((*_k)_G) = \{A|A$ is $G-$graded and no nonzero graded homomorphic image of A in $(*_k)_G\}$.*

We end this paper by arising the following question.

Question 1. *For any group G, whether β_G coincide with $\mathcal{U}((*_k)_G)$?, where β_G is the $G-$graded prime radical and $\mathcal{U}((*_k)_G) = \{A|A$ is a $G-$graded ring and A has no nonzero homomorphic image in $(*_k)_G$ (the class which is previously described in the Example 3.3 and Remark 3.2.*

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Approximation by means of Fourier trigonometric series in weighted Lebesgue spaces with variable exponent

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Abstract

We investigate the approximation of the functions by trigonometric polynomials $N_n^{\lambda}(f;x)$ of degree n in the weighted variable exponent Lebesgue spaces.

1 Introduction, some auxiliary results and main results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$ and $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable functions on $\mathbb{T}$.

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Let us denote by $\wp$ the class of Lebesgue measurable functions $p : \mathbb{T} \longrightarrow (1, \infty)$ such that $1 < p_* := \underset{x \in \mathbb{T}}{ess\,\inf} p(x) \leq p^* := \underset{x \in T}{ess\,\sup}\, p(x) < \infty$. The conjugate exponent of $p(x)$ is shown by $p'(x) := \frac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(.)}(\mathbb{T})$ of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{R}$ satisfying the condition

$$\int_{\mathbf{T}} |f(x)|^{p(x)}\, dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\{\ \lambda > 0 : \int_{\mathbf{T}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\}.$$

The spaces $L^{p(\cdot)}(\mathbb{T})$ are called generalized Lebesgue spaces with variable exponent. It is know that for $p(x) := p\ (1 < p < \infty)$, the space $L^{p(x)}(\mathbb{T})$ coincides with the Lebesgue space $L^p(\mathbb{T})$. If $p^* < \infty$ then the spaces $L^{p(\cdot)}(\mathbb{T})$ represent a special case of the so-called Orlicz-Musielak spaces [32]. For the first time Lebesgue spaces with variable exponent were introduced by Orlicz [34]. Note that the generalized Lebesgue spaces with variable exponent are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of diferential operators, and in variational calculus [7], [8], [9], [36] and [38]. Detailed information about properties of the Lebesque spaces with variable exponent can be found in [10], [26], [30], [31], [37] and [39]. Note that, some of the fundamental problems of the approximation theory in the generalized Lebesgue spaces with variable exponent of periodic and non-periodic functions were studied and solved by Sharapudinov [39]-[44] .

A function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function if* ω is a measurable and almost everywhere (a.e.) positive.

Let ω be a 2π periodic weight function. We denote by $L^p_\omega(\mathbb{T})$ the weighted Lebesgue space of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $f\omega^{\frac{1}{p}} \in L^p(\mathbb{T})$. For $f \in L^p_\omega(\mathbb{T})$ we set

$$\|f\|_{L^p_\omega(\mathbb{T})} := \left\|f\omega^{\frac{1}{p}}\right\|_{L^p(\mathbb{T})}.$$

$L^{p(.)}_\omega(\mathbb{T})$ stands for the class of Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $\omega f \in L^{p(\cdot)}(\mathbb{T})$. $L^{p(.)}_\omega(\mathbb{T})$ is called the weighted Lebesgue space with variable exponent. The space $L_{\omega}^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} := \|f\omega\|_{L^{p(\cdot)}(\mathbb{T})}.$$

Its known [25] that the set of trigonometric polynomials is dense in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, if $[\omega(x)]^{p(x)}$ is integrable on $\mathbb{T}$.

Let $\mathcal{B}$ be the class of all intervals in $\mathbb{T}$. For $B \in \mathcal{B}$ we set

$$p_B := (\frac{1}{|B|}\int\limits_B \frac{1}{p(x)}dx)^{-1}.$$

For given $p \in \wp$ the class of weights ω satisfying the condition

$$\left\|\omega^{p(x)}\right\|_{A_{p(\cdot)}} := \sup_{B\in\mathcal{B}} \frac{1}{|B|^{p_B}} \left\|\omega^{p(x)}\right\|_{L^1(B)} \left\|\frac{1}{\omega^{p(x)}}\right\|_{L^{(p'(\cdot)/p(\cdot))}(B)} < \infty$$

will be denoted by $A_{p(.)}$ [1].

We say that the variable exponent $p(x)$ satisfies *local log-Hölder continuity condition,* if there is a positive constant c_1 such that

$$| \, p(x) - p(y) \, | \leq \frac{c_1}{\log(\frac{1}{|x-y|})}, \tag{1.1}$$

for all $x, y \in \mathbb{T}$.

A function $p \in \wp$ is said to belong to the class $\wp^{\log}$, if the condition (1.1) is satisfied.

We denote by $E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})}$ the best approximation of $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} = \inf\{\| \, f - T_n \, \|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} : T_n \in \Pi_n\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let us suppose that $p \in \wp$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$, for some $p_0 \in (1, p_*)$. For

$f \in L_{\omega}^{p(.)}(\mathbb{T})$ we set

$$(\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\, dt,\ 0 < h < \pi,\ x \in \mathbb{T}.$$

If $p \in \wp^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$ and $f \in L_{\omega}^{p(.)}(\mathbb{T})$, then the shift operator ν_{h_i} is a bounded linear operator on $L_{\omega}^{p(.)}(\mathbb{T})$ [27]:

$$\|\nu_{h_i}(f)\|_{L_{\omega}^{p(.)}(\mathbb{T})} \le c_2 \|f\|_{L_{\omega}^{p(.)}(\mathbb{T})}.$$

Let $p \in \wp$ and $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$. The function

$$\Omega_{p(.),\omega}(\delta,\ f) := \sup_{0<h\le\delta} \|f - (\nu_h f)\|_{L_{\omega}^{p(.)}(\mathbb{T})},\ \delta > 0$$

is called the *moduli of continuity* of $f \in L_{\omega}^{p(.)}(\mathbb{T})$.

It can easily be shown that $\Omega_{p(.),\omega}(\cdot, f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_{p(.),\omega}(\delta, f) = 0,\ \Omega_{p(.),\omega}(\delta, f+g) \le \Omega_{p(.),\omega}(\delta, f) + \Omega_{p(.),\omega}(\delta, g),\ \delta > 0$$

for $f, g \in L_{\omega}^{p(.)}(\mathbb{T})$. Note that detailed information about properties of moduli of continuity $\Omega_{p(.),\omega}(\cdot, f)$ can be found in the paper [1]. Also, moduli of this type was considered by E. A. Hadjieva [16] in Lebesgue space with Muckenhoupt $A_p,\ 1 < p < \infty$ weight.

Let $0 < \alpha \le 1$. The set of functions $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ such that

$$\Omega_{p(.),\omega}(f, \delta) = O(\delta^{\alpha}),\ \ \delta > 0$$

is called the *Lipschitz class* $Lip(\alpha, p(\cdot), \omega)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \tag{1.2}$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $a_k(f)$ are $b_k(f)$ the Fourier coefficients of the function f. The *n-th partial sum* of series (1.2) is defined, as

$$\begin{aligned} S_n(f;x) &= \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k(f)\cos kx + b_k(f)\sin kx\right), \\ &= \sum_{k=0}^{n} Q_k(f;x). \end{aligned}$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. The sequence $\{p_n\}_0^\infty$ is called *almost monotone decreasing (increasing)*, denoted by $\{p_n\}_0^\infty \in AMDS$ ($\{p_n\}_0^\infty \in AMIS$), if there exist a constant c, depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the following inequality holds:

$$p_n \leq cp_m, \qquad (p_m \leq cp_n).$$

In proof of the main result we will use the notations

$$\Delta\beta_n := \beta_n - \beta_{n+1}, \;\; \Delta_m\beta(n,m) := \beta(n,m) - \beta(n,m+1).$$

As in [33] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$. Following [4], [35] the Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \; n = 1, 2, ...,$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the $C_\lambda-$method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. We suppose that $\{p_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (1.2), as

$$N_n^\lambda(f;x) \;\; = \;\; \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{n} p_{\lambda(n)-m} s_m(f;x)$$

where $P_n := \sum_{m=0}^{n} p_m \neq 0 \;\; (n \geq 0), \;\; p_{-1} = P_{-1} = 0$. Note that in the case $p_n = 1, n \geq 0, N(f;x)$ is equal to the mean

$$\sigma_n^{\lambda}(f;x)=\frac{1}{\lambda(n)+1}\sum_{m=0}^{\lambda(n)}S_m(f;x).$$

In the present paper we study the approximation of the functions by trigonometric polynomials $N_n^{\lambda}(f;x)$ in weighted Lebesgue spaces with variable exponent. The results obtained in this work are generalization of the results [33] to the weighted Lebesgue spaces with variable exponent. Similar problems about approximations of the functions by trigonometric polynomials in the different spaces have been investigated by several authors (see, for example, [2-6], [11-15], [17-24], [28], [29], [33] and [45-47]).

