The Aligarh Bulletin of Mathematics
Volume 41, Numbers 1 (2022), 27-40

ISSN: 0304-9787

Strong commutativity preserving endomorphisms in prime rings with involution

Nadeem Ahmad Dar[1], **Cihat Abdioglu**[2*], **Shakir Ali**[3] **and Vaishali Varshney**[4]
[1]Government HSS, Kaprin, Shopian-192231
Jammu & Kashmir, India
[2]Department of Mathematics and Science Education
Karamanoglu Mehmetbey University
Karaman, Turkey.
[3,4]Department of Mathematics, Aligarh Muslim University
Aligarh -202002, India.
Email: ndmdarlajurah@gmail.com, cabdioglu@kmu.edu.tr,
shakir.ali.mm@amu.ac.in, vaishalivarshney7625@gmail.com

(Received: April 3, 2022, Accepted: June 5, 2022)

Abstract

Let $\mathscr{R}$ be a noncommutative prime ring with involution of the second kind and $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ be the set of symmetric and skew symmetric elements of $\mathscr{R}$. The aim of the present paper is to show that every strong commutativity preserving endomorphism on $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ is strong commutativity preserving on $\mathscr{R}$.

1. Introduction

Let $\mathscr{R}$ be a ring with centre $\mathcal{Z}(\mathscr{R})$. The symbol $[x, y] = xy - yx$ denotes the commutator of $x, y \in \mathscr{R}$. A mapping $\phi : \mathscr{R} \to \mathscr{R}$ preserves commutativity

[0]**Keywords and phrases :** Prime ring, endomorphism, involution
[0]**2010 AMS Subject Classification :** 16N60, 16W10, 16W25
[0*]**Corresponding Author**

if $[\phi(x), \phi(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in \mathscr{R}$. The commutativity preserving maps has been studied intensively in matrix theory, operator theory and ring theory (see [5, 11]). Following [4], let $\mathcal{S}$ be a subset of $\mathscr{R}$, a map $\phi : \mathscr{R} \to \mathscr{R}$ is said to be strong commutativity preserving (SCP) on $\mathcal{S}$ if $[\phi(x), \phi(y)] = [x, y]$ for all $x, y \in \mathcal{S}$. In the course of time several techniques have been developed to investigate the behaviour of strong commutativity preserving maps using restrictions on polynomials invoking derivations, generalized derivations etcetera.
In [3], Bell and Daif investigated the commutativity in rings admitting a derivation which is strong commutativity preserving on a nonzero right ideal. More precisely, they proved that if a semiprime ring $\mathscr{R}$ admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of $\mathscr{R}$, then $I \subseteq \mathcal{Z}(\mathscr{R})$. In particular, $\mathscr{R}$ is commutative if $I = \mathscr{R}$. Later, Deng and Ashraf [8] proved that if there exists a derivation d of a semiprime ring $\mathscr{R}$ and a map $f : I \to \mathscr{R}$ defined on a nonzero ideal I of $\mathscr{R}$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then $\mathscr{R}$ contains a nonzero central ideal. Thus, $\mathscr{R}$ is commutative in the special case when $I = \mathscr{R}$. Further Al and Huang [2] showed that if $\mathscr{R}$ is a 2-torsion free semi prime ring and d is a derivation of $\mathscr{R}$ satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of $\mathscr{R}$, then $\mathscr{R}$ contains a nonzero central ideal. Many other results in this direction can be found in [1, 5–7, 9] and references therein.
Recall that a ring $\mathscr{R}$ is called $*$-ring or ring with involution if there is an additive map $* : \mathscr{R} \to \mathscr{R}$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in \mathscr{R}$. Let $\mathcal{H}(\mathscr{R}) = \{x \in \mathscr{R} | x^* = x\}$ and $\mathcal{S}(\mathscr{R}) = \{x \in \mathscr{R} | x^* = x\}$ denote the set of symmetric and skew symmetric elements of $\mathscr{R}$. The involution is said to be of the first kind if $\mathcal{Z}(\mathscr{R}) \subseteq \mathcal{H}(\mathscr{R})$, otherwise it is said to be of the second kind. In the later case, $\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$ (e.g. involution in the case of ring of quaternions).

