The Aligarh Bulletin of Mathematics
Volume 41, Number 1 (2022), 63-80
ISSN: 0304-9787

Approximation by means of Fourier trigonometric series in weighted Lebesgue spaces with variable exponent

Sadulla Z. Jafarov
Department of Mathematics and Science Education
Faculty of Education, Muş Alparslan University
49250, Muş, Turkey
and
Institute of Mathematics and Mechanics
National Academy of Sciences of Azerbaijan
9 B. Vahabzadeh STR., AZ 1141, Baku, Azerbaijan
Email:s.jafarov@alparslan.edu.tr

(Received: September 21, 2021, Accepted: April 19, 2022)

Abstract

We investigate the approximation of the functions by trigonometric polynomials $N_n^{\lambda}(f;x)$ of degree n in the weighted variable exponent Lebesgue spaces.

1 Introduction, some auxiliary results and main results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$ and $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable functions on $\mathbb{T}$.

Keywords and phrases: Trigonometric approximation, Muckenhoupt weight, modulus of continuity, Lipschitz class, variable exponent Lebesgue space.
2020 AMS Subject Classification: Primary: 40D05, 41A10, 42A05, 42A10.

Let us denote by $\wp$ the class of Lebesgue measurable functions $p : \mathbb{T} \longrightarrow (1,\infty)$ such that $1 < p_* := \underset{x\in\mathbb{T}}{ess\inf} p(x) \leq p^* := \underset{x\in T}{ess\sup}\, p(x) < \infty$. The conjugate exponent of $p(x)$ is shown by $p'(x) := \frac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(.)}(\mathbb{T})$ of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{R}$ satisfying the condition

$$\int\limits_{\mathbf{T}} |f(x)|^{p(x)}\, dx < \infty.$$

This class $L^{p(.)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(.)}(\mathbb{T})} := \inf\{\, \lambda > 0 : \int\limits_{\mathbf{T}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\}.$$

The spaces $L^{p(.)}(\mathbb{T})$ are called generalized Lebesgue spaces with variable exponent. It is know that for $p(x) := p \ (1 < p < \infty)$, the space $L^{p(x)}(\mathbb{T})$ coincides with the Lebesgue space $L^p(\mathbb{T})$. If $p^* < \infty$ then the spaces $L^{p(.)}(\mathbb{T})$ represent a special case of the so-called Orlicz-Musielak spaces [32]. For the first time Lebesgue spaces with variable exponent were introduced by Orlicz [34]. Note that the generalized Lebesgue spaces with variable exponent are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of diferential operators, and in variational calculus [7], [8], [9], [36] and [38]. Detailed information about properties of the Lebesque spaces with variable exponent can be found in [10], [26], [30], [31], [37] and [39]. Note that, some of the fundamental problems of the approximation theory in the generalized Lebesgue spaces with variable exponent of periodic and non-periodic functions were studied and solved by Sharapudinov [39]-[44] .

A function $\omega : \mathbb{T} \to [0,\infty]$ is called a *weight function if* ω is a measurable and almost everywhere (a.e.) positive.

Let ω be a 2π periodic weight function. We denote by $L^p_\omega(\mathbb{T})$ the weighted Lebesgue space of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $f\omega^{\frac{1}{p}} \in L^p(\mathbb{T})$. For $f \in L^p_\omega(\mathbb{T})$ we set

$$\|f\|_{L^p_\omega(\mathbb{T})} := \left\|f\omega^{\frac{1}{p}}\right\|_{L^p(\mathbb{T})}.$$

$L^{p(.)}_\omega(\mathbb{T})$ stands for the class of Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $\omega f \in L^{p(.)}(\mathbb{T})$. $L^{p(.)}_\omega(\mathbb{T})$ is called the weighted Lebesgue space with variable exponent. The space $L_{\omega}^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} := \|f\omega\|_{L^{p(\cdot)}(\mathbb{T})}.$$

Its known [25] that the set of trigonometric polynomials is dense in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, if $[\omega(x)]^{p(x)}$ is integrable on $\mathbb{T}$.