Note that, in the proof of the main results we use the method as in the proof of [33]. Our main result is the following:

Theorem 1.1.

1. *Let* $p\in\wp$, $\omega^{-p_0}\in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0\in(1,p_*)$, *if* $f\in Lip(\alpha,p(.),\omega)$, $0<\alpha<1$ *and if one of the following conditions*

$$\text{(i)}\quad \{p_n\}_0^{\infty}\in AMDS$$

$$\text{(ii)}\quad \{p_n\}_0^{\infty}\in AMIS,$$

and

$$(\lambda(n)+1)p_{\lambda(n)}=O(P_{\lambda(n)}) \tag{1.3}$$

holds, then

$$\left\|f-N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}=O\left((\lambda\left(n\right))^{-\alpha}\right).$$

2. *Let* $p\in\wp$, $\omega^{-p_0}\in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0\in(1,p_*)$, *if* $f\in Lip(1,p(.),\omega)$ *and if one of the following conditions*

$$\text{(iii)}\quad \sum_{k=1}^{\lambda(n)-1}k\left|\Delta p_k\right|=O(P_{\lambda(n)})$$

$$\text{(iv)}\quad \sum_{k=0}^{\lambda(n)-1}\left|\Delta p_k\right|=O(P_{\lambda(n)}/\lambda(n)),\ \textit{and (1.3) holds,}$$

then the estimate

$$\left\|f - N_n^{\lambda}(f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} = O\left((\lambda\left(n\right))\right)^{-1}.$$

holds.

In the proof of the main result we need the following Lemmas:

Lemma 1.1. (see [19]). *Let* $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$. Then for $f \in Lip(\alpha, p(.), \omega),\ 0 < \alpha \leq 1$ and $n =$ 1, 2,3.... the estimate

$$\|f - S_n(f)\|_{L_\omega^{p(\cdot)}(\mathbb{T})} = O(n^{-\alpha})$$

holds.

Lemma 1.2. (see [19]). *Let* $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0 \in (1, p_*)$. *Then for* $f \in Lip(1, p(.), \omega)$ *and* $n =$ 1, 2,3, ... *the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{L_\omega^{p(\cdot)}(\mathbb{T})} = O(n^{-1})$$

holds.

Lemma 1.3. (see [33]). *If* $\{p_n\}_0^\infty \in AMDS$ *or* $\{p_n\}_0^\infty \in AMIS$ *and* (1.3) *holds, then*

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} = O\left((\lambda(n))^{-\alpha} P_{\lambda(n)}\right)$$

for $0 < \alpha < 1$.

2 Proofs of the main results

Proof of Theorem 1.1. We prove the cases (i) and (ii) together. It is clear that

$$N_n^{\lambda}(f;x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m} \left\{s_m(f;x) - f(x)\right\}. \tag{2.1}$$

Then using Lemma 1.1 and Lemma 1.3 and (2.1) and condition (1.3) we have

$$\begin{aligned}
\left\|N_n^{\lambda}(f)-f\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} &\leq \frac{1}{P_{\lambda(n)}}\sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m}\left\|f-s_m(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m}\left\|f-s_m(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\quad +\left\|f-s_0(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m}O(m^{-\alpha})+O(\frac{p_{\lambda(n)}}{P_{\lambda(n)}}) \\
&= O((\lambda(n))^{-\alpha}).
\end{aligned}$$

Case (iv): We suppose that $\alpha=1$. Using Abel's transformation, we find that

$$N_n^{\lambda}(f;x)=\frac{1}{P_{\lambda(n)}}\sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m}\left\{s_m(f;x)-f(x)\right\}Q_m(f;x).$$

Thus we have

$$s_n^{\lambda}(f;x)-N_n^{\lambda}(f;x)=\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}(P_{\lambda(n)}-P_{\lambda(n)-m})\left\{s_m(f;x)-f(x)\right\}Q_m(f;x).$$

Use of Abel's transformation leads to

$$\begin{aligned}
s_n^{\lambda}(f;x)-N_n^{\lambda}(f;x) &= \frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m})) \\
&\quad \times\sum_{k=1}^{m}kQ_k(f;x)+\frac{1}{(\lambda(n)+1)}\sum_{k=1}^{\lambda(n)}kQ_k(f;x)
\end{aligned} \tag{2.2}$$

Taking account of (2.2) we have

$$\begin{aligned}\left\|s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} &\leq \left|\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}))\right| \\ &\quad\times\left\|\sum_{k=1}^{m}kQ_k(f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} \\ &\quad+\frac{1}{(\lambda(n)+1)}\left\|\sum_{k=1}^{\lambda(n)}kQ_k(f;x)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})}. \quad (2.3)\end{aligned}$$

It is clear that

$$s_n(f,x)-\sigma_n(f;x)=\frac{1}{n+1}\sum_{k=1}^{n}kQ_k(f;x). \quad (2.4)$$

Then from Lemma 1.2 and (2.4) we have

$$\left\|\sum_{k=1}^{n}kQ_k(f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})}=(n+1)\left\|s_n(f)-\sigma_n(f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})}=O(1). \quad (2.5)$$

Thus use of (2.3) and (2.5) gives us

$$\begin{aligned}\left\|s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} &= O(\frac{1}{P_{\lambda(n)}})\sum_{m=1}^{\lambda(n)}\left|\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}))\right| \\ &\quad+O((\lambda(n))^{-1}). \quad (2.6)\end{aligned}$$

By [33] the following relations hold :

$$\begin{aligned}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}) &= \frac{1}{m}\Delta_m(P_{\lambda(n)}-P_{\lambda(n)-m}) \\ &\quad+\frac{P_{\lambda(n)}-P_{\lambda(n)-m-1}}{m(m+1)} \\ &= \frac{P_{\lambda(n)-m-1}-P_{\lambda(n)-m}}{m}\end{aligned}$$

$$
\begin{aligned}
& +\frac{P_{\lambda(n)}-P_{\lambda(n)-m-1}}{m(m+1)} \\
= \quad & \frac{P_{\lambda(n)}-P_{\lambda(n)-m-1}}{m(m+1)}-\frac{p_{\lambda(n)-m}}{m} \\
= \quad & \frac{1}{m(m+1)}\left[P_{\lambda(n)}-P_{\lambda(n)-m-1}\right] \\
& -\frac{1}{m(m+1)}(m+1)p_{\lambda(n)-m}, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_m\left(\frac{P_{\lambda(n)}-P_{\lambda(n)-m}}{m}\right) \quad = \quad & \frac{1}{m(m+1)} \\
& \times\left[\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k-(m+1)p_{\lambda(n)-m}\right]. \qquad (2.8)
\end{aligned}
$$

Next we will prove by the induction the inequality

$$
\begin{aligned}
& \left|\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k-(m+1)p_{\lambda(n)-m}\right| \\
\leq \quad & \sum_{k=1}^{m} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|. \qquad (2.9)
\end{aligned}
$$

Let $m=1$. Then we obtain

$$
\left|\sum_{k=\lambda(n)-1}^{\lambda(n)} p_k-2p_{\lambda(n)-1}\right|=\left|p_{\lambda(n)}-p_{\lambda(n)-1}\right|.
$$

That is, the relation (2.9) holds, for $m=1$. Now we suppose that the relation (2.9) holds for $m=j$. We prove the inequality for $m=j+1$ ($\leq \lambda(n)$). The inequality

$$
\begin{aligned}
&\left|\sum_{k=\lambda(n)-(j+1)}^{\lambda(n)} p_k-(j+2)p_{\lambda(n)-(j+1)}\right| \\
= &\left|\sum_{k=\lambda(n)-j}^{\lambda(n)} p_k-(j+1)p_{\lambda(n)-(j+1)}\right| \\
= &\left|\sum_{k=\lambda(n)-j}^{\lambda(n)} p_k-(j+1)p_{\lambda(n)-j}+(j+1)p_{\lambda(n)-j}-(j+1)p_{\lambda(n)-(j+1)}\right| \\
\leq &\left|\sum_{k=\lambda(n)-j}^{\lambda(n)} p_k-(j+1)p_{\lambda(n)-1}\right|+\left|(j+1)p_{\lambda(n)-j}-(j+1)p_{\lambda(n)-(j+1)}\right| \\
\leq &\sum_{k=1}^{j} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|+(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-(j+1)}\right| \\
= &\sum_{k=1}^{j+1} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|.
\end{aligned}
$$

holds. That is, (2.9) is true for $m=j+1$. Thus the relation (2.9) is proved for any $1 \leq m \leq \lambda(n)$. Consideration of (2.8) and (2.9) gives us

$$
\begin{aligned}
&\sum_{m=1}^{\lambda(n)}\left|\Delta_m\left(\frac{P_{\lambda(n)}-P_{\lambda(n)-m}}{m}\right)\right| \\
\leq &\sum_{m=1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right| \\
\leq &\sum_{k=1}^{\lambda(n)} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right| \sum_{m=k}^{\infty} \frac{1}{m\,(m+1)} \\
= &\sum_{k=0}^{\lambda(n)-1} \left|\Delta p_k\right|. \qquad (2.10)
\end{aligned}
$$

