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Some common fixed point theorems on partial metric spaces involving auxiliary function

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Abstract

In this paper, we prove some common fixed point theorems in the framework of partial metric spaces by using auxiliary function and give some consequences of the main result. Also we give some examples in support of the result. The presented results in this paper extend and generalize several results from the existing literature.

1 Introduction and Preliminaries

Metric fixed point theory has been the centre of extensive research for several researchers. Fixed point theory has become an important tool for solving many nonlinear problems related to science and engineering because of its applications. The Banach contraction mappings principle is the opening and vital result in the direction of fixed point theory. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $\mathcal{T}x = x$ (where $\mathcal{T}$ is a self mapping defined on a nonempty set $\mathcal{X}$), is the most widely used fixed point theorem in all of analysis. In a metric space setting it can be briefly stated as follows.

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Theorem 1.1. *([7]) Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{S}\colon \mathcal{X} \to \mathcal{X}$ be a map satisfying*

$$d(\mathcal{S}(p), \mathcal{S}(q)) \leq m\, d(p, q), \text{ for all } p, q \in \mathcal{X}, \tag{1.1}$$

where $0 < m < 1$ is a constant. Then

(1) *$\mathcal{S}$ has a unique fixed point z in $\mathcal{X}$;*

(2) *The Picard iteration $\{y_n\}_{n=0}^{\infty}$ defined by*

$$y_{n+1} = \mathcal{S} y_n, \quad n = 0, 1, 2, \dots \tag{1.2}$$

converges to z, for any $y_0 \in \mathcal{X}$.

Remark 1.1. *(i) A self-map satisfying* (1) *and* (2) *is said to be a Picard operator (see, [28, 29]).*

(ii) Inequality (1.1) also implies the continuity of $\mathcal{S}$.

In literature, there are many generalizations of Banach contraction principle in metric and generalized metric spaces. These generalizations are made either by using different contractive conditions or by imposing some additional condition on the ambient spaces. On the other hand, a number of generalizations of metric spaces have been done and one of such generalization is partial metric space introduced in 1992 by Matthews [22, 23]. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews proved the partial metric version of Banach fixed point theorem ([7]). Then, many authors gave some generalizations of the result of Matthews and proved some fixed point theorems in this space (see, i.e., [1], [2], [3], [16], [17], [18], [19], [25], [27], [30], [36]-[39], [40] and many others).

Recently, many authors proved fixed point and common fixed point results via contractive type conditions in various ambient spaces (see, e.g., [4, 5, 8, 9, 10, 11, 14, 15, 19, 20, 26, 31, 32, 33, 34, 35] and many others).

The purpose of this work is to prove some common fixed point theorems for contractive type condition involving auxiliary function in the setting of partial metric spaces.

Now, we recall some basic concepts on partial metric spaces defined as follows.

Definition 1.1. *([23]) Let* $\mathcal{X}$ *be a nonempty set and* $p\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ *be a self mapping of* $\mathcal{X}$ *such that for all* $u,v,w\in\mathcal{X}$ *the followings are satisfied:*

$(P1)$ $u=v\Leftrightarrow p(u,u)=p(u,v)=p(v,v)$,

$(P2)$ $p(u,u)\le p(u,v)$,

$(P3)$ $p(u,v)=p(v,u)$,

$(P4)$ $p(u,v)\le p(u,w)+p(w,v)-p(w,w)$.

Then p is called partial metric on $\mathcal{X}$ *and the pair* $(\mathcal{X},p)$ *is called partial metric space (in short PMS).*

Remark 1.2. *It is clear that if* $p(u,v)=0$, *then* $u=v$. *But, on the contrary* $p(u,u)$ *need not be zero.*

Example 1.1. *([6]) Let* $\mathcal{X}=\mathbb{R}^+$ *and* $p\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ *be given by* $p(u,v)=\max\{u,v\}$ *for all* $u,v\in\mathbb{R}^+$. *Then* $(\mathbb{R}^+,p)$ *is a partial metric space.*

Example 1.2. *([6]) Let* I *denote the set of all intervals* $[a,b]$ *for any real numbers* $a\le b$. *Let* $p\colon I\times I\to[0,\infty)$ *be a function such that*

$$p\Big([a,b],[c,d]\Big)=\max\{b,d\}-\min\{a,c\}.$$

Then (I,p) *is a partial metric space.*

Example 1.3. *([12]) Let* $\mathcal{X}=\mathbb{R}$ *and* $p\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ *be given by* $p(u,v)=e^{\max\{u,v\}}$ *for all* $u,v\in\mathbb{R}$. *Then* $(\mathbb{X},p)$ *is a partial metric space.*

Various applications of this space has been extensively investigated by many authors (see, K*ü*nzi [21] and Valero [40] for details).

