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On the graded $^*-$rings

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Abstract

A prime ring A is called a $*-$ring if $\beta(A/I) = A/I$ for every nonzero ideal proper I of A, where β is the prime radical. Gardner in 1988 asked whether the upper radical $\mathcal{U}(*_k)$ of the essential closure $*_k$ of the class of all $*-$rings coincide with the prime radical β. Until now, this problem remains open. In this paper, we construct a graded $*-$ring that motivates a further research to bring the Gardner problem into a graded version.

1 Introduction

In this paper, R (respectively, A) will denote a ring which has identity (respectively, a ring which is not necessary to have identity). A nonzero ideal of J of a ring R

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is essential if $I \cap J \neq 0$ for every nonzero ideal I of R and it will be denoted by $I \triangleleft \circ R$.

A class γ of rings is called a radical class in the sense of Amitsur and Kurosh if γ satisfies the following property [7]:

1. $A \in \gamma \Rightarrow \forall A \to B \neq 0$, there exists an ideal C of B such that $0 \neq C \in \gamma$.
2. A is arbitrary ring and $\forall A \to B \neq 0$, there exists an ideal C of B such that $0 \neq C \in \gamma \Rightarrow A \in \gamma$.

Example 1.1. *The following examples are radical class of rings:*

1. *The class $\mathcal{N} = \{A \mid \forall a \in A$, then there exists a positive integer n such that $a^n = 0\}$ of all nil rings is a radical class.*
2. *The class $\mathcal{J} = \{A$ is a ring $|(A, \circ)$ forms a group, where $a \circ b = a + b - ab, \forall\, a, b \in A\}$ forms a radical class.*
3. *Let π be the class of all prime rings. The class $\mathcal{U}(\pi) = \{A|A$ has no nonzero homomorphic image is $\pi\}$ is a radical class. It is famously named the prime radical class and it is denoted by β.*

For any radical class γ. The symbol $\gamma(A)$ represents the largest ideal of a ring A which belongs to γ. A prime ring A is called a $*-$ring if $\beta(A/I) = A/I$ for every nonzero ideal proper I of A, where β is the prime radical. The class of all $*-$ring is denoted by $*$ [2]. The definition of $*-$ring was introduced in [11] and some further properties of $*-$rings and their implementation in radical theory can be seen in [3, 4, 5, 6].

Example 1.2. *Consider the following examples of $*-$rings.*

1. *Every field is a $*-$ring.*
2. *The set $J = \{j \in \mathbb{Q}|j$ has even numerator and odd denumerator$\}$ forms a $*-$ring.*
3. *Let W be a simple idempotent ring of characteristic 0, but with no unity. The ring W is also a $*-$ring.*
4. *Every simple ring with unity is a $*-$ring.*

On the other hand, we have also the following conditions (S1 and S2) which motivate the existence of the definition of semisimple class of rings

1. (S1) If $A \in \sigma$, then for every nonzero $B \lhd A$ there exists a nonzero homomorphic image C of B such that $C \in \sigma$.

2. (S2) If A is a ring of the universal class $\mathbb{A}$ and for every nonzero $B \lhd A$ there exists a nonzero homomorphic image C of B such that $C \in \sigma$, then A is in σ.

A class σ of rings which satisfies the condition S1 is called a regular class of rings. Furthermore, if the class σ of rings satisfies S1 and S2, σ is called a semisimple class of rings [7].

The relationship between radical class and semisimple class is described in the Proposition 1.1 below.

Proposition 1.1. *[7] Let γ be a radical class of rings. The class $S\gamma = \{A$ is a ring $|\gamma(A) = 0\}$ is the semisimple class of γ.*

Proof. Please see the Proof of Proposition 2.3.2 in [7]. □

Furthermore, for any regular class ϱ of rings. We have the following property:

Theorem 1.1. *[7] If ϱ is a regular class of rings, then the class $\mathcal{U}(\varrho) = \{A|A$ has no nonzero homomorphic image in $\varrho\}$ is a radical class, $\mathcal{U}(\varrho) \cap \varrho = \{0\}$ and $\mathcal{U}(\varrho)$ is the largest radical having zero intersection with ϱ.*

Proof. Please see the proof of Theorem 2.2.3 in [7]. □

A class μ of prime rings is called a special class of rings if μ satisfies the following properties [7]:

1. $A \in \mu \Rightarrow I \in \mu$ for every ideal I of A.