Taking into account the condition of the Theorem 1.1 the relation, we have

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n)) \tag{2.11}$$

holds. Then taking the relations (2.10), (2.11) and (2.6) into account we get

$$\left\| s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_{L_\omega^p} = O((\lambda(n))^{-1}). \tag{2.12}$$

Thus from (2.12) and Lemma 1.1 for $\alpha = 1$ we have

$$\left\| f - N_n^{\lambda}(f) \right\|_{L_\omega^p} = O((\lambda(n))^{-1}).$$

Case (iii): First of all we prove the estimate

$$\sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) = O\left(\frac{P_{\lambda(n)}}{\lambda(n)} \right). \tag{2.13}$$

According to condition in the case (iii) of Theorem 1.1 the following relations holds:

$$\sum_{k=1}^{\lambda(n)-1} k\,|\Delta p_k| = O(P_{\lambda(n)}). \tag{2.14}$$

Consideration of (2.8) and (2.9) gives us

$$\begin{aligned} & \sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \\ \leq \; & \sum_{m=1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ = \; & \sum_{m=1}^{r} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ & + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ : \; & = S_1 + S_2. \end{aligned} \tag{2.15}$$

Let r denote the integral part of $(\lambda(n)/2)$. Using Abel's transformation and (2.14), we find that

$$\begin{aligned} S_1 &= \sum_{m=1}^{r} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ &\leq \sum_{k=1}^{r} \left| \Delta_k p_{\lambda(n)-k} \right| \leq \sum_{j=r-2}^{\lambda(n)-1} |\Delta p_j| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \end{aligned} \tag{2.16}$$

For S_2, we can write the following:

$$\begin{aligned} S_2 &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta p_{\lambda(n)-k} \right| \\ &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{r} k \left| \Delta p_{\lambda(n)-k} \right| \\ &\quad + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=r}^{m} k \left| \Delta p_{\lambda(n)-k} \right| \\ : &= S_{21} + S_{22}. \end{aligned} \tag{2.17}$$

If using again the condition (2.14) we get

$$\begin{aligned} S_{21} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{j=r-2}^{\lambda(n)-1} |\Delta p_j| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right), \\ S_{22} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{k=r}^{m} \left| \Delta p_{\lambda(n)-k} \right| \\ &= O\left(\frac{1}{\lambda(n)}\right) \left[|\Delta p_0| + 2|\Delta p_1| + ... + (r+1)|\Delta p_{r+1}| \right] \\ &= O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \end{aligned} \tag{2.18}$$

By (2.15)-(2.18) this implies that (2.13). Using (2.6), (2.13) and Lemma 1.1 we reach

$$\begin{aligned} &\left\| f - N_n^{\lambda}(f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &\qquad = \left\| f - s_n^{\lambda}(f) + s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \end{aligned}$$

$$\leq \left\| f - s_n^{\lambda}(f) \right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + \left\| s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}$$
$$\leq O((\lambda(n))^{-1}).$$

The proof of Theorem 1.1 is completed.

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ANALYSIS OF TRANSMISSION DYNAMICS AND MITIGATION SUCCESS OF COVID-19 IN NIGERIA: AN INSIGHT FROM A MATHEMATICAL MODEL

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Abstract

The first case of COVID-19 was confirmed in Nigeria on February 27 2020. The government of Nigeria took drastic steps in the form of enforcement of lockdown and social distancing order which necessitated closure of schools, worship centres, markets, offices, leisure spots and businesses to curtail the spread of the virus which have resulted in low confirmed cases and mortality with a case fatality ratio of 2% as at September 3 2020. While experts had predicted doom for Africa in the wake of COVID-19 pandemic, the ability of the Nigerian government to contain the pandemic in the first six months of emergence remained the biggest surprise. Therefore, a mathematical model was designed to analyse effectiveness of government mitigation

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measures in minimising the spread and mortality of COVID-19 in Nigeria. The positivity and boundedness of solutions were validated for the model. The analytical parameter, effective reproductive ratio, that governed the virus transmissibility in terms of the model parameters was derived and the stability analysed locally and globally around disease-free and disease-endemic equilibria. The disease-free equilibrium of the model is locally and globally asymptotically stable if the effective reproduction number is less than one. Otherwise, it is the endemic equilibrium, that is, asymptotically stable, locally and globally whenever the effective reproduction number is greater than one. Simulation was conducted to justify the theoretical results. Results from the simulation showed that low rates of transmission and mortality from COVID-19 in Nigeria were attributed to the effectiveness of mitigation measures. The results also indicate that implementation of non-pharmaceutical interventions can put Nigeria and other African countries in a good position for combatting subsequent emergence of any form of infectious diseases.

1 Introduction

The ongoing ravaging COVID-19 is a contagious disease instigated by SARS-CoV-2. The first case of the disease was reported in December 2019 in Wuhan, China, and has, within a few weeks, spread across the globe, leading to COVID-19 pandemic [1]. The coronavirus disease 2019 has been regarded as the largest global health crisis in human history as a result of the magnitude of confirmed cases, accompanied with the degree of fatalities across the continents [2]. Reliable data had it that by April 2020, COVID-19 pandemic had led to over 3 million confirmed cases and 230 000 deaths and the disease has spread to over 210 nations globally [3]. As of 1st October, 2020, 10:31 GMT, the figure has skyrocketed to 34, 192, 734 reported cases with 867 347 fatalities [4].

The symptoms and signs of COVID-19 develop within 2 to 14 days [5]. When the disease is fully incubated, the infected individuals may exhibit fever, fatigue, cough and breathing disorder that is similar to those infections instigated by SARS-CoV and MERS-CoV [6, 7]. However, many COVID-19 acute cases and fatalities come from the elderly people (from the age of 65 upward) and individuals with severe health challenges (such as people with kidney disease, hypertension, diabetes, obesity and other health issues that deteriorate the immune system) [3].

The global scourge of COVID-19 pandemic has elicited the attention of scholars in different disciplines, prompting several proposals to examine and envisage the development of the pandemic [8, 9]. Ndairov *et al.* [10] propose a model for the transmissibility of COVID-19 in the presence of super-spreaders. They performed the stability and sensitivity analyses of the model and discovered that daily reduction in the number of confirmed cases of COVID-19 is a function of the number of hospitalisations. Yang and Wang [11] proposed a model to study the transmission pathways of COVID-19 in terms of human-to-human and environment-to-human spread. Their analysis confirms the tendency of COVID-19 to remain endemic even with prevention and intervention measures. A model for the dynamics of COVID-19 with parameter estimations, sensitivity analysis and data fitting was investigated in [12] while a model for COVID-19 infection that describes the impact of slow diagnosis on the dynamics of COVID-19 is also studied in [13]. In [14], the researchers employ a statistical study of coronavirus disease to calculate time-regulated risk for fatality from the COVID-19 in Wuhan. Their results indicate that movement restrictions and adequate social distancing procedures are capable of reducing the spread of the disease. Furthermore, a data-oriented model that includes behavioural impacts of humans and governmental efforts on the dynamics of COVID-19 in Wuhan was proposed in [15]. A good number of mathematics and non-mathematics studies have also been conducted on COVID-19 [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 41].

Generally, the experts' prediction that COVID-19 would overrun precarious health systems in many African countries and eliminate millions has not come true as the continent has the second lowest COVID-19 fatality ratio globally as of 31st August 2020 [27]. Particularly, the situation report for the pandemic in Nigeria is better than most African countries especially South Africa with the biggest economy in the continent.

Table 1. Reported Cases and Death of COVID-19 for some African Countries

Countries	Population	Total COVID-19 Cases	Total Recovered	Total Deaths
Mali	20, 250,000	2, 087	-	126
Cameroon	26, 550, 000,	19, 604	-	415
Kenya	53, 770, 000	34, 705	-	585
Ivory Coast	26, 380, 000	18, 208	-	119
Nigeria	206, 984, 347	54, 463	42, 439	1, 027
South Africa	59, 436, 725	630, 595	553, 456	14, 389
Egypt	102, 300, 000	99, 425	-	5, 479
Morocco	36, 910, 000	66, 855	-	1, 253
Algeria	43, 850, 000	45, 469	-	1, 529
Ghana	31, 070, 000	32, 969	-	168

Retrieved 2020-09-03 from https://en.wikipedia.org/wiki/COVID-19_pandemic_in_each country

After the first case of the pandemic was reported, the government of Nigeria enforced lockdown and social distancing which necessitated closure of schools, worship centres, markets, offices, leisure spots and businesses to check the spread of the virus. COVID-19 phenomenon has caught the attentions of many researchers both mathematicians and non-mathematicians and several studies have been conducted to monitor and report the situation in Nigeria [28, 29, 30, 31]. Also, the effects of government interventions, policy stringency, policy on demand or application of personal protective equipment (PPEs) have been considered by some researchers [32, 33, 34, 35]. However, the analysis of how mitigation measures succeeded in averting massive infections and mortality from COVID-19 in Nigeria through the use of mathematical modelling is new in the literature. Although COVID-19 is gradually dying off at present, and some countries have started relaxing mitigation measures, while other countries have completely relaxed all measures, this study gives an account of how Nigerian government managed the pandemic in the first six months of emergence and would provide important information about mitigation strategies to adequately equip the country and other African countries towards future outbreak of infectious diseases.