Let $\mathcal{B}$ be the class of all intervals in $\mathbb{T}$. For $B \in \mathcal{B}$ we set

$$p_B := (\frac{1}{|B|}\int\limits_B \frac{1}{p(x)}dx)^{-1}.$$

For given $p \in \wp$ the class of weights ω satisfying the condition

$$\left\|\omega^{p(x)}\right\|_{A_{p(.)}} := \sup_{B\in\mathcal{B}} \frac{1}{|B|^{p_B}} \left\|\omega^{p(x)}\right\|_{L^1(B)} \left\|\frac{1}{\omega^{p(x)}}\right\|_{L^{(p'(\cdot)/p(\cdot))}(B)} < \infty$$

will be denoted by $A_{p(.)}$ [1].

We say that the variable exponent $p(x)$ satisfies *local log-Hölder continuity condition,* if there is a positive constant c_1 such that

$$\mid p(x) - p(y) \mid \leq \frac{c_1}{\log(\frac{1}{|x-y|})}, \tag{1.1}$$

for all $x, y \in \mathbb{T}$.

A function $p \in \wp$ is said to belong to the class $\wp^{\log}$, if the condition (1.1) is satisfied.

We denote by $E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})}$ the best approximation of $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} = \inf\{\| f - T_n \|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} : T_n \in \Pi_n\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let us suppose that $p \in \wp$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$, for some $p_0 \in (1, p_*)$. For

$f \in L_{\omega}^{p(.)}(\mathbb{T})$ we set

$$(\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\, dt,\ 0 < h < \pi,\ x \in \mathbb{T}.$$

If $p \in \wp^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$ and $f \in L_{\omega}^{p(.)}(\mathbb{T})$, then the shift operator ν_{h_i} is a bounded linear operator on $L_{\omega}^{p(.)}(\mathbb{T})$ [27]:

$$\|\nu_{h_i}(f)\|_{L_{\omega}^{p(.)}(\mathbb{T})} \leq c_2 \|f\|_{L_{\omega}^{p(.)}(\mathbb{T})}.$$

Let $p \in \wp$ and $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$. The function

$$\Omega_{p(.),\omega}(\delta,\ f) := \sup_{0<h\leq\delta} \|f - (\nu_h f)\|_{L_{\omega}^{p(.)}(\mathbb{T})},\ \delta > 0$$

is called the *moduli of continuity* of $f \in L_{\omega}^{p(.)}(\mathbb{T})$.

It can easily be shown that $\Omega_{p(.),\omega}(\cdot, f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_{p(.),\omega}(\delta, f) = 0,\ \Omega_{p(.),\omega}(\delta, f+g) \leq \Omega_{p(.),\omega}(\delta, f) + \Omega_{p(.),\omega}(\delta, g),\ \delta > 0$$

for $f, g \in L_{\omega}^{p(.)}(\mathbb{T})$. Note that detailed information about properties of moduli of continuity $\Omega_{p(.),\omega}(\cdot, f)$ can be found in the paper [1]. Also, moduli of this type was considered by E. A. Hadjieva [16] in Lebesgue space with Muckenhoupt A_p, $1 < p < \infty$ weight.

Let $0 < \alpha \leq 1$. The set of functions $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ such that

$$\Omega_{p(.),\omega}(f, \delta) = O(\delta^{\alpha}),\ \ \delta > 0$$

is called the *Lipschitz class* $Lip(\alpha, p(\cdot), \omega)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx) \tag{1.2}$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $a_k(f)$ are $b_k(f)$ the Fourier coefficients of the function f. The *n-th partial sum* of series (1.2) is defined, as

$$\begin{aligned} S_n(f;x) &= \frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k(f)\cos kx+b_k(f)\sin kx\right), \\ &= \sum_{k=0}^{n} Q_k(f;x). \end{aligned}$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. The sequence $\{p_n\}_0^\infty$ is called *almost monotone decreasing (increasing)*, denoted by $\{p_n\}_0^\infty \in AMDS$ ($\{p_n\}_0^\infty \in AMIS$), if there exist a constant c, depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the following inequality holds:

$$p_n \leq c p_m, \qquad (p_m \leq c p_n).$$

In proof of the main result we will use the notations

$$\Delta\beta_n := \beta_n - \beta_{n+1}, \;\; \Delta_m \beta(n,m) := \beta(n,m) - \beta(n,m+1).$$

As in [33] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$. Following [4], [35] the Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)} x_k, \; n = 1,2,...,$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the $C_\lambda-$method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. We suppose that $\{p_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (1.2), as