2. for every essential ideal J of R such that $I \in \mu$ implies $R \in \mu$.

Example 1.3. *The class π of all prime rings is a special class. Moreover, the upper radical $\mathcal{U}(\pi)$ is precisely the prime radical β.*

In 1988, Gardner in his paper [8] asked whether the prime radical β coincides with the upper radical $\mathcal{U}(*_k)$, where $\mathcal{U}(*_k)$ will be explained more detail in the Section 2. However, this question remains open in general. Hence, it is important to investigate this problem at least on a specific condition. In this research, we give an insight of the problem specifically on the restricted graded version for radical class of rings.

2 The structure of $'*'$

We start this section revisiting the properties and the structure of $*$.

Remark 2.1. *[2] The class $*$ is neither a radical class nor a semisimple class.*

Consider the following counter examples.

Example 2.1. *Let W be a simple idempotent ring of characteristic 0, but with no unity. It is clear that the ring J in the Example 1.2 is a $*-$ring. The ring W is also a $*-$ring. Define $R = \{(j, w)|j \in J, w \in W\}$ with $(j, w)+(k, x) = (j+k, w+x)$ and $(j, w)(k, x) = (jk, jx + kw + wx)$ for all $j, k \in J$ and $w, x \in W$. Then, R is a subdirectly irreducible ring with the heart $H(R) \cong W$ such that $R/H(R) \cong J$ and any Noetherian homomorphic image of R (other then the identical one and $R/H(R)$ is nilpotent. Therefore R is not a $*-$ring in spite of the fact that both $R/H(R) \cong J$ and $H(R)$ are a $*-$rings.*

Remark 2.2. *The existence of a simple idempotent ring of characteristic 0, but with no unity can be seen in [10] and [14].*

Example 2.2. *Let $M_\infty(R)$ be the ring of all infinite matrices which has infinite row over a ring R. In the other words, every matrix in $M_\infty(R)$ has countably infinite number of rows but almost all entries in each row are equal to 0. If R is a simple ring, then so $M_\infty(R)$ and, clearly, the center of $M_\infty(R)$ is $\{0\}$. Therefore, $M_\infty(R)$ does not contain the identity element. So, in particular, if R is any simple ring with characteristic 0, then so is $M_\infty(R)$.*

Remark 2.3. *[2] The class $*$ is not essentially closed. Therefore, the class $*$ is not a special class.*

Example 2.3. *Example 2.1 shows that the ring $R = \{(j, w)|j \in J, w \in W\}$, where $J = \{j \in \mathbb{Q}|j$ has even numerator and odd dinominator$\}$ and W is a simple idempotent ring of characteristic 0 without unity, is a subdirectly irreducible. The heart $H(R)$ of R is $W' \cong W$. So, we may deduce that the ideal W' is an essential ideal of R and R is an essential extension of W'. In fact, $W' \cong W$ is a $*-$ring. However, the ring R is not a $*-$ring by Example 2.1. Therefore, the class $*$ is not a special class.*

Let δ be any class of rings which is not essentially closed. The class $\varepsilon(\delta) = \{A|\exists B \in \delta$ such that $B \lhd \circ A\}$ is the essential cover of δ, where $B \lhd \circ A$ expresses

B is the essential ideal of A. The definition of an essential ideal is previously described in the beginning Section 1. But, the class $\varepsilon(\delta)$ is not generally closed under essential extensions. This motivates the existence of essential closure of δ.

Definition 2.1. *[9] Let $\delta^{(0)} = \delta$ and $\delta^{(i+1)} = \varepsilon(\delta^{(i)})$. The class $\delta_k = \bigcup_{i=0}^{\infty} \delta^{(i)}$ is the essential closure of δ.*

The essential closure of class of rings is always essentially closed.