2 Model Formulation and Basic Properties

The model is made up of two components as in [11] - the human population and the environment. The human population is divided into five compartments: the susceptible S, the exposed E, the infected I, the isolated Q and the removed individuals R. Individuals in the infected and the exposed classes are both infectious though infected individuals are more infectious because of the relatively low level of virus in the exposed individuals' systems. Susceptible individuals can contract the virus from the infectious individuals E and I and from the already contaminated environment V at rates α_1, α_2 and α_3 respectively. However, infectivity is reduced at a constant rate m by various measures that have been put on ground by the government (e.g. lockdown, social distancing, handwashing, use of face masks, environmental sanitation, etc.). Also, some individuals are detected at the exposed stage at rate $(1 - \sigma)$ through measures such as contact tracing and put in isolation Q while the rest who are not detected become infectious at rate σ. Individuals in isolation are receiving treatment and are not allowed to interact with the general public. Therefore, they do not spread COVID-19 to the individuals in the susceptible category. Besides, the exposed and infectious individuals increase the viral loads in the environment at rates β_1 and β_2 respectively. However, the rate of increase in the virus concentration in the environment V by the exposed and infectious individuals is reduced at a constant rate n, the awareness created by the government regarding environmental health. The effects of parameter n on the compartments S, Q and R are not considered because it is assumed that the compartments do not increase the viral load in the environment. It is only the exposed and infectious individuals that can pass the virus to the environment. Individuals in the infected and isolation classes are treated and recovered at different rates p_1 and p_2 respectively while death due to COVID-19 occurs for infectious and isolated individuals at rates k_1 and k_2 respectively. Death unrelated to COVID-19 occurs for every individual in the population at the same rate μ while virus removal unrelated to human interference occurs at rate τ. The flow of transmission across the compartments is displayed in Figure 1.

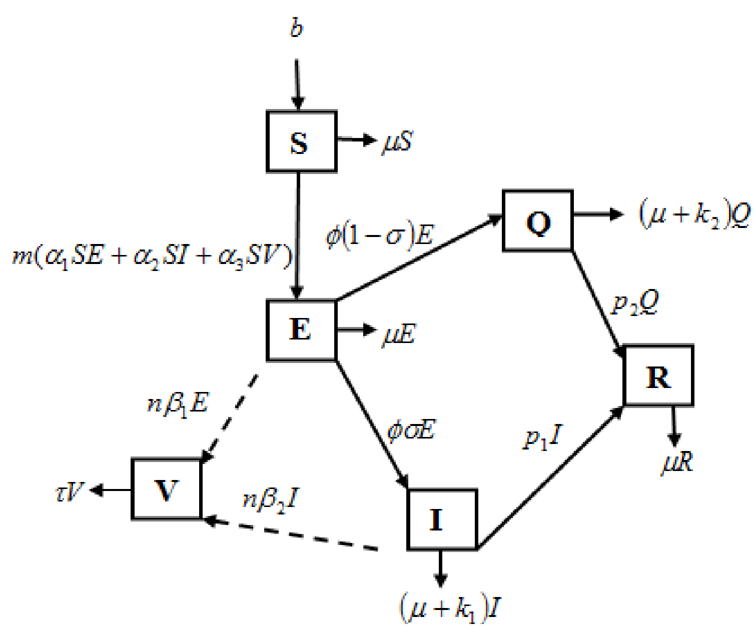

Figure 1: The flow chart of the model

Following the aforementioned assumptions, a model that describes the dynamics and effort to mitigate COVID-19 in Nigeria is introduced thus

$$\frac{dS}{dt} = b - m\alpha_1 SE - m\alpha_2 SI - m\alpha_3 SV - \mu S, \quad (2.1)$$
$$\frac{dE}{dt} = m\alpha_1 SE + m\alpha_2 SI + m\alpha_3 SV - (\phi + \mu)E, \quad (2.2)$$
$$\frac{dI}{dt} = \phi\sigma E - (k_1 + p_1 + \mu)I, \quad (2.3)$$
$$\frac{dQ}{dt} = \phi(1 - \sigma)E - (k_2 + p_2 + \mu)Q, \quad (2.4)$$
$$\frac{dR}{dt} = p_1 I + P_2 Q - \mu R, \quad (2.5)$$
$$\frac{dV}{dt} = n\beta_1 E + n\beta_2 I - \tau V, \quad (2.6)$$

with non-negative initial conditions

$$S(0) = S_\circ, E(0) = E_\circ, I(0) = I_\circ, Q(0) = Q_\circ, R(0) = R_\circ, V(0) = V_\circ.$$

Where $k_1 > k_2$, $\beta_2 > \beta_1$, $\alpha_3 < \alpha_1 < \alpha_2$ and $p_1 < p_2$ with parameters nomenclatures stated in Table 2.

Table 2. Description of parameters for the model

Parameters	Nomenclatures	Units
b	human recruitment rate	day^{-1}
m	reduction factor in disease transmission potential	-
α_1	effective contact rate between S and E	day^{-1}
α_2	effective contact rate between S and I	day^{-1}
α_3	effective contact rate between S and V	day^{-1}
μ	natural death rate in all human compartments	day^{-1}
ϕ	transition rate from the exposed stage	day^{-1}
σ	probability of transition from the exposed class into the infected class	-
k_1	death rate due to infection in I class	day^{-1}
k_2	death rate due to infection in Q class	day^{-1}
p_1	recovery rate in I class	day^{-1}
p_2	recovery rate in Q class	day^{-1}
β_1	exposed individuals' rate of contribution to the growth of pathogen	day^{-1}
β_2	infected individuals' rate of contribution to the growth of pathogen	day^{-1}
n	reduction factor in the human contribution to the growth of pathogen	-
τ	removal rate of pathogen	$ml\ day^{-1}$

Basic features of the model

For the system to be valid socially, environmentally, mathematically and epidemiologically, it is necessary to establish that its variables are positive for all $t > 0$ and besides, the region Ω, where the solutions for the model exist, is bounded. The following statements are therefore declared.

Theorem 2.1. *Since* $S_\circ \geq 0$, $E_\circ \geq 0$, $I_\circ \geq 0$, $Q_\circ \geq 0$, $R_\circ \geq 0$ *and* $V_\circ \geq 0$.*Then the coordinate (S,E,I,Q,R,V), which represents the solutions of the model, is positive.*

Proof. Suppose $t_1 = \sup\{t > 0 : S > 0, E > 0, I > 0, Q > 0, R > 0, V > 0 \in [0,t]\}$. Then $t_1 > 0$ and it is obvious from (2.1), if the positive term is neglected that

$$\frac{dS}{dt} \geq -\Gamma S,$$

where

$$\Gamma = m\alpha_1 E + m\alpha_2 I + m\alpha_3 V + \mu.$$

Hence,

$$\int \frac{dS}{dt} \geq -\int \Gamma S,$$

which implies that

$$S(t_1) = S(0)exp\left[-\int_0^{t_1} \lambda(r)dr\right] > 0 \ \forall \ t_1 > 0$$

from (2.1). Hence, $S(0) > 0$ since $e^p > 0 \ \forall$ real values of p. Following the same procedure, it can be established that:

$$E > 0, I > 0, Q > 0, R > 0, V > 0.$$

□

Lemma 2.1. *The region*
$\Omega = \left\{(S,E,I,Q,R,V) \in \Re^6_+; S,E,I,Q,R,V \geq 0; N(t) \leq \frac{b}{\mu}; V(t) \leq \frac{b\lambda}{\tau\mu}\right\}$
is not only positively invariant but attracts all nonnegative solutions of the system.