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{n} p_{\lambda(n)-m} s_m(f;x)$$

where $P_n := \sum_{m=0}^{n} p_m \neq 0 \;\; (n \geq 0), \;\; p_{-1} = P_{-1} = 0$. Note that in the case $p_n = 1$, $n \geq 0$, $N(f;x)$ is equal to the mean

$$\sigma_n^{\lambda}(f;x)=\frac{1}{\lambda(n)+1}\sum_{m=0}^{\lambda(n)}S_m(f;x).$$

In the present paper we study the approximation of the functions by trigonometric polynomials $N_n^{\lambda}(f;x)$ in weighted Lebesgue spaces with variable exponent. The results obtained in this work are generalization of the results [33] to the weighted Lebesgue spaces with variable exponent. Similar problems about approximations of the functions by trigonometric polynomials in the different spaces have been investigated by several authors (see, for example, [2-6], [11-15], [17-24], [28], [29], [33] and [45-47]).

Note that, in the proof of the main results we use the method as in the proof of [33]. Our main result is the following:

Theorem 1.1.
1. *Let* $p\in\wp$, $\omega^{-p_0}\in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0\in(1,p_*)$, *if* $f\in Lip(\alpha,p(.),\omega)$, $0<\alpha<1$ *and if one of the following conditions*

$$\text{(i)}\quad \{p_n\}_0^{\infty}\in AMDS$$

$$\text{(ii)}\quad \{p_n\}_0^{\infty}\in AMIS,$$

and

$$(\lambda(n)+1)p_{\lambda(n)}=O(P_{\lambda(n)}) \tag{1.3}$$

holds, then

$$\left\|f-N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}=O\left((\lambda\left(n\right))^{-\alpha}\right).$$

2. *Let* $p\in\wp$, $\omega^{-p_0}\in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0\in(1,p_*)$, *if* $f\in Lip(1,p(.),\omega)$ *and if one of the following conditions*

$$\text{(iii)}\quad \sum_{k=1}^{\lambda(n)-1}k\left|\Delta p_k\right|=O(P_{\lambda(n)})$$

$$\text{(iv)}\quad \sum_{k=0}^{\lambda(n)-1}\left|\Delta p_k\right|=O(P_{\lambda(n)}/\lambda(n)),\ \textit{and (1.3) holds,}$$

then the estimate

$$\left\| f - N_n^{\lambda}(f) \right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} = O\left((\lambda(n)) \right)^{-1}.$$

holds.

In the proof of the main result we need the following Lemmas:

Lemma 1.1. (see [19]). *Let* $p \in \wp$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$. Then for $f \in Lip(\alpha, p(.), \omega)$, $0 < \alpha \leq 1$ and $n = 1$, 2,3.... the estimate

$$\| f - S_n(f) \|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} = O(n^{-\alpha})$$

holds.

Lemma 1.2. (see [19]). *Let* $p \in \wp$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0 \in (1, p_*)$. *Then for* $f \in Lip(1, p(.), \omega)$ *and* $n = 1$, *2,3, ... the estimate*

$$\| S_n(f) - \sigma_n(f) \|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} = O(n^{-1})$$

holds.

Lemma 1.3. (see [33]). *If* $\{p_n\}_0^{\infty} \in AMDS$ *or* $\{p_n\}_0^{\infty} \in AMIS$ *and* (1.3) *holds, then*

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} = O\left((\lambda(n))^{-\alpha} P_{\lambda(n)} \right)$$

for $0 < \alpha < 1$.

2 Proofs of the main results

Proof of Theorem 1.1. We prove the cases (i) and (ii) together. It is clear that

$$N_n^{\lambda}(f;x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m} \left\{ s_m(f;x) - f(x) \right\}. \tag{2.1}$$

Then using Lemma 1.1 and Lemma 1.3 and (2.1) and condition (1.3) we have

$$\begin{aligned}
\left\| N_n^{\lambda}(f) - f \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} &\leq \frac{1}{P_{\lambda(n)}} \sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m} \left\| f - s_m(f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} \left\| f - s_m(f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&\quad + \left\| f - s_0(f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} O(m^{-\alpha}) + O(\frac{p_{\lambda(n)}}{P_{\lambda(n)}}) \\
&= O((\lambda(n))^{-\alpha}).
\end{aligned}$$

Case (iv): We suppose that $\alpha = 1$. Using Abel's transformation, we find that

$$N_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m} \left\{ s_m(f;x) - f(x) \right\} Q_m(f;x).$$

Thus we have

$$s_n^{\lambda}(f;x) - N_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} (P_{\lambda(n)} - P_{\lambda(n)-m}) \left\{ s_m(f;x) - f(x) \right\} Q_m(f;x).$$