Proposition 2.1. *[9] Let δ be any class of rings which is not essentially closed. The essential closure $\delta_k = \bigcup_{i=0}^{\infty} \delta^{(i)}$ of δ is essentially closed*

Proposition 2.2. *[2] Let $*$ be the class of all $*-$rings. The class $*_k = \bigcup_{i=0}^{\infty} *^{(i)}$ is the essential closure of $*$ and $*_k$ is a special class of rings.*

Definition 2.2. *[7] The upper radical $\mathcal{U}(\mu)$ of a special class μ of rings is called a special radical.*

In fact, we have both the prime radical β and the upper radical $\mathcal{U}(*_k)$ are special radical classes, where $\mathcal{U}(*_k)$ is the upper radical of the essential closure $*_k$ of the class of all $*-$rings. Hence, it is important to scrutinize the properties of $\mathcal{U}(*_k)$ and the structure of $*-$ring especially for restricted graded version.

3 Graded $*$-rings

Since the question which asked whether $\beta = \mathcal{U}(*_k)$ is still open, we will be trying to bring this open problem to the graded version. We start from the following definition.

Let $\mathbb{Z}$ be the set of all integers. A ring R is called a $\mathbb{Z}-$graded if there is a family of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R which satisfies

1. $R = \oplus_n R_n$, and
2. $R_n R_m \subseteq R_{n+m}, \forall n, m \in \mathbb{Z}$.

In general, we have the following definition.

Definition 3.1. *[1] Let G be an additive group. A ring R is called a $G-$graded ring if $R = \oplus_{g\in G} R_g$ where the set $\{R_g | g \in G\}$ is the additive subgroups of R such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.*

Example 3.1. *For every ring A. The polynomial ring $A(x)$ over A is a graded ring by its degree.*

Further implementation of graded rings in radical theory motivates the restricted graded radical class. In specific case, restricted graded Jacobson radical class can be explored in [13].

Definition 3.2. *A ring A is called a graded $*-$ring if $A = \oplus_{g\in G} A_g$, where the set $\{A_g | g \in G\}$ is the additive subgroups of A such that $A_g R_h \subseteq A_{gh}$ for all $g, h \in G$ and it is a $*-$ring or simply A is a graded ring which is a $*-$ring.*

Remark 3.1. *It is clear that every polynomial ring $R(x)$ over a ring R can be seen as a graded ring by its degree. Furthermore, if R is a semiprime ring, then so is $R(x)$. Now, a natural question asks whether the polynomial ring $R(x)$ over $*-$ring R is also a $*-$ring?.*

The answer of the question described in the Remark 3.1 will be explained in the following proposition.

Proposition 3.1. *Let R be a $*-$ring with unity. The polynomial ring $R(x)$ is not a $*-$ring.*

Proof. Let R be a ring with unity which is $*-$ring. Then it follows from the definition of $*-$ring that R is a prime ring and $\beta(R/I) = R/I$ for every nonzero ideal proper I of R. Now, suppose $R(x)$ is the polynomial ring over R. It follows from [12] that $R(x)/\langle x\rangle \cong R$ which implies that $\beta(R(x)/\langle x\rangle) = 0$ since, R is a prime ring. Thus, $R(x)$ is not a $*-$ring. □

So, we do not have any natural graded $*-$rings constructed by polynomial rings.

In this part, we provide a graded $*-$ring.

Theorem 3.1. *Let $\mathbb{R}$ be set of all real numbers. Then the ring $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices over $\mathbb{R}$ is a graded $*-$ring.*

Proof. Let $\mathbb{R}$ be set of all real numbers and let δ be the class of all simple rings with unity. Then $R \in \delta$. It follows from Example 3.8.14 in [7] that δ is a special class of rings which implies δ consists of prime rings. Thus, the ring R is a prime ring. It is clear that R is a $*-$ring since R is simple. Furthermore, it follows from Example 11.2.15 in [12] that the ring $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices over $\mathbb{R}$ is a simple ring with unity. Thus $M_{2\times 2}(\mathbb{R}) \in \delta$. Hence, $M_{2\times 2}(\mathbb{R})$ is a $*-$ring. □