Proof. Adding system (2.1)-(2.5) for human population,

$$\frac{dN}{dt} = b - (S + E + I + Q + R)\mu - k_1 I - k_2 Q, \Rightarrow$$

$$\frac{dN}{dt} \leq b - \mu N, \Rightarrow$$

$$\frac{dN}{b - \mu N} \leq dt, \Rightarrow$$

$$b - \mu N(t) \geq c_1 e^{-\mu t}.$$

As $t = 0$, $N(t) = N(0)$
then,

$$c_1 = b - \mu N(0).$$

Hence,

$$b - \mu N(t) \geq (\pi - \mu N(0))e^{-\mu t}, \Rightarrow$$

$$N(t) \leq \frac{b}{\mu} - \left(\frac{b - \mu N(0)}{\mu}\right) e^{-\mu t}.$$

As $t \to \infty$,

$$0 \leq N(t) \leq \frac{b}{\mu}. \tag{2.7}$$

Hence, the feasible solution set for human population enter the region

$$\Delta = \left\{(S, E, I, Q, R) \in \Re^5_+; N(t) \leq \frac{b}{\mu}\right\}.$$

Also, the population of the pathogen at time t in the system (2.6) is given by

$$\frac{dV}{dt} = (E + I)\lambda - \tau V, \tag{2.8}$$

with $\lambda = \min(n\beta_1, n\beta_2)$.
λ is the net contribution of infectious individuals (both exposed and infected) to the growth of pathogen in the environment. Notice that both E and I are subset of human population which has been proved to be less than or equal to $\frac{b}{\mu}$ in inequality (2.7).
Therefore, $E + I \leq \frac{b}{\mu}$.
Hence, from (2.8),

$$\frac{dV}{dt} \leq \frac{b\lambda}{\mu} - \tau V, \Rightarrow$$

$$V(t) \leq \frac{b\lambda}{\tau\mu}\left(1 - c_2 e^{-\tau t}\right).$$

As $t \to \infty$ then,

$$V(t) \leq \frac{b\lambda}{\tau\mu}.$$

Therefore, the feasible solutions for the dynamics of coronavirus population in the model exists in the region

$$\Psi = \left\{ V \in \Re^{+}; V(t) \leq \frac{b\lambda}{\tau\mu} \right\}.$$

Hence, the region Ω that contains $\{\Delta \cup \Psi\}$ attracts every solution in $\Re^6_+$. □

3 Model Analysis

The dynamical behaviour of the model can be studied since the solutions are positive and bounded.

3.1 Equilibria

Two equilibria will be discussed - disease-free and endemic equilibria. Disease-free will be discussed first then followed by the reproduction number before discussing endemic equilibrium. Before February 27 2020, Nigeria was free from COVID-19. The COVID-19-free equilibrium then could be expressed mathematically as

$$\begin{aligned} \mathcal{E}_0 &= (S^\circ, E^\circ, I^\circ, Q^\circ, R^\circ, V^\circ), \\ &= \left(\frac{b}{\mu}, 0, 0, 0, 0, 0 \right). \end{aligned}$$

The agents of infection are in the compartments E, I and V. Therefore, F and V, the new infection and the transition matrices, are derived following Driessche and Watmough [36] as

$$F = \begin{pmatrix} m\alpha_1 S^\circ & m\alpha_2 S^\circ & m\alpha_3 S^\circ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

$$V = \begin{pmatrix} \phi + \mu & 0 & 0 \\ -\phi\sigma & k_1 + p_1 + \mu & 0 \\ -n\beta_1 & -n\beta_2 & \tau \end{pmatrix}. \tag{3.2}$$

Following Driessche and Watmough [36],

$$\begin{aligned} \mathcal{R}_e &= \frac{m\alpha_1 S^\circ}{(\phi + \mu)} + \frac{\phi\sigma m\alpha_2 S^\circ}{(\phi + \mu)(k_1 + p_1 + \mu)} + \frac{[(k_1 + p_1 + \mu)n\beta_1 + \phi\sigma n\beta_2] m\alpha_3 S^\circ}{\tau(\phi + \mu)(k_1 + p_1 + \mu)}, \\ &= \mathcal{R}_a + \mathcal{R}_b + \mathcal{R}_c. \end{aligned} \tag{3.3}$$

$\mathcal{R}_e$ measures disease transmissibility with mitigation measures on ground. It is called effective reproduction number and the first two terms in it quantify disease spread from human-to-human while the third term measures disease transmission into the human population from the environment. If all the mitigation measures are not on ground, the effective reproduction number reduces to the basic reproduction number $\mathcal{R}_0$ given as

$$\begin{aligned}\mathcal{R}_0 &= \frac{\alpha_1 S^\circ}{(\phi+\mu)} + \frac{\phi\alpha_2 S^\circ}{(\phi+\mu)(k_1+p_1+\mu)} + \frac{[(k_1+p_1+\mu)\beta_1+\phi\beta_2]\alpha_3 S^\circ}{\tau(\phi+\mu)(k_1+p_1+\mu)},\\ &= \mathcal{R}_x+\mathcal{R}_y+\mathcal{R}_z.\end{aligned} \tag{3.4}$$

After February 27 2020 when coronavirus cases have been confirmed, the endemic equilibrium of the disease could be given as $\mathcal{E}$ with coordinates (S, E, I, Q, R, V). To obtain the value of each coordinate, the system of equations (2.1)-(2.6) is solved and

$$S = \frac{1}{\mu}[b-(\phi+\mu)E], \tag{3.5}$$

$$I = \left(\frac{\phi\sigma}{k_1+p_1+\mu}\right)E, \tag{3.6}$$

$$Q = \left(\frac{\phi(1-\sigma)}{k_2+p_2+\mu}\right)E, \tag{3.7}$$

$$R = \frac{p_1 I + p_2 Q}{\mu}, \tag{3.8}$$

$$V = \frac{[(k_1+p_1+\mu)n\beta_1+\phi\sigma n\beta_2]E}{\tau(k_1+p_1+\mu)}. \tag{3.9}$$

Combining (3.6), (3.9) and (2.2) $\Rightarrow$

$$m\alpha_1 SE + \frac{m\alpha_2\phi\sigma SE}{(k_1+p_1+\mu)} + \frac{m\alpha_3 S[(k_1+p_1+\mu)n\beta_1+\phi\sigma n\beta_2]E}{\tau(k_1+p_1+\mu)} - (\phi+\mu)E = 0. \tag{3.10}$$

In (3.10), $E = 0$ corresponds to the disease-free equilibrium before the outbreak of COVID-19.

Also, from (3.10),

$$S = \frac{(\phi+\mu)}{m\alpha_1 + \frac{m\alpha_2\phi\sigma}{(k_1+p_1+\mu)} + \frac{m\alpha_3[(k_1+p_1+\mu)n\beta_1+\phi\sigma n\beta_2]}{\tau(k_1+p_1+\mu)}}. \tag{3.11}$$

Multiply the numerator and denominator of (3.11) by $\dfrac{S^\circ}{(\phi+\mu)}$ then

$$S = \frac{S^\circ}{\mathcal{R}_0}, \tag{3.12}$$

$$E = \frac{1}{(\phi+\mu)}\left[\pi - \frac{\mu S^\circ}{\mathcal{R}_0}\right], \tag{3.13}$$

$$I = \frac{\phi\sigma}{(\phi+\mu)(k_1+p_1+\mu)}\left[\pi - \frac{\mu S^\circ}{\mathcal{R}_0}\right], \tag{3.14}$$

$$Q = \frac{\phi(1-\sigma)}{(\phi+\mu)(k_2+p_2+\mu)}\left[\pi - \frac{\mu S^\circ}{\mathcal{R}_0}\right], \tag{3.15}$$

$$R = \frac{1}{\mu}\left[\frac{p_1\phi\sigma}{(\phi+\mu)(k_1+p_1+\mu)}\left\{\pi - \frac{\mu S^\circ}{\mathcal{R}_0}\right\} + \frac{p_2\phi(1-\sigma)}{(\phi+\mu)(k_2+p_2+\mu)}\left\{\pi - \frac{\mu S^\circ}{\mathcal{R}_0}\right\}\right], \tag{3.16}$$

$$V = \frac{[(k_1+p_1+\mu)n\beta_1 + \phi\sigma n\beta_2]}{\tau(\phi+\mu)(k_1+p_1+\mu)}\left\{\pi - \frac{\mu S^\circ}{\mathcal{R}_0}\right\}. \tag{3.17}$$

3.2 Stability of Equilibria

The local and global stability of the model around disease-free equilibrium can be investigated using linearisation approach and Lyapunov functional, respectively.

Theorem 3.1. *The DFE $\mathcal{E}_0$ of the system (2.1)-(2.6) is locally asymptotically stable if $\mathcal{R}_e < 1$ otherwise it is unstable if $\mathcal{R}_e > 1$.*

Proof. If the system (2.1)-(2.6) is linearised around disease-free equilibrium $\mathcal{E}_0$, the variational matrix is obtained as

$$J(\mathcal{E}_0) = \begin{pmatrix} -\mu & -m\alpha_1 S^\circ & -m\alpha_2 S^\circ & 0 & 0 & -m\alpha_3 S^\circ \\ 0 & m\alpha_1 S^\circ - (\phi+\mu) & m\alpha_2 S^\circ & 0 & 0 & m\alpha_3 S^\circ \\ 0 & \phi\sigma & -(k_1+p_1+\mu) & 0 & 0 & 0 \\ 0 & \phi(1-\sigma) & 0 & -(k_2+p_2+\mu) & 0 & 0 \\ 0 & 0 & p_1 & p_2 & -\mu & 0 \\ 0 & n\beta_1 & n\beta_2 & 0 & 0 & -\tau \end{pmatrix} \tag{3.18}$$

$\mathcal{R}_0$ will be less than one and the disease-free equilibrium will be locally asymptotically stable if it can be established that all the eigenvalues of (3.18) are nonpositive.