One can observe that every strong commutativity preserving endomorphism on $\mathscr{R}$ is strong commutativity preserving on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$ but the converse is not true in general (see Example 3.1). Now if we take the statement that an endomorphism θ is strong commutativity preserving on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$, does it follow that θ is strong commutativity preserving on $\mathscr{R}$. The answer is obviously affirmative in case $\mathscr{R}$ is commutative or θ is the identity map. However some restrictions must certainly be imposed here for the answer is negative in case of noncommutative rings, if θ is not the identity map. Thus we will assume that $\mathscr{R}$ is a 2-torsion free noncommutative prime ring with involution of the second kind and $\theta \neq I$, the identity map in order that the said question makes sense.

2. Preliminary Results

In the present section, we present following facts which are very crucial for developing the proofs of our main results.

Fact 2.1. *If the involution is of the second kind, then $\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$. which indeed implies that $\mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$.*

Fact 2.2. *Let $\mathscr{R}$ be a 2-torsion free prime ring with involution of the second kind. Then every $x \in \mathscr{R}$ can uniquely be represented as $2x = h + k$, where $h \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$.*

Fact 2.3. *Let $\mathscr{R}$ be a 2-torsion free prime ring with involution of the second kind such that*
(1) If $[h, h^{'}] = 0$ for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$, then $\mathscr{R}$ is commutative.
(2) If $[k, k^{'}] = 0$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$, then $\mathscr{R}$ is commutative.

Proof. (1) Suppose that $[h, h^{'}] = 0$. Replacing h by kk_0, with $k \in \mathcal{S}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we obtain $[k, h^{'}]k_0 = 0$ for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$, which because of primeness yields that $[k, h^{'}] = 0$ for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2, we obtain $2[x, h^{'}] = [2x, h^{'}] = [h+k, h^{'}] = [h, h^{'}] + [k, h^{'}] = 0$. Hence $[x, h^{'}] = 0$ for all $x \in \mathscr{R}$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Again replacing $h^{'}$ by $k^{'}k_0$, where $k^{'} \in \mathcal{S}(\mathscr{R})$, $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we find that $[x, k^{'}] = 0$ for all $x \in R$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again making use of Fact 2.2, we finally arrive at $[x, y] = 0$ for all $x, y \in \mathscr{R}$. Hence $\mathscr{R}$ is commutative.
(2) Assume that $[k, k^{'}] = 0$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we obtain $[h, k^{'}]k_0 = 0$ for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$, which because of primeness yields that $[h, k^{'}] = 0$ for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2, we obtain $2[x, k^{'}] = [2x, k^{'}] = [h + k, k^{'}] = [h, k^{'}] + [k, k^{'}] = 0$. Hence $[x, k^{'}] = 0$ for all $x \in \mathscr{R}$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again replacing $k^{'}$ by $h^{'}k_0$, where $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we find that $[x, h^{'}] = 0$ for all $x \in R$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Again making use of Fact 2.2, we finally arrive at $[x, y] = 0$ for all $x, y \in \mathscr{R}$. Hence $\mathscr{R}$ is commutative. □

Fact 2.4. *Let $\mathscr{R}$ be a 2-torsion free noncommutative prime ring with involution of the second kind.*
(1) If $a[h,h^{'}]b=0$ for all $h,h^{'}\in\mathcal{H}(\mathscr{R})$, $a,b\in\mathscr{R}$, then $a=0$ or $b=0$.
(2)If $a[k,k^{'}]b=0$ for all $k,k^{'}\in\mathcal{S}(\mathscr{R})$, $a,b\in\mathscr{R}$, then $a=0$ or $b=0$.

Proof. (1) Suppose that $a[h,h^{'}]b=0$ for all $h,h^{'}\in\mathcal{H}(\mathscr{R})$. Arguing on similar lines as in the proof of Fact 2.3, we obtain $a[x,y]b=0$ for all $x,y\in\mathscr{R}$. Substituting yb for y, we get $ay[x,b]b=0$ so that $a=0$ or $b\in\mathcal{Z}(\mathscr{R})$. In the later case, our hypothesis leads to $a=0$ or $b=0$.
(2) Suppose that $a[k,k^{'}]b=0$ for all $k,k^{'}\in\mathcal{S}(\mathscr{R})$. Again arguing on similar lines as in the proof of Fact 2.3, we obtain $a[x,y]b=0$ for all $x,y\in\mathscr{R}$. Hence $a=0$ or $b=0$ as shown above. □

Fact 2.5. *Let $\mathscr{R}$ be a 2-torsion free prime ring. If $[[a,y],a]=0$ for all $y\in\mathscr{R}$, then $a\in\mathcal{Z}(\mathscr{R})$.*