Use of Abel's transformation leads to

$$\begin{aligned}
s_n^{\lambda}(f;x) - N_n^{\lambda}(f;x) &= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} \Delta_m (m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})) \\
&\quad \times \sum_{k=1}^{m} k Q_k(f;x) + \frac{1}{(\lambda(n)+1)} \sum_{k=1}^{\lambda(n)} k Q_k(f;x) \quad (2.2)
\end{aligned}$$

Taking account of (2.2) we have

$$\begin{aligned}\left\|s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \leq & \left|\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}))\right| \\ & \times\left\|\sum_{k=1}^{m}kQ_k(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\ & +\frac{1}{(\lambda(n)+1)}\left\|\sum_{k=1}^{\lambda(n)}kQ_k(f;x)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}. \quad (2.3)\end{aligned}$$

It is clear that

$$s_n(f,x)-\sigma_n(f;x)=\frac{1}{n+1}\sum_{k=1}^{n}kQ_k(f;x). \quad (2.4)$$

Then from Lemma 1.2 and (2.4) we have

$$\left\|\sum_{k=1}^{n}kQ_k(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}=(n+1)\left\|s_n(f)-\sigma_n(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}=O(1). \quad (2.5)$$

Thus use of (2.3) and (2.5) gives us

$$\begin{aligned}\left\|s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} = & O(\frac{1}{P_{\lambda(n)}})\sum_{m=1}^{\lambda(n)}\left|\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}))\right| \\ & +O((\lambda(n))^{-1}). \quad (2.6)\end{aligned}$$

By [33] the following relations hold :

$$\begin{aligned}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}) = & \frac{1}{m}\Delta_m(P_{\lambda(n)}-P_{\lambda(n)-m}) \\ & +\frac{P_{\lambda(n)}-P_{\lambda(n)-m-1}}{m(m+1)} \\ = & \frac{P_{\lambda(n)-m-1}-P_{\lambda(n)-m}}{m}\end{aligned}$$

$$
\begin{aligned}
& +\frac{P_{\lambda(n)}-P_{\lambda(n)-m-1}}{m(m+1)} \\
= \; & \frac{P_{\lambda(n)}-P_{\lambda(n)-m-1}}{m(m+1)}-\frac{p_{\lambda(n)-m}}{m} \\
= \; & \frac{1}{m(m+1)}\left[P_{\lambda(n)}-P_{\lambda(n)-m-1}\right] \\
& -\frac{1}{m(m+1)}(m+1)p_{\lambda(n)-m}, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_m\left(\frac{P_{\lambda(n)}-P_{\lambda(n)-m}}{m}\right) = \; & \frac{1}{m(m+1)} \\
& \times\left[\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k-(m+1)p_{\lambda(n)-m}\right]. \qquad (2.8)
\end{aligned}
$$

Next we will prove by the induction the inequality

$$
\begin{aligned}
& \left|\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k-(m+1)p_{\lambda(n)-m}\right| \\
\leq \; & \sum_{k=1}^{m} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|. \qquad (2.9)
\end{aligned}
$$

Let $m=1$. Then we obtain

$$
\left|\sum_{k=\lambda(n)-1}^{\lambda(n)} p_k-2p_{\lambda(n)-1}\right|=\left|p_{\lambda(n)}-p_{\lambda(n)-1}\right|.
$$

That is, the relation (2.9) holds, for $m=1$. Now we suppose that the relation (2.9) holds for $m=j$. We prove the inequality for $m=j+1$ $(\leq \lambda(n))$. The inequality

$$\begin{aligned}
&\left|\sum_{k=\lambda(n)-(j+1)}^{\lambda(n)} p_k-(j+2)p_{\lambda(n)-(j+1)}\right| \\
=&\left|\sum_{k=\lambda(n)-j}^{\lambda(n)} p_k-(j+1)p_{\lambda(n)-(j+1)}\right| \\
=&\left|\sum_{k=\lambda(n)-j}^{\lambda(n)} p_k-(j+1)p_{\lambda(n)-j}+(j+1)p_{\lambda(n)-j}-(j+1)p_{\lambda(n)-(j+1)}\right| \\
\leq&\left|\sum_{k=\lambda(n)-j}^{\lambda(n)} p_k-(j+1)p_{\lambda(n)-1}\right|+\left|(j+1)p_{\lambda(n)-j}-(j+1)p_{\lambda(n)-(j+1)}\right| \\
\leq&\sum_{k=1}^{j} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|+(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-(j+1)}\right| \\
=&\sum_{k=1}^{j+1} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|.
\end{aligned}$$