Evaluating $|J(\mathcal{E}_0) - \lambda I| = 0$, four of the solutions are obtained as
$\lambda_1 = \lambda_2 = -\mu, \lambda_3 = -\tau$ and $\lambda_4 = -(k_2+p_2+\mu)$.
The remaining two solutions can be obtained from the equation

$$\lambda^2 + a_1\lambda + a_2 = 0, \tag{3.19}$$

where

$$a_1 = k_1 + p_1 + \phi + 2\mu - m\alpha_1 S^{\circ}$$

and,

$$a_2 = (k_1 + p_1 + \mu)\{(\phi + \mu) - m\alpha_1 S^{\circ}\} - m\alpha_2 \phi\sigma S^{\circ}.$$

It is obvious that the two solutions in (3.19) are nonpositive if $a_1 > 0$ and $a_2 > 0$. Hence, $\mathcal{R}_0 < 1$ and the disease-free equilibrium, $\mathcal{E}_0$ of the model is locally asymptotically stable if the condition $a_1 > 0$ and $a_2 > 0$ are satisfied. □

Having established the necessary and sufficient conditions for the disease-free equilibrium of the model to be locally asymptotically stable, attempt would be made to study the global asymptotic stability behaviour of the model around disease-free equilibrium. Following the popular Lyapunov functional approach as in [28, 29, 37], Lyapunov function W is constructed thus

$$W(t) = A_1 E + A_2 I + A_3 V, \tag{3.20}$$

with the time derivative $\dot{W}(t)$ given as

$$\dot{W}(t) = A_1 \dot{E}(t) + A_2 \dot{I}(t) + A_3 \dot{V}(t). \tag{3.21}$$

Theorem 3.2. *The DFE $\mathcal{E}_0$ of the model is globally asymptotically stable if the time derivative of the Lyapunov functional $\dot{W}(t) \leq 0 \in \Omega$*

Proof. Putting $\dot{E}(t)$, $\dot{I}(t)$ and $\dot{V}(t)$ in (2.2), (2.3) and (2.6) into (3.21)

$$\begin{aligned} \dot{W}(t) &= A_1[m\alpha_1 SE + m\alpha_2 SI + m\alpha_3 SV - (\phi + \mu)E] \\ &+ A_2[\phi\sigma E - (k_1 + p_1 + \mu)I] \\ &+ A_3[n\beta_1 E + n\beta_2 I - \tau V] \Rightarrow \end{aligned} \tag{3.22}$$

$$\begin{aligned} \dot{W}(t) &= [A_1 m\alpha_1 S - A_1(\phi + \mu) + A_2\phi\sigma + A_3 n\beta_1]E \\ &+ [A_1 m\alpha_2 S - A_2(k_1 + p_1 + \mu) + A_3 n\beta_2]I \\ &+ [A_1 m\alpha_3 S + \tau]V. \end{aligned} \tag{3.23}$$

At DFE when $\mathcal{R}_e < 1$, $S = S^{\circ}$, and $E = I = Q = R = V = 0$ but if $\mathcal{R}_e = 1$, the equality $\dot{W}(t) = 0 \Rightarrow$

$$\begin{aligned} &[A_1 m\alpha_1 S - A_1(\phi + \mu) + A_2\phi\sigma + A_3 n\beta_1]E \\ &+ [A_1 m\alpha_2 S - A_2(k_1 + p_1 + \mu) + A_3 n\beta_2]I \\ &+ [A_1 m\alpha_3 S + \tau]V = 0. \end{aligned} \tag{3.24}$$

It is reasonable in (3.24) for

$$[A_1 m\alpha_1 S - A_1(\phi+\mu) + A_2\phi\sigma + A_3 n\beta_1]E \leq 0 \tag{3.25}$$

In view of (3.10) and given that the analysis is around the DFE then (3.25) implies that

$$\left[m\alpha_1 S^\circ + \frac{m\alpha_2\phi\sigma S^\circ}{(k_1+p_1+\mu)} + \frac{m\alpha_3 S^\circ[(k_1+p_1+\mu)n\beta_1 + \phi\sigma n\beta_2]}{\tau(k_1+p_1+\mu)} - (\phi+\mu)\right] E \leq 0, \tag{3.26}$$

$$(\phi+\mu)\left[\frac{m\alpha_1 S^\circ}{(\phi+\mu)} + \frac{m\alpha_2\phi\sigma S^\circ}{(\phi+\mu)(k_1+p_1+\mu)} + \frac{m\alpha_3 S^\circ[(k_1+p_1+\mu)n\beta_1 + \phi\sigma n\beta_2]}{\tau(\phi+\mu)(k_1+p_1+\mu)} - 1\right] E \leq 0, \tag{3.27}$$

$$\Rightarrow (\phi+\mu)[\mathcal{R}_e - 1]E \leq 0. \tag{3.28}$$

Hence, $\dot{W}(t) = 0$ if $E = 0$. Conversely, $\dot{W}(t) \leq 0$ provided $\mathcal{R}_e \leq 1$. This implies that as
$t \to \infty$, $(S(t), E(t), I(t), Q(t), R(t), V(t)) \to \left(\frac{b}{\mu}, 0, 0, 0, 0, 0\right)$. Therefore, the largest invariant set in $\{S(t), E(t), I(t), Q(t), R(t), V(t) \in \Omega : W(t) = 0\}$ is the singleton $\{\mathcal{E}_0\}$. Hence, by LaSalle's invariance principle [38], the DFE $\mathcal{E}_0$ is globally asymptotically stable in Ω if $\mathcal{R}_e \leq 1$. □

To assess the effectiveness of the government mitigation measures in the light of the disease-endemic equilibrium, the stability analysis is extended to the endemic equilibrium so that government mitigation measures are adequately evaluated under a disease- pandemic scenario.

Theorem 3.3. *The disease-endemic equilibrium exists and is locally asymptotically stable if the effective reproductive ratio* $\mathcal{R}_e > 1$.

Proof. Linearising the system (2.1)-(2.6) around disease-pandemic equilibrium then

$$J(P^*) = \begin{pmatrix} -b_1 & -m\alpha_1 S^* & -m\alpha_2 S^* & 0 & 0 & -m\alpha_3 S^* \\ b_3 & b_2 & m\alpha_2 S^* & 0 & 0 & 0 \\ 0 & \phi\sigma & -(k_1+p_1+\mu) & 0 & 0 & 0 \\ 0 & \phi(1-\sigma) & 0 & -(k_2+p_1+\mu) & 0 & 0 \\ 0 & 0 & p_1 & p_2 & -\mu & 0 \\ 0 & n\beta_1 & n\beta_2 & 0 & 0 & -\tau \end{pmatrix}, \tag{3.29}$$

where $b_1 = m(\alpha_1 E^* + \alpha_2 I^* + \alpha_3 V^* + \mu), b_2 = m\alpha_1 S^* - (\mu + \phi)$ and $b_3 = m(\alpha_1 E^* + \alpha_2 I^* + \alpha_3 V^*$.
Three eigenvalues of (3.29) are $\lambda_1 = -\tau, \lambda_2 = -\mu$ and $\lambda_3 = -(k_2 + p_1 + \mu)$. Using row reduced operation, the remaining eigenvalues can be obtained from submatrix U

$$J(U) = \begin{pmatrix} -b_1 & -m\alpha_1 S^* & -m\alpha_2 S^* \\ 0 & \dfrac{a_1 a_2}{a_3} - m\alpha_1 S^* & \dfrac{a_1 m\alpha_2 S^*}{a_3} - m\alpha_2 S^* \\ 0 & \phi\sigma & -(k_1+p_1+\mu) \end{pmatrix}. \tag{3.30}$$

The matrix in (3.30) has one of the eigenvalues being $\lambda_4 = -b_1$ while the remaining eigenvalues can be obtained from the characteristic equation

$$k_0\lambda^2 + k_1\lambda + k_2 = 0, \tag{3.31}$$

where

$$\begin{aligned} k_0 &= 1, \\ k_1 &= m\alpha_1 S^* - \frac{b_1 b_2}{b_3} - (k_1 + p_1 + \mu), \\ k_2 &= (k_1 + p_1 + \mu)\left[\frac{b_1 b_2}{b_3} - m\alpha_1 S^*\right] - \phi\sigma\left[\frac{b_1 m\alpha_2 S^*}{b_3} - m\alpha_2 S^*\right]. \end{aligned}$$

It is straightforward to conclude that the two eigenvalues in (3.31) are negative if $k_1 > 0$ and $k_2 > 0$ so that the effective reproductive ratio $\mathcal{R}_e$ is more than unity. Therefore, the disease-pandemic equilibrium is locally asymptotically stable if $k_1 > 0$ and $k_2 > 0$. □

Theorem 3.4. *The disease-pandemic equilibrium is globally asymptotically stable if the effective reproductive ratio $\mathcal{R}_e > 1$ such that the derivative of the Lyapunov function $\mathcal{J}$ is negative, i.e., $\dfrac{d\mathcal{J}}{dt} < 0$.*

Proof. To establish the existence of $\mathcal{R}_e > 1$ under global stability for disease-endemic equilibrium, the derivative of the Lyapunov function $\mathcal{J}$ must be negative.