Proof. Let $a\in\mathscr{R}$ is such that $[a,[a,y]]=0$ for all $y\in\mathscr{R}$. First applying $2[a,y][a,x]=[a,[a,yx]]-y[a,[a,x]]-[a,[a,y]]x$, we conclude that $[a,y][a,x]=0$ for all $y,x\in\mathscr{R}$. Replacing x by xy in this identity and using $[a,xy]=[a,x]y+x[a,y]$, we get $[a,y]\mathscr{R}[a,y]=0$ for all $y\in\mathscr{R}$. Thus $[a,y]=0$ by the primeness of $\mathscr{R}$. □

3. When θ is SCP on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$

We begin this section with the following examples which show that a strong commutativity preserving endomorphism on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ need not be strong commutativity preserving on $\mathscr{R}$.

Example 3.1. *Let $\mathscr{R}=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \Bigm| a,b,c,d\in\mathbb{Q}\right\}$. Of course, $\mathscr{R}$ with matrix addition and matrix multiplication is a prime ring. Define $*:\mathscr{R}\longrightarrow\mathscr{R}$ such that $\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{*}=\begin{pmatrix} d & b\\ c & a\end{pmatrix}$. Let $\mathcal{S}(\mathscr{R})$ be the set of skew symmetric elements of $\mathscr{R}$. If $\theta:\mathscr{R}\to\mathscr{R}$ is an inner automorphism of $\mathscr{R}$ defined by $\theta(X)=PXP^{-1}$, where $P=\begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix}$. Clearly $\theta(K)=K$ for all $K\in\mathcal{S}(\mathscr{R})$. Thus one can easily see that $[\theta(K),\theta(K^{'})]=[K,K^{'}]$*

for all $K, K^{'} \in \mathcal{S}(\mathscr{R})$. *But* $[\theta(X), \theta(Y)] \neq [X, Y]$ *for all* $X, Y \in \mathscr{R}$. *For instance if* $X = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, *It is easy to verify that* $[\theta(X), \theta(Y)] \neq [X, Y]$.

Further if we define the involution $* : \mathscr{R} \longrightarrow \mathscr{R}$ *such that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. *Then clearly* $\theta(H) = H$ *for all* $H \in \mathcal{H}(\mathscr{R})$, *symmetric elements of* $\mathscr{R}$. *Hence* $[\theta(H), \theta(H^{'})] = [H, H^{'}]$ *for all* $H, H^{'} \in \mathcal{H}(\mathscr{R})$. *But again* $[\theta(X), \theta(Y)] \neq [X, Y]$ *for all* $X, Y \in \mathscr{R}$.

Regarding the converse part, we prove the following result.

Theorem 3.1. *Let* $\mathscr{R}$ *be a 2-torsion free noncommutative prime ring with involution of the second kind. If* θ *is a nontrivial endomorphism of* $\mathscr{R}$, *then the following assertions are equivalent;*
(1) $[\theta(h), \theta(h^{'})] = [h, h^{'}]$ *for all* $h, h^{'} \in \mathcal{H}(\mathscr{R})$;
(2) $[\theta(k), \theta(k^{'})] = [k, k^{'}]$ *for all* $k, k^{'} \in \mathcal{S}(\mathscr{R})$;
(3) $[\theta(x), \theta(y)] = [x, y]$ *for all* $x, y \in \mathscr{R}$.

Proof. It is obvious that (3) implies both (1) and (2). Hence we need to prove that (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3).
(1) $\Longrightarrow$ (3) Suppose that

$$[\theta(h), \theta(h^{'})] - [h, h^{'}] = 0 \tag{3.1}$$

for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. Replacing h by hh_0, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, we obtain

$$[\theta(h), \theta(h^{'})]\theta(h_0) - [h, h^{'}]h_0 = 0 \tag{3.2}$$

for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. Right multiplying (3.1) by $\theta(h_0)$, we have

$$[\theta(h), \theta(h^{'})]\theta(h_0) - [h, h^{'}]\theta(h_0) = 0. \tag{3.3}$$

On comparing equations (3.2) and (3.3) one can easily see that

$$[h,h^{'}](\theta(h_0)-h_0)=0 \tag{3.4}$$

for all $h,h^{'}\in\mathcal{H}(\mathscr{R})$. Since $\theta(h_0)\in\mathcal{Z}(\theta(\mathscr{R}))$, the above equation implies that