holds. That is, (2.9) is true for $m=j+1$. Thus the relation (2.9) is proved for any $1\leq m\leq\lambda(n)$. Consideration of (2.8) and (2.9) gives us

$$\begin{aligned}
&\sum_{m=1}^{\lambda(n)}\left|\Delta_m\left(\frac{P_{\lambda(n)}-P_{\lambda(n)-m}}{m}\right)\right| \\
\leq&\sum_{m=1}^{\lambda(n)}\frac{1}{m\,(m+1)}\sum_{k=1}^{m} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right| \\
\leq&\sum_{k=1}^{\lambda(n)} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|\sum_{m=k}^{\infty}\frac{1}{m\,(m+1)} \\
=&\sum_{k=0}^{\lambda(n)-1}\left|\Delta p_k\right|. \qquad (2.10)
\end{aligned}$$

Taking into account the condition of the Theorem 1.1 the relation, we have

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n)) \tag{2.11}$$

holds. Then taking the relations (2.10), (2.11) and (2.6) into account we get

$$\left\| s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_{L_\omega^p} = O((\lambda(n))^{-1}). \tag{2.12}$$

Thus from (2.12) and Lemma 1.1 for $\alpha = 1$ we have

$$\left\| f - N_n^{\lambda}(f) \right\|_{L_\omega^p} = O((\lambda(n))^{-1}).$$

Case (iii): First of all we prove the estimate

$$\sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) = O\left(\frac{P_{\lambda(n)}}{\lambda(n)} \right). \tag{2.13}$$

According to condition in the case (iii) of Theorem 1.1 the following relations holds:

$$\sum_{k=1}^{\lambda(n)-1} k\,|\Delta p_k| = O(P_{\lambda(n)}). \tag{2.14}$$

Consideration of (2.8) and (2.9) gives us

$$\begin{aligned} & \sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \\ \leq\ & \sum_{m=1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ =\ & \sum_{m=1}^{r} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ & + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\ :\ & = S_1 + S_2. \end{aligned} \tag{2.15}$$

Let r denote the integral part of $(\lambda(n)/2)$. Using Abel's transformation and (2.14), we find that

$$\begin{aligned} S_1 &= \sum_{m=1}^{r} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left|\Delta_k p_{\lambda(n)-k}\right| \\ &\leq \sum_{k=1}^{r} \left|\Delta_k p_{\lambda(n)-k}\right| \leq \sum_{j=r-2}^{\lambda(n)-1} |\Delta p_j| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \end{aligned} \tag{2.16}$$

For S_2, we can write the following:

$$\begin{aligned} S_2 &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left|\Delta p_{\lambda(n)-k}\right| \\ &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{r} k \left|\Delta p_{\lambda(n)-k}\right| \\ &\quad + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=r}^{m} k \left|\Delta p_{\lambda(n)-k}\right| \\ &: = S_{21} + S_{22}. \end{aligned} \tag{2.17}$$

If using again the condition (2.14) we get

$$\begin{aligned} S_{21} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{j=r-2}^{\lambda(n)-1} |\Delta p_j| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right), \\ S_{22} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{k=r}^{m} \left|\Delta p_{\lambda(n)-k}\right| \\ &= O\left(\frac{1}{\lambda(n)}\right) \left[|\Delta p_0| + 2|\Delta p_1| + ... + (r+1)|\Delta p_{r+1}|\right] \\ &= O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \end{aligned} \tag{2.18}$$

By (2.15)-(2.18) this implies that (2.13). Using (2.6), (2.13) and Lemma 1.1 we reach

$$\begin{aligned} &\left\|f - N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\ &\qquad = \left\|f - s_n^{\lambda}(f) + s_n^{\lambda}(f) - N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \end{aligned}$$

$$\leq \left\|f - s_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + \left\|s_n^{\lambda}(f) - N_n^{\lambda}(f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}$$
$$\leq O((\lambda(n))^{-1}).$$

The proof of Theorem 1.1 is completed.

Acknowledgement:
The author is grateful to the learned refree for his careful reading and useful suggestions on the manuscripts.