$$\begin{aligned}J(S^*,E^*,I^*,Q^*,R^*,V^*) &= \left(S - S^* - S^* \ln \frac{S^*}{S}\right) + \left(E - E^* - E^* \ln \frac{E^*}{E}\right) \\ &+ \left(I - I^* - S^* \ln \frac{I^*}{I}\right) + \left(Q - Q^* - Q^* \ln \frac{Q^*}{Q}\right) \\ &+ \left(R - R^* - R^* \ln \frac{R^*}{R}\right) + \left(V - V^* - V^* \ln \frac{V^*}{V}\right).\end{aligned}$$

So that,

$$\begin{aligned}\frac{d\mathcal{J}}{dt} &= \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{E^*}{E}\right)\frac{dE}{dt} + \left(1 - \frac{I^*}{I}\right)\frac{dI}{dt} \\ &+ \left(1 - \frac{Q^*}{Q}\right)\frac{dQ}{dt} + \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt} + \left(1 - \frac{V^*}{V}\right)\frac{dV}{dt}.\end{aligned} \tag{3.32}$$

(3.32) can be written as

$$\frac{d\mathcal{J}}{dt} = M + N, \tag{3.33}$$

where

$$\begin{aligned}M &= b\left(1 - \frac{S^*}{S}\right) + m(\alpha_1 SE + \alpha_2 IS + \alpha_3 SV)\left(1 - \frac{E^*}{E}\right) \\ &+ \phi\sigma E\left(1 - \frac{I^*}{I}\right) + \phi(1-\sigma)E\left(1 - \frac{Q^*}{Q}\right) \\ &+ (p_1 I + p_2 Q)\left(1 - \frac{R^*}{R}\right) + (n\beta_1 E + n\beta_2 I)\left(1 - \frac{V^*}{V}\right)\end{aligned}$$

and,

$$\begin{aligned}N &= (m\alpha_1 E + m\alpha_2 I + m\alpha_3 V + \mu)\left(1 - \frac{S}{S^*}\right)S^* + (\mu + \phi)\left(1 - \frac{E}{E^*}\right)E^* \\ &+ (k_1 + p_1 + \mu)\left(1 - \frac{I}{I^*}\right)I^* + (k_2 + p_2 + \mu)\left(1 - \frac{Q}{Q^*}\right)Q^* \\ &+ \mu\left(1 - \frac{R}{R^*}\right)R^* + \tau\left(1 - \frac{V}{V^*}\right)V^*.\end{aligned}$$

It is observed that $M > 0$ while $N < 0$ because $S^* < S, E^* < E, I^* < I, Q^* < Q, R^* < R$ and $V^* < V$. Therefore, there exists disease-pnademic equilibrium and by LaSalle's invariance principle [38], the disease-endemic equilibrium is globally asymptotically stable if $M < N$ so that $\frac{d\mathcal{J}}{dt} < 0$. □

3.3 Local Sensitivity Analysis

The local sensitivity of the key system parameters to COVID-19 spread is computed in (3.34)-(3.37) by using the sensitivity index formula [39, 40].

$$\mathcal{S}_b = \frac{m\alpha_1}{\mu(\phi+\mu)} + \frac{\phi\sigma m\alpha_2}{\mu(\phi+\mu)(k_1+p_1+\mu)} + \frac{[(k_1+p_1+\mu)n\beta_1+\phi\sigma n\beta_2]m\alpha_3}{\mu\tau(\phi+\mu)(k_1+p_1+\mu)} \times \frac{b}{\mathcal{R}_e}, \tag{3.34}$$

$$\mathcal{S}_{\alpha_1} = \frac{mb}{\mu(\phi+\mu)} \times \frac{\alpha_1}{\mathcal{R}_e}, \tag{3.35}$$

$$\mathcal{S}_{\alpha_2} = \frac{\phi\sigma mb}{\mu(\phi+\mu)(k_1+p_1+\mu)} \times \frac{\alpha_2}{\mathcal{R}_e}, \tag{3.36}$$

$$\mathcal{S}_{\alpha_3} = \frac{[(k_1+p_1+\mu)n\beta_1+\phi\sigma n\beta_2]mb}{\mu\tau(\phi+\mu)(k_1+p_1+\mu)} \times \frac{\alpha_3}{\mathcal{R}_e}. \tag{3.37}$$

4 Simulation and Discussion

To quantify the analytical results, numerical simulation is carried out using parameters values whose sources are from [28] as well as assumption. Graphical profiles are then displayed to visualise the transmission dynamics and mitigation success of COVID-19 within the first six months of the emergence of the pandemic in Nigeria. To obtain the numerical values for the theoretical results in section 3.0, fix $m = 0.5, p_1 = 0.13978, p_2 = 0.23978, b = 0.349, \alpha_1 = 0.03, \alpha_2 = 0.08, \alpha_3 = 0.01, \phi = 0.2, \sigma = 0.01, \mu = 1/54.5, k_1 = 0.015, k_2 = 0.01, n = 0.3, \tau = 10, \beta_1 = 0.00011, \beta_2 = 0.075, S(0) = 100000, E(0) = 8000, I(0) = 1500, Q(0) = 6000, R(0) = 4262, V(0) = 60$. Then the effective reproduction number $\mathcal{R}_e$ in (3.3) is evaluated as $\mathcal{R}_e = 1.35$ from which

$$\mathcal{R}_a = 1.31, \mathcal{R}_b = 0.04 \text{ and } \mathcal{R}_c = 0.00.$$

$\mathcal{R}_a, \mathcal{R}_b$ and $\mathcal{R}_c$ measure the risk of infection from each transmission pathway. Of all the transmission routes, the largest risk $(\mathcal{R}_a)$ is from the exposed to the susceptible individuals. This is because the exposed individuals are asymptomatic but can spread the virus to the susceptible individuals usually in unintentional modes. However, the transmission from other routes $(\mathcal{R}_a)$ and $(\mathcal{R}_b)$ are insignificant. While the insignificant transmission of the virus from the infected individuals might be attributed to the conspicuousness of disease symptoms in the infected individuals which would make other people to keep away from them, the insignificant contribution from the environment to the general infection risk might be attributed to the presence of parameter n, the reduction factor in the net contribution of infectious individuals (both E and I) to the growth of pathogen in the environment. The sensitivity indices for the transmission parameters α_1, α_2 and α_3 in (3.35)-(3.37) are 0.97, 0.03 and 0.00 respectively. The sensitivity indices for α_1, α_2 and α_3 correlate with the results for $\mathcal{R}_a, \mathcal{R}_b$ and $\mathcal{R}_c$ and a confirmation that transmission from the exposed to the susceptible individuals poses the greatest threat.

In the same manner, the numerical value of the basic reproduction number $(\mathcal{R}_0)$ in (3.4) is also evaluated and $\mathcal{R}_0 = 10.67$ from which

$$\mathcal{R}_x = 2.61, \mathcal{R}_y = 8.05 \text{ and } \mathcal{R}_z = 0.01.$$

Unlike in effective reproduction number $\mathcal{R}_e$ where the transmission from the infected individuals $\mathcal{R}_b$ is insignificant, the contribution of the infected individuals is enormous in the absence of control with $\mathcal{R}_y = 8.05$. The infection risk from the exposed to the susceptible individuals is also higher with $\mathcal{R}_x = 2.61$ than in the effective reproduction number where $\mathcal{R}_a = 1.31$. Generally, the reproduction number of COVID-19, based on scholars' estimation, is in the region 2.0-3.0 which makes it higher than the reproduction numbers of SARS and MERS coronaviruses [21]. The numerical values of the effective reproduction number $\mathcal{R}_e$ and the basic reproduction number $\mathcal{R}_0$ in the present analysis for Nigeria, 1.35 and 10.67 respectively, indicate that Nigerian government took the prevention and control of COVID-19 with all seriousness at the start of the pandemic. The timely enforcement of lockdown and social distancing order records an outstanding success in averting massive infection and mortality with $\mathcal{R}_e = 1.35$ below the region estimated by the scholars (2.0-3.0).

The experts' projection that Africa will experience the worst impacts of COVID-19 pandemic given the fragile health system in many African countries would have come to pass in Nigeria [29, 31]. The reproduction number $\mathcal{R}_0 = 10.67$ would have been experienced in Nigeria if the mitigation measures were not taken seriously. COVID-19 first case was confirmed in Nigeria on February 27, 2020. The government instituted lockdown on March 30, 2020 and the lockdown did not relax until July 1, 2020 when the government approved partial ease of lockdown by allowing partial reopening of schools for students in terminal classes and removing of ban on inter-state travels. The schools did not open fully until September 21, 2020 while the international flights did not resume until September 5, 2020. Besides, the government through its agency, Nigeria Centre for Disease Control (NCDC), embarks on massive sensitisation of the public on the signs, symptoms, dangers and preventions of COVID-19 through various media especially text messages from time to time. Individuals who are infected with the virus are supported and well catered for promptly. They are fed and treated free of charge.