$$[h,h^{'}]\theta(\mathscr{R})(\theta(h_0)-h_0)=0 \tag{3.5}$$

In particular

$$[h,h^{'}][\theta(u),\theta(v)](\theta(h_0)-h_0)=0 \tag{3.6}$$

for all $h,h^{'},u,v\in\mathcal{H}(\mathscr{R})$. Thus our hypothesis forces that

$$[h,h^{'}][u,v](\theta(h_0)-h_0)=0 \tag{3.7}$$

for all $h,h^{'},u,v\in\mathcal{H}(\mathscr{R})$. Applying Fact 2.4, we get either $[h,h^{'}]=0$ for all $h,h^{'}\in\mathcal{H}(\mathscr{R})$ or $\theta(h_0)=h_0$ for all $h_0\in\mathcal{H}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$. Now $[h,h^{'}]=0$ forces $\mathscr{R}$ to be commutative in view of Fact 2.3, which leads us to contradiction.
So $\theta(h_0)=h_0$ for all $h_0\in\mathcal{H}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$; hence $\theta(k_0^2)=k_0^2$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$, therefore $(\theta(k_0)+k_0)(\theta(k_0)-k_0)=0$. This implies that $(\theta(k_0)+k_0)\theta(\mathscr{R})(\theta(k_0)-k_0)=0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$. In particular $(\theta(k_0)+k_0)[\theta(u),\theta(v)](\theta(k_0)-k_0)=0$ for all $u,v\in\mathcal{H}(\mathscr{R})$. Making use of our hypothesis, we obtain $(\theta(k_0)+k_0)[u,v](\theta(k_0)-k_0)=0$ for all $u,v\in\mathcal{H}(\mathscr{R})$. Thus invoking Fact 2.4, it follows that either $\theta(k_0)=k_0$ or $\theta(k_0)=-k_0$. Using Brauer's trick, we conclude that $\theta(k_0)=k_0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$ or $\theta(k_0)=-k_0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$.
Suppose $\theta(k_0)=-k_0$ for all $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})$. Replacing h by kk_0, where $k\in\mathcal{S}(\mathscr{R})$ and $k_0\in\mathcal{S}(\mathscr{R})\cap\mathcal{Z}(\mathscr{R})/\{0\}$ in (3.1), we obtain $([\theta(k),\theta(h^{'})]+[k,h^{'}])k_0=0$. Thus primeness of $\mathscr{R}$ forces that

$$[\theta(k),\theta(h^{'})]+[k,h^{'}]=0 \tag{3.8}$$

for all $h^{'}\in\mathcal{H}(\mathscr{R})$ and $k\in\mathcal{S}(\mathscr{R})$. Now since for $x\in\mathscr{R}$, $x+x^*\in\mathcal{H}(\mathscr{R})$ and $x-x^*\in\mathcal{S}(\mathscr{R})$, one can easily derive from equation (3.8) that

$$[\theta(x),\theta(x^*)]+[x,x^*]=0 \tag{3.9}$$

for all $x \in \mathscr{R}$. Linearizing equation (3.9), one can find that

$$[\theta(x), \theta(y^*)] + [\theta(y), \theta(x^*)] + [x, y^*] + [y, x^*] = 0 \quad (3.10)$$

for all $x, y \in \mathscr{R}$. Substituting yk_0 for y in (3.10), where $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we have $([\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] - [x, y^*] + [y, x^*])k_0 = 0$ which leads us to

$$[\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] - [x, y^*] + [y, x^*] = 0 \quad (3.11)$$

for all $x, y \in \mathscr{R}$. Combining equation (3.10) with (3.11), we get $[\theta(x), \theta(y^*)] = [x^*, y]$, which further implies that $\theta[x, y] = [y, x]^*$ for all $x, y \in \mathscr{R}$. Replacing y by yx, we obtain $[y, x]^*\theta(x) = x^*[y, x^*]$ for all $x, y \in \mathscr{R}$. Taking $x = [r, s]$, where $r, s \in \mathscr{R}$, one can verify that $[s, r][y, [r, s]] = [y, [r, s]][r, s]$ for all $r, s, y \in \mathscr{R}$. Thus obtaining $[[r, s], y] \circ [r, s] = 0$ for all $r, s, y \in \mathscr{R}$. This further implies that $[[r, s]^2, y] = 0$ for all $r, s, y \in \mathscr{R}$ and thus $[r, s]^2 \in \mathcal{Z}(\mathscr{R})$ for all $r, s \in \mathscr{R}$. On linearizing one can see that