References

[1] R. Akgun, V. Kokilashvili, On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces, Banach J. Math. Anal. 5 (1), 70-82.

[2] R. Akgün, *Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent,* Ukrain. Math. J. 63 (1)(2011), 1-26.

[3] R. Akgün, *Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with non-standart growth*, Georgian Math. J. 18 (2) (2011), 203-235.

[4] D. H. Armitage, I. J. Maddox, *A new type of Cesáro mean,* Analysis, 9 (1-2)(1989), 195-204.

[5] H. S. Arslan, *A note on $|N, p_n^{\alpha}|$ summability factors,* Soochow J. Math., 27 (1) (2001), 45-51

[6] P. Chandra, *Trigonometric approximation of functions in L_p-norm,* J. Math. Anal. Appl., 277 (1)(2002), 13-26.

[7] D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces foundation and harmonic analysis*, Heidelberg:Springer; 2013.

[8] L. Diening and M. Ruzicka, *Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics*, Preprint 04.07.2002, Albert-Ludwings-University, Freiburg.

[9] L. Diening, P. Harjulehto, P. Hästö, Michael Ruzicka, *Lebesgue and Sobolev spaces with exponents,* Heidelberg:Springer; 2011.

[10] L. Diening, P. Hästö and A. Nekvinda, *Open problems in variable exponent and Sobolev spaces*, In: Function Spaces, Differential Operators and Nonlinear Analysis, Proc. Conf. held in Milovy, Bohemian-Moravian Uplands, May 29-June 2, 2004, Math. Inst.Acad. Sci. Czhech. Repyblic. Praha, 2005, 38-58.

[11] U. Deger, I. Dagadur and M. Kücükaslan, *Approximation by trigonometric polynomials to functions in L_p norm,* Proc. Jangjeon Math. Soc. 15 (2)(2012), 203-213

[12] V. G. Gavriljuk, *Linear summability methods for Fourier series and best approximation,* Ukrain. Math. Zh. 15 (4)(1963), 412-418 (in Russian).

[13] A. Guven, *Trigonometric,approximation of functions in weighted L^p spaces*, Sarajevo J. Math. 5 (17) (2009), 99-108.

[14] A. Guven, D. M. Israfilov, *Approximation by Means of Fourier trigonometric series in weighted Orlicz space*s, Adv. Stud. Contemp. Math. (Kyundshang), 19 (2)(2009), 283-295.

[15] A. Guven, D. M. Israfilov, *Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$*, J. Math. Inequal. 4 (2010), 285-299.

[16] E. A. Hadjieva, *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii-Besov spaces,* Author's summary of a dissertation, Tbilisi, 1986 (in Russian)

[17] N. A. Il'yasov, *Approximation of periodic functions by Zygmund means*, Mat. Zametki 39 (3)(1986), 367-382 (in Russian).

[18] D. M. Israfilov, *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,w)$ and the Bieberbach polynomials,* Constr. Approx. 17 (2001), 335-351.

[19] D. M. Israfilov, A. Testici, *Approximation by matrix transforms in weighted Lebesgue spaces with variable exponent,* Results Math.73(8)(2018), 1-25.

[20] D. M. Israfilov, A. Testici, *Approximation in Smirnov classes with variable exponent,* Complex Var. Elliptic Equ. 60 (9)(2015), 1243-1253.

[21] S. Z. Jafarov, *Approximation by rational functions in Smirnov-Orlicz classes,* J. Math. Anal. Appl. 379 (2011), 870-877.

[22] S. Z. Jafarov, *Linear methods of summing Fourier series and approximation in weighted variable exponent Lebesgue spaces,* Ukrainian Math. J. 66(10) (2015), 1509-1518.

[23] S. Z. Jafarov, *Approximation by Fejér sums of Fourier trigonometric seies in weighted Orlicz spaces,* Hacet. J. Math. Stat. 42 (3) (2013), 259-258.

[24] S.Z. Jafarov, *Linear methods of summing Fourier series and approximation in weighted Orlicz spaces,* Turk J. Math. 42 (2018), 2916-2925.

[25] V. Kokilashvili and S. G. Samko, *Singular integrals weighted Lebesgue spaces with variable exponent,* Georgian Math. J. 10 (1) (2003), 145-156.

[26] V. Kokilashvili and S. G. Samko, *Singular integrals and potentials in some Banach function spaces with variable exponent,* J. Funct. Spaces Appl. 1 (1) (2003), 45-59.