As regards the necessary and sufficient conditions for the disease-free equilibrium of the model to be locally asymptotically stable that are established in subsection 3.2, the initial values of a_1 and a_2 are computed with the usual parameters values to ascertain whether the model is locally asymptotically stable around disease-free equilibrium or not. The value of m, the reduction factor in disease transmission potential is then varied to examine the stability behaviour of the model around the disease-free equilibrium, the results of which are stated in Table 3.

Table 3. Stability behaviour of the model

S/No	k_1	p_1	μ	ϕ	m	α_1	b	α_2	σ	a_1	a_2	DFE
1	0.015	0.13978	1/54.5	0.2	0.5	0.03	0.349	0.08	0.01	+0.11	−0.01	*Unstable*
2	0.015	0.13978	1/54.5	0.2	0.6	0.03	0.349	0.08	0.01	+0.05	−0.02	*Unstable*
3	0.015	0.13978	1/54.5	0.2	0.7	0.03	0.349	0.08	0.01	−0.01	−0.03	*Stable*
4	0.015	0.13978	1/54.5	0.2	0.8	0.03	0.349	0.08	0.01	−0.07	−0.04	*Stable*
5	0.015	0.13978	1/54.5	0.2	0.9	0.03	0.349	0.08	0.01	−0.12	−0.05	*Stable*

In Table 3, it is observed that the disease-free equilibrium of the model is stable when the reduction factor m in the disease transmission potential is greater than 0.6. The stability of the disease-free equilibrium implies that COVID-19 would not have spread in Nigeria from the initial infection if adequate measures had been put on ground before February 27 2020 when the first case was reported. Experts argued that Nigeria ought to have closed her borders and stopped international flights before February 27 2020. If the steps had been taken, the Italian businessman who brought the virus to Nigeria would have been prevented and Nigeria would have maintained a stable region in Table 3 for as long as possible. However, since the disease was able to spread from the initial infection, the disease-endemic equilibrium of the model is stable and the stability in endemic equilibrium is observable in the region where the reduction factor m in the disease transmission potential is less than 0.7 In Table 3. The disease-endemic equilibrium becomes stable when the disease-free equilibrium is unstable. It is therefore evident that government mitigation strategies (closure of schools, worship centres, markets, offices, leisure spots, businesses as well as awareness campaigns) play crucial roles in shaping the dynamics of COVID-19 in Nigeria during the first six months of emergence. Although the disease spread, the government of Nigeria was able to avert massive infections and mortality from the pandemic through effective mitigation measures. With the same stated parameters values and initial conditions for the state variables, plots are generated for the model in Figure 2 and Figure 3 to visualise the effect of the parameters on the dynamics of the disease.

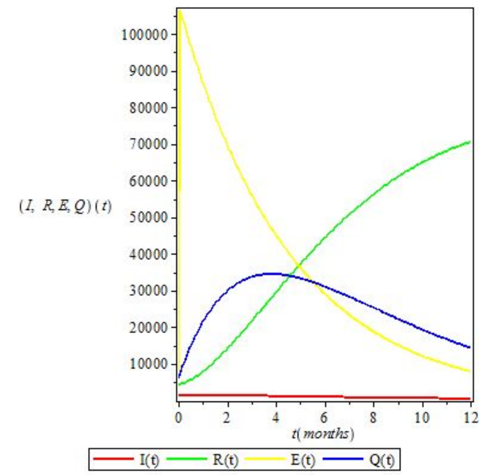

Figure 2: Result of simulation with mitigation measures on ground. $\mathcal{R}_e = 1.35$, $I(t)$ = 1,038

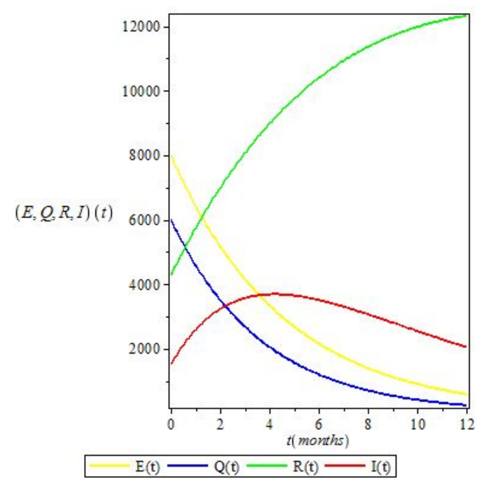

Figure 3: Result of simulation in the absence of mitigation measures. $\mathcal{R}_e = 10.67$, $I(t)$ = $1,815$

Comparing the initial value of I $(i.e.,\ I(0) = 1500)$ with the numerical results of $I(t)$ in Figure 2 and Figure 3, it is evident that things would have gone out of hand if the mitigation measures had not been taken seriously in Nigeria. In Figure 2, the population of infected individuals falls continuously but it rises before it begins to fall in Figure 3. The escalation of the figure of individuals who are infected with COVID-19 would have spelt doom for Nigeria given the precarious health system but the Nigerian government is able to be on top of the situation with effective mitigation measures that minimise disease transmission and spread to the barest minimum. Key model parameters (transmission parameters α_1, α_2 and α_3) are varied to perform further numerical simulations to visualise the effect of partial and total mitigation measures on disease spread parameters α_1, α_2 and α_3 and the population of infectious individuals $I(t)$ in Figure 4 and Figure 5

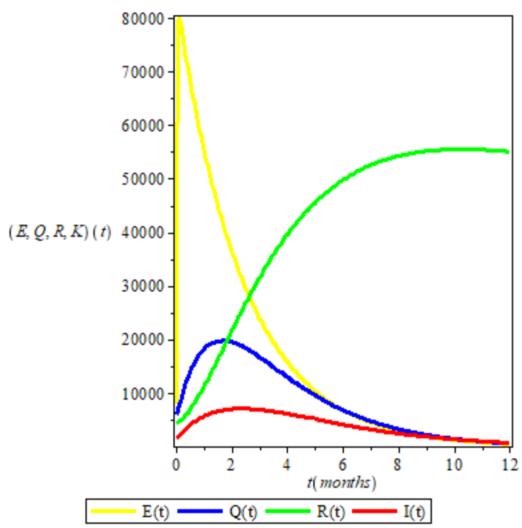

Figure 4: Result of simulation for partial measures.
$\alpha_1 = 0.003, \alpha_2 = 0.008, \alpha_3 = 0.001$

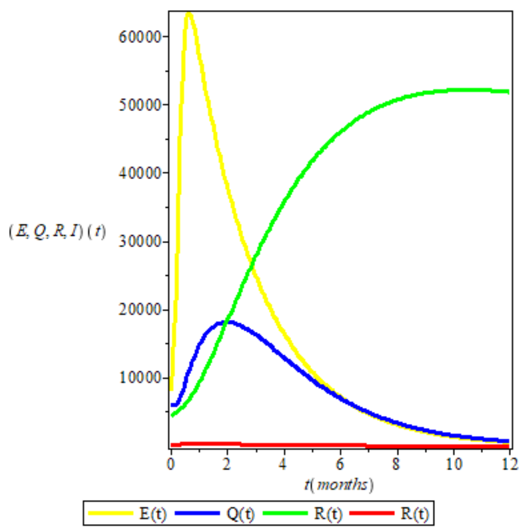

Figure 5: Result of simulation for total measures.
$\alpha_1 = 0.0003, \alpha_2 = 0.0008, \alpha_3 = 0.0001$

Whether partial or total, Figure 4 and Figure 5 indicate that government mitigation measures against COVID-19 limit the spread of the disease and the population of individuals who become infectious. The mitigation measures in the form of lockdown and social distancing enforced by the Nigerian government are comprehensive and able to avert massive infections and mortality from COVID-19.

5 Conclusion

The outbreak of COVID-19 brought a fear that the African continent is on the verge of ruin due to widespread poverty and fragile health systems. Nigerians are particularly worried about the fate of the country should the outbreak escalate given the population and the vulnerability index of infectious disease for the county which are in excess of 200 million and 0.27 respectively [31]. However, the pandemic has been effectively managed and the impacts of the outbreak have been largely minimised in Nigeria as of September 3 2020. A mathematical model has been formulated to analyse how massive infections and mortality from COVID-19 have been averted in Nigeria through mitigation measures. The validity of the model is confirmed by establishing positivity and boundedness of its solutions. A thorough analysis is then conducted qualitatively and quantitatively. The qualitative analysis is performed by obtaining the equilibria of the model and deriving the reproduction numbers both effective reproduction number $\mathcal{R}_e$ and the basic reproduction number $\mathcal{R}_0$. The model stability is also examined and it is shown that the disease- free and the disease-endemic equilibria of the model are locally and globally asymptotically stable whenever $\mathcal{R}_e < 1$ and $\mathcal{R}_e > 1$ respectively. The quantitative analysis is finally conducted to justify the theoretical results. Both the theoretical and numerical results indicate that low infections and mortality from COVID-19 during the first six months of COVID-19 emergence in Nigeria are attributed to the effectiveness of government mitigation measures. The effective implementation of non-pharmaceutical interventions can therefore put Nigeria and other African countries in a good position for combatting subsequent emergence of any form of infectious diseases.

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