$$[r, s][r, t] + [r, t][r, s] \in \mathcal{Z}(\mathscr{R}) \quad (3.12)$$

for all $r, s, t \in \mathscr{R}$. If $d_r(x) = [r, x]$, then d_r is an inner derivation and $d_r(s) \circ d_r(t) \in \mathcal{Z}(\mathscr{R})$. Thus in view of [[10], Corollary 3.6], either $\mathscr{R}$ is commutative or $d_r = 0$, which again implies commutativity of $\mathscr{R}$, a contradiction.
Therefore we have $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Substituting kk_0, where $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ for h in equation (3.1), we get $[\theta(k), \theta(h^{'})]\theta(k_0) - [k, h^{'}]k_0 = 0$. This gives $([\theta(k), \theta(h^{'})] - [k, h^{'}])k_0 = 0$. Using the primeness of $\mathscr{R}$, we obtain

$$[\theta(k), \theta(h^{'})] - [k, h^{'}] = 0 \quad (3.13)$$

for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2 and using equations (3.1) and (3.13), we find that

$$\begin{aligned} 2([\theta(x), \theta(h^{'})] - [x, h^{'}]) &= [\theta(2x), \theta(h^{'})] - [2x, h^{'}] \quad (3.14) \\ &= [\theta(h + k), \theta(h^{'})] - [h + k, h^{'}] \\ &= [\theta(h), \theta(h^{'})] + [\theta(k), \theta(h^{'})] - [h, h^{'}] - [k, h^{'}] \\ &= 0. \end{aligned}$$

Using 2-torsion freeness of R, we have

$$[\theta(x), \theta(h^{'})] - [x, h^{'}] = 0 \tag{3.15}$$

for all $x \in \mathscr{R}$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Again replacing $h^{'}$ by $k^{'}k_0$, where $k^{'} \in \mathcal{S}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$, we get

$$[\theta(x), \theta(k^{'})] - [x, k^{'}] = 0 \tag{3.16}$$

for all $x \in \mathscr{R}$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Thus invoking Fact 2.2 and using (3.15) and (3.16), one can easily derive that $[\theta(x), \theta(y)] = [x, y]$ for all $x, y \in \mathscr{R}$, as desired.

(2) $\implies$ (3) Assume that

$$[\theta(k), \theta(k^{'})] - [k, k^{'}] = 0 \tag{3.17}$$

for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$. Replacing k by kh_0, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ and proceeding on similar lines as in the first case, one can easily find that $[k, k^{'}](\theta(h_0) - h_0) = 0$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$ and $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Since $\theta(h_0) \in \mathcal{Z}(\theta(\mathscr{R}))$, the above equation implies that $[k, k^{'}]\theta(\mathscr{R})(\theta(h_0) - h_0) = 0$. In particular $[k, k^{'}][\theta(u), \theta(v)](\theta(h_0) - h_0) = 0$ for all $k, k^{'}, u, v \in \mathcal{S}(\mathscr{R})$. Thus our hypothesis forces that

$$[k, k^{'}][u, v](\theta(h_0) - h_0) = 0 \tag{3.18}$$

for all $k, k^{'}, u, v \in \mathcal{S}(\mathscr{R})$. Applying Fact 2.4, we get either $[k, k^{'}] = 0$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$ or $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Now $[k, k^{'}] = 0$ implies $\mathscr{R}$ is commutative in view of Fact 2.3, a contradiction.
So $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. This gives $\theta(k_0^2) = k_0^2$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, therefore $(\theta(k_0) + k_0)(\theta(k_0) - k_0) = 0$. This implies that $(\theta(k_0) + k_0)\theta(\mathscr{R})(\theta(k_0) - k_0) = 0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. In particular $(\theta(k_0)+k_0)[\theta(u), \theta(v)](\theta(k_0)-k_0) = 0$ for all $u, v \in \mathcal{S}(\mathscr{R})$. Making use of our hypothesis, we obtain $(\theta(k_0) + k_0)[u, v](\theta(k_0) - k_0) = 0$ for all $u, v \in \mathcal{S}(\mathscr{R})$. Thus invoking Fact 2.4 again, it follows that either $\theta(k_0) = k_0$ or $\theta(k_0) = -k_0$. Using Brauer's trick, we conclude that $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ or $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$.