[27] V. Kokilashvili and S. G. Samko, *Operators of harmonic analysis in weighted spaces with non-standard growth*, J. Math. Anal. Appl. 352 (2009), 15-34.

[28] V. Kokilashvili and Ts. Tsanava, *On the norm estimate of deviation by linear summability means and an extension of the Bernstein inequality,* Proc. A. Razmadze Math Inst. 154 (2010), 144-146.

[29] V. Kokilashvili and Ts. Tsanava, *Approximation by linear summability means in weighted variable exponent Lebesgue spaces*, Proc. A. Razmadze Math. Inst. 154 (2010), 147-150.

[30] V. Kokilashvili, *On a progress in the theory of integral operators in weighted Banach function spaces,* In: Function spaces, Differential Operators and Nonlinear Analysis, Proc. Conf. held in Milovy, Bohemian- Moravian Uplands, May 29 - June 2, 2004, Math. Inst. Acad. Sci. Czech Republi, Praha, 2005, 152-174.

[31] O. Kováčik and J. Rákosnik, *On spaces $L^{p(x)}$and $W^{k,p(x)}$,* Czechoslovak Math. J. 41 (116) (1991), 592-618.

[32] J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer, Berlin, 1983.

[33] M. L. Mittal, Mradul Veer Singh, *Approximation of signals (functions) by trigonometric polynomials in L_p-norm,* Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences., Volume 2014, Article ID 267383, 6 pages,

[34] W. Orlicz, *Überconjugierte Exponentenfolgen*, Studia Math., 3(1931), 200-211.

[35] J. A. Osikiewicz, *Equivalence results for Cesáro submethods*, Analysis (Münich) 20 (1) (2000), 35-43.

[36] M. Ruzicka, *Elektrorheological fluids: modeling and mathematical theory*, Vol. 1748, Lecture notes in mathematics, Berlin: Springer-Verlag; 2000.

[37] S. G. Samko, *Diferentiation and integration of variable order an the spaces* $L^{p(x)}$, Operator theory for complex and hypercomplex analysis (Mexico City, 1994), 203-219, Contemp,. Math., 212, Amer. Math. Soc., Providence, RI, 1998.

[38] S. G. Samko, *On a progres in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*, Integral Transforms Spec. Funct. 16 (5-6)(2005), 461-482.

[39] I. I. Sharapudinov, *The topology of the space* $L^{p(t)}([0,1])$, Matem. Zametki, 26 (4) (1979), 613-632 (in Russian); English transl.: Math. Notes. 26 (3-4)(1979), 796-806.

[40] I. I. Sharapudinov, *Approximation of functions in the metric of the space* $L^{p(t)}([a,b])$ *and quadrature* (in Russian). Constructive function theory 81. (Varna, 1981), 189-193, Publ. House Bulgar. Acad. Sci., Sofia, 1983.

[41] I.I. Sharapudinov, *Some questions of approximation theory in the Lebesgue spaces with variable exponent*, Vladikavkaz:Russian Academy of Sciences, South Mathematical Institute; 2012.

[42] I.I. Sharapudinov, *Some questions of approximation theory in the spaces* $L^{p(x)}(E)$, Anal. Math. 33 (2007), 135-153.

[43] I.I. Sharapudinov, *Approximation of functions in* $L^{p(x)}_{2\pi}$ *by trigonometric polynomials,* Izvestiya RAN: Ser. Math. 77 (2013), 197-244. English transl. Izvestiya: Mathematics, 77 (2013), 407-434.

[44] I. I. Sharapudinov, *Approximatiıon of functions in variable-exponent Lebesgue and Sobolev spaces by finite Fourier- Haar series*, Mat. Sb. 205 (1-2)(2014), 145-160; translation in Sb. Math. 205 (2014) 291-306.

[45] S. Sonker, U. Singh, *Approximation of signals (functions)belonging to* $Lip(\alpha, p, \omega)-$*class using trigonometric polynomials*, Procedia Engineering, 38 (2012), 1575-1585.

[46] M. F. Timan, *Best approximation of a function and linear methods of summing Fourier series* , Izv. Akad. Nauk SSSR Ser: Math. 29 (1965), 587- 604 (in Russian).

[47] M. F. Timan, *The approximation of continuous periodic functions by linear operators which are constructed on the basis of their Fourier series*, Dokl. Akad. Nauk SSSR 181 (1968), 1339-1342 (in Russian).