In fact, the speciality of the class π of all prime rings implies that the essential closure π_k of π is the class π itself since π is closed under essential extension. A radical class γ is said to have the Amitsur property if $\gamma(A(x)) = (\gamma(A))(x)$ for every ring A [5]. Furthermore, it follows from the Amitsur property of β that if A is a prime ring, then the polynomial $A(x) \in S\beta$, where $S\beta$ is the semisimple class of the prime radical β. We continue to compare with property of the class $*$ of all $*-$rings.

Proposition 3.2. *The class $*$ of all $*-$rings is properly contained in the semisimple class $S\mathcal{U}(*_k)$ of $\mathcal{U}(*_k)$.*

Proof. It is clear that $*$ is contained in $*_k$. Now let R be any $*-$ring. This means that $R \in *$. It follows from Proposition 3.1 that the polynomial ring $R(x)$ is not a $*-$ring. Thus, $R(x) \notin *$. We will show that $R(x) \in S\mathcal{U}(*_k)$.

Since $R \in *, R \in *_k$. It follows from Corollary 13 in [5] that $R(x) \in S\mathcal{U}(*_k)$. This implies $*$ is properly contained in $S\mathcal{U}(*_k)$ which ends the proof. □

Now, we shall follow the construction of the graded radical in $G-$category introduced by [1]. Let G be any arbitrary group and let A be a $G-$graded ring. The graded prime radical of A is denoted by $\beta_G(A)$ and it is defined as the intersection of all the $G-$graded prime ideals of A. Furthermore, the class of rings

$$\beta_G = \{A|A \text{ is a } G-\text{graded and } \beta_G(A) = A\}$$

is the graded prime radical class determined by the class of $G-$graded prime rings [1].

Definition 3.3. *[1] A nonempty class μ_G of $G-$graded rings is called a $G-$graded special class if satisfies the following conditions:*

1. *γ_G consists of $G-$graded prime rings.*
2. *If $A \in \mu_G$, then every nonzero graded ideal I of A is in μ_G.*
3. *If A is any $G-$graded ring such that there exists a graded two-sided essential ideal I of A and $I \in \mu_G$ implies $A \in \mu$.*

Let G be any group. Examples of $G-$graded special class of rings can be seen in the Example 3.2 and the Example 3.3 below.

Example 3.2. *Let G be any group. It is clear that the class π_G of all $G-$graded prime rings is a $G-$graded special class of rings.*

It is clear that the class π_G of all $G-$graded prime rings is the largest $G-$graded special class of rings. The Example 3.3 explains a specific example of $G-$graded special class of rings which is properly contained in the class π_G of all $G-$graded prime rings.

Example 3.3. *Define $(*_k)_G = \{A|A$ is a $G-$graded ring such that $A^u \in *_k\}$, where $*_k$ is the essential closure of the class of all $*-$rings which is previously described in the Proposition 2.2. Then class $(*_k)_G$ is graded special.*

Remark 3.2. *In specific case, when $G = \mathbb{Z}_4$, the class $(*_k)_{\mathbb{Z}_4}$ is not an empty set since $M_{2\times 2}(\mathbb{R})$ is contained in $(*_k)_{\mathbb{Z}_4}$. In general, we can follows the construction of graded special radical [1] which is the graded upper radical of graded special class. Hence, for any group G. The graded radical $\mathcal{U}((*_k)_G) = \{A|A$ is $G-$graded and no nonzero graded homomorphic image of A in $(*_k)_G\}$.*

We end this paper by arising the following question.

Question 1. *For any group G, whether β_G coincide with $\mathcal{U}((*_k)_G)$?, where β_G is the $G-$graded prime radical and $\mathcal{U}((*_k)_G) = \{A|A$ is a $G-$graded ring and A has no nonzero homomorphic image in $(*_k)_G$ (the class which is previously described in the Example 3.3 and Remark 3.2.*

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