If $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ in (3.17), we obtain

$$[\theta(h), \theta(k^{'})] + [h, k^{'}] = 0 \tag{3.19}$$

for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. For $x \in \mathscr{R}$, $x + x^* \in \mathcal{H}(\mathscr{R})$ and $x - x^* \in \mathcal{S}(\mathscr{R})$, therefore one can easily derive from equation (3.19) that

$$[\theta(x), \theta(x^*)] + [x, x^*] = 0 \tag{3.20}$$

for all $x \in \mathscr{R}$ which is same as equation (3.9), thus on similar lines one can get $\mathscr{R}$ is commutative, a contradiction.
Therefore, we have $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ in equation (3.17), we obtain

$$[\theta(h), \theta(k^{'})] - [h, k^{'}] = 0 \tag{3.21}$$

for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Invoking Fact 2.2 and making use of the equations (3.17) and (3.21), we find that

$$[\theta(x), \theta(k^{'})] - [x, k^{'}] = 0 \tag{3.22}$$

for all $x \in \mathscr{R}$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again replacing $k^{'}$ by $h^{'}k_0$, where $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ and arguing as above, one can find that

$$[\theta(x), \theta(h^{'})] - [x, h^{'}] = 0 \tag{3.23}$$

for all $x \in \mathscr{R}$ and $h^{'} \in \mathcal{H}(\mathscr{R})$. Thus in view of Fact 2.2 and equations (3.22) and (3.23), one can obtain $[\theta(x), \theta(y)] = [x, y]$ for all $x, y \in \mathscr{R}$. Thus the proof is complete. □

In view of the the above result and Theorem 1 [6] ,we have the following corollary:

Corollary 3.1. *Let $\mathscr{R}$ be a 2-torsion free noncommutative prime ring with involution of the second kind. Let $\mathcal{S}$ be the set of symmetric elements of $\mathscr{R}$. Suppose $\theta : \mathcal{S} \to \mathscr{R}$ is a nontrivial endomorphism such that $[\theta(x), \theta(y)] = [x, y]$ for all $x, y \in \mathcal{S}$, then $\theta(x) = \lambda x + \mu(x)$ where $\lambda \in \mathscr{C}$, $\lambda^2 = 1$ and μ is an additive map of $\mathscr{R}$ into $\mathscr{C}$.*

4. When θ is SSCP on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$

In [2], Ali and Huang established that if $\mathscr{R}$ is a 2-torsion free semiprime ring and d is a derivation of $\mathscr{R}$ such that $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of $\mathscr{R}$, then $\mathscr{R}$ contains a nonzero central ideal. To be more general in the class of such mappings. We call a mapping $f : \mathscr{R} \to \mathscr{R}$ strong skew-commutativity preserving (SSCP) if $[f(x), f(y)] = -[x, y]$ for all $x, y \in \mathscr{R}$. Here again one can observe that every SSCP endomorphism θ on $\mathscr{R}$ is SSCP on the subsets $\mathcal{H}(\mathscr{R})$ and $\mathcal{S}(\mathscr{R})$ of $\mathscr{R}$. But the converse is not true in general.

Example 4.1. *Let $\mathscr{R}$ be the ring of real quaternions. If we define $* : R \longrightarrow R$ by $(\alpha+\beta i+\gamma j+\delta k)^* = \alpha-\beta i+\gamma j+\delta k$. Let $\mathcal{S}(\mathscr{R})$ be the set of skew symmetric elements of $\mathscr{R}$. Clearly one can see that all skew symmetric elements commute with one another. Therefore if θ is any non trivial endomorphism of $\mathscr{R}$, the condition $[\theta(k), \theta(k^{'})] = -[k, k^{'}]$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$ holds. However $[\theta(x), \theta(y)] \neq -[x, y]$ for all $x, y \in \mathscr{R}$, because $\mathscr{R}$ is non commutative.*

However if $*$ is the usual conjugation $(\alpha+\beta i+\gamma j+\delta k)^* = \alpha-\beta i-\gamma j-\delta k$, all symmetric elements are central and hence the property $[\theta(h), \theta(h^{'})] = -[h, h^{'}]$ for all symmetric elements $h, h^{'}$ holds. However $[\theta(x), \theta(y)] \neq -[x, y]$ for all $x, y \in \mathscr{R}$,

Again one can observe that if $\mathscr{R}$ is commutative, then the converse is also true. Moreover, if $\theta = I$, the identity map, then in our case $[\theta(h), \theta(h^{'})] = -[h, h^{'}]$ implies that $[h, h^{'}] = 0$ for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. That is, $\mathscr{R}$ is commutative in view of Lemma 2.3. Hence we will again assume $\mathscr{R}$ is a 2-torsion free noncommutative prime ring with involution of the second kind and $\theta \neq I$, the identity map.

Theorem 4.1. *Let $\mathscr{R}$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If θ is a nontrivial endomorphism of $\mathscr{R}$, then the following assertions are equivalent;*
(1) $[\theta(h), \theta(h^{'})] = -[h, h^{'}]$ for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$;
(2) $[\theta(k), \theta(k^{'})] = -[k, k^{'}]$ for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$;
(3) $[\theta(x), \theta(y)] = -[x, y]$ for all $x, y \in \mathscr{R}$.

Proof. Clearly (3) implies both (1) and (2). Hence we need to prove that (1) $\implies$ (3) and (2) $\implies$ (3).

(1) $\implies$ (3) Suppose that

$$[\theta(h), \theta(h^{'})] + [h, h^{'}] = 0 \tag{4.1}$$

for all $h, h^{'} \in \mathcal{H}(\mathscr{R})$. Replacing h by hh_0, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ and reasoning as in the case of Theorem 3.1, we obtain $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. This further implies that $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ or $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Suppose that $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing h by kk_0, where $k \in \mathcal{S}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ in (4.1), we obtain

$$[\theta(k), \theta(h^{'})] - [k, h^{'}] = 0 \tag{4.2}$$

for all $h^{'} \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Now for $x \in \mathscr{R}$, $x + x^* \in \mathcal{H}(\mathscr{R})$ and $x - x^* \in \mathcal{S}(\mathscr{R})$, therefore equation (4.2) leads us to

$$[\theta(x), \theta(x^*)] - [x, x^*] = 0 \tag{4.3}$$

for all $x \in \mathscr{R}$. Linearizing (4.3), we find that

$$[\theta(x), \theta(y^*)] + [\theta(y), \theta(x^*)] - [x, y^*] - [y, x^*] = 0 \tag{4.4}$$

for all $x, y \in \mathscr{R}$. Substituting yk_0 for y in (4.4), where $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ and using $\theta(k_0) = -k_0$, we have $([\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] + [x, y^*] - [y, x^*])k_0 = 0$ for all $x, y \in \mathscr{R}$ which proves that

$$[\theta(x), \theta(y^*)] - [\theta(y), \theta(x^*)] + [x, y^*] - [y, x^*] = 0 \tag{4.5}$$

for all $x, y \in \mathscr{R}$. Comparing equations (4.4) and (4.5), it follows that $[\theta(x), \theta(y)] - [y^*, x^*] = 0$. Hence $\theta[x, y] = [x, y]^*$ for all $x, y \in \mathscr{R}$. Replacing y by yx in the last expression and using it, one can find that $[x, y]^*\theta(x) = x^*[x, y]^*$ for all $x, y \in \mathscr{R}$. Taking $x = [r, s]$, $r, s \in \mathscr{R}$, we get $[r, s][[r, s], y] = [[r, s], y][r, s]$ for all $r, s, y \in \mathscr{R}$. This implies that $[[[r, s], y], [r, s]] = 0$ for all $r, s, y \in \mathscr{R}$. Applying Fact 2.5, we get $[r, s] \in \mathcal{Z}(\mathscr{R})$ for all $r, s \in \mathscr{R}$, therefore $\mathscr{R}$ is commutative, a contradiction.
So $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Following the similar steps as in the proof Theorem 3.1, One finally obtains

$$[\theta(x), \theta(y)] = -[x, y] \tag{4.6}$$

for all $x, y \in \mathscr{R}$, as desired.
(2) $\implies$ (3) Suppose that

$$[\theta(k), \theta(k^{'})] + [k, k^{'}] = 0 \tag{4.7}$$

for all $k, k^{'} \in \mathcal{S}(\mathscr{R})$. Taking $k = kh_0$, where $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ and arguing as in Theorem 3.1, one obtains $\theta(h_0) = h_0$ for all $h_0 \in \mathcal{H}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$, since $\mathscr{R}$ is noncommutative.
Again following the proof of Theorem 3.1, one can easily show that $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$ or $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Assume that $\theta(k_0) = -k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Replacing k by hk_0, where $h \in \mathcal{H}(\mathscr{R})$ and $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})/\{0\}$ in (4.7), we obtain

$$[\theta(h), \theta(k^{'})] - [h, k] = 0 \tag{4.8}$$

for all $h \in \mathcal{H}(\mathscr{R})$ and $k^{'} \in \mathcal{S}(\mathscr{R})$. Again taking $h = x + x^{'}$ and $k^{'} = x - x^{'}$, where $x \in \mathscr{R}$, we get

$$[\theta(x), \theta(x^*)] - [x, x^*] = 0 \tag{4.9}$$

for all $x \in \mathscr{R}$. which is same as equation (4.3). Thus arguing on similar lines, one obtain $\mathscr{R}$ is commutative, a contradiction. Now assume $\theta(k_0) = k_0$ for all $k_0 \in \mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Thus following the same steps as in Theorem 3.1, one can easily derive that

$$[\theta(x), \theta(y)] = -[x, y] \tag{4.10}$$

for all $x, y \in \mathscr{R}$, thereby completing the proof of the theorem.

□

We end our paper by providing an example which shows that the said question does not hold in case θ is simply an additive map. Hence we conclude that for the said question to hold, θ needs to be of some special type such as endomorphism in our case.

Example 4.2. *Let* $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{C} \right\}$. *Define* $* : R \longrightarrow R$ *such that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$. *Clearly* $*$ *is of the second kind. Let*

$\mathcal{H}(\mathscr{R})$ be the set of symmetric elements of $\mathscr{R}$. If $\theta : \mathscr{R} \to \mathscr{R}$ is an additive map of $\mathscr{R}$ defined by $\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \bar{b} \\ c & d \end{pmatrix}$. Then one can see that $\theta(H) = H$ for all $H \in \mathcal{H}(\mathscr{R})$ and hence the condition $[\theta(H), \theta(H^{'})] = [H, H^{'}]$ holds for all $H, H^{'} \in \mathcal{H}(\mathscr{R})$. However $[\theta(X), \theta(Y)] \neq [X, Y]$ for all $X, Y \in \mathscr{R}$.

Acknowledgements: The authors wish sincere thank to the anonymous referees for their useful comments and valuable suggestions which improved the paper immensely. This work was supported by **TUBITAK**, the Scientific and Technological Research Council of Turkey, under the program 2221-Fellowship for Visiting Professor/Scientists at Karamanoglu Mehmetbey University (KMU), Karaman Turkey during June 21 to July 8, 2022. So, the second and third authors, gratefully acknowledge the TUBITAK for technical and financial support.

References

[1] Ali, S., Dar, N. A., Khan, A. N., On strong commutativity preserving like maps in rings with involution. *Miskolc Math. Notes* 16(1): (2015), 17-24.

[2] Ali, S., Huang, S. On derivation in semiprime rings. *Alg. Represent Theor.* 15(6): (2012), 1023-1033.

[3] Bell, H. E., Daif, M., N. On commutativity and strong commutativity preserving maps. *Canad. Math. Bull.* 37: (1994), 443-447.

[4] Bell, H. E., Mason, G., On derivations in near rings and rings. *Math. J. Okayama Univ.* 34:(1992), 135-144.

[5] Brešar, M., Commuting traces of biadditive mappings, commutativity preserving mappings and Lie mappings. *Trans. Amer. Math. Soc.* 335:(1993), 525-546.

[6] Brešar, M., Miers, C .R., Strong commutativity preserving mappings of semiprime rings. *Canad. Math. Bull.* 37: (1994), 457-460.

[7] Dar, N. A., Khan, A. N., Generalized derivations in rings with involution. *Algebra Colloquium* 24(3): (2017), 393-399.

[8] Deng, Q., Ashraf, M., On strong commutativity preserving maps. *Results Math.* 30:(1996), 259-263.

[9] Mamouni, A., Oukhtite, L., Elmir, H., New classes of endomorphisms and some classification theorems. *Comm. Algebra*, (2020). DOI:10.100/00927872.2019.1632330.

[10] Nejjar, B., Kacha, A., Mamouni, A., Oukhtite, L., Commutativity theorems in rings with Involution. *Comm. Algebra.* 45(2):(2017), 698-708.

[11] Šemrl, P., Commutativity preserving maps. *Linear Algebra Appl.* 429:(2008), 1051-1070.