Remark 1.3. *([17]) Let* $(\mathcal{X},p)$ *be a partial metric space.*

(1) *The function* $d_p\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ *defined as* $d_p(u,v)=2p(u,v)-p(u,u)-p(v,v)$ *is a metric on* $\mathcal{X}$ *and* $(\mathcal{X},d_p)$ *is a metric space.*

(2) *The function* $d_s\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ *defined as* $d_s(u,v)=\max\{p(u,v)-p(u,u),p(u,v)-p(v,v)\}$ *is a metric on* $\mathcal{X}$ *and* $(\mathcal{X},d_s)$ *is a metric space.*

Note also that each partial metric p on $\mathcal{X}$ generates a T_0 topology τ_p on $\mathcal{X}$, whose base is a family of open p-balls $\{B_p(u,\varepsilon):u\in\mathcal{X},\varepsilon>0\}$ where,

$$B_p(u,\varepsilon)=\{v\in\mathcal{X}:p(u,v)<p(u,u)+\varepsilon\},$$

for all $u \in \mathcal{X}$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as

$$B_p[u, \varepsilon] = \{v \in \mathcal{X} : p(u, v) \leq p(u, u) + \varepsilon\},$$

for all $u \in \mathcal{X}$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [22].

Definition 1.2. *([22]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is said to be convergent to a point $r \in \mathcal{X}$ if and only if $p(r, r) = \lim_{n\to\infty} p(r_n, r)$;*

(2) *a sequence $\{r_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(r_m, r_n)$ exists and finite;*

(3) *$(\mathcal{X}, p)$ is said to be complete if every Cauchy sequence $\{r_n\}$ in $\mathcal{X}$ converges to a point $r \in \mathcal{X}$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(r_m, r_n) = \lim_{n\to\infty} p(r_n, r) = p(r, r).$$

(4) *A mapping $f : \mathcal{X} \to \mathcal{X}$ is said to be continuous at $r_0 \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f\Big(B_p(r_0, \delta)\Big) \subset B_p\Big(f(r_0), \varepsilon\Big)$.*

Definition 1.3. *([24]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is called 0-Cauchy if $\lim_{m,n\to\infty} p(r_m, r_n) = 0$;*

(2) *$(\mathcal{X}, p)$ is said to be 0-complete if every 0-Cauchy sequence $\{r_n\}$ in $\mathcal{X}$ converges to a point $r \in \mathcal{X}$, such that $p(r, r) = 0$.*

Lemma 1.1. *([22, 23]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(1) a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{X}, d_p)$,

(2) $(\mathcal{X}, p)$ is complete if and only if the metric space $(\mathcal{X}, d_p)$ is complete,

(3) a subset E of a partial metric space $(\mathcal{X}, p)$ is closed if a sequence $\{r_n\}$ in E such that $\{r_n\}$ converges to some $r \in \mathcal{X}$, then $r \in E$.

Lemma 1.2. *([2]) Assume that $r_n \to r$ as $n \to \infty$ in a partial metric space $(\mathcal{X}, p)$ such that $p(r,r) = 0$. Then $\lim_{n\to\infty} p(r_n, u) = p(r, u)$ for every $u \in \mathcal{X}$.*

Lemma 1.3. *(see [19]) Let $(\mathcal{X}, p)$ be a partial metric space.*
(i) If $p(u, v) = 0$, then $u = v$;
(ii) If $u \neq v$, then $p(u, v) > 0$.

2 Main Results

In this section, we shall prove some unique common fixed point theorems in the framework of partial metric spaces by using auxiliary function.

We shall denote Ψ the set of functions ψ: $[0, \infty) \to [0, \infty)$ satisfying the following conditions:
(Ψ_1) ψ is continuous; (Ψ_2) $\psi(t) < t$ for all $t > 0$.
Obviously, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) \leq t$ for all $t \geq 0$.

Theorem 2.1. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two self-maps on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{R}_1 y, \mathcal{R}_2 z) \leq \alpha_1 \Lambda_1^p(y, z) + \alpha_2 \Lambda_2^p(y, z), \tag{2.1}$$

for all $y, z \in \mathcal{X}$, where

$$\Lambda_1^p(y, z) = \psi\Big(p(y, \mathcal{R}_1 y)\frac{1 + p(z, \mathcal{R}_2 z)}{1 + p(y, z)}\Big), \tag{2.2}$$

and

$$\begin{aligned}\Lambda_2^p(y, z) = \max\Big\{&\psi(p(y, z)), \psi(p(y, \mathcal{R}_1 y)), \psi\Big(\frac{1}{2}[p(z, \mathcal{R}_1 y) + p(y, \mathcal{R}_2 z)]\Big),\\ &\psi\Big(\frac{p(y, \mathcal{R}_1 y)[1 + p(z, \mathcal{R}_2 z)]}{1 + p(y, z)}\Big)\Big\},\end{aligned} \tag{2.3}$$

for all $\psi \in \Psi$, where $\alpha_1, \alpha_2 \in [0,1)$ with $\alpha_1 + \alpha_2 < 1$. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ have a unique common fixed point in $\mathcal{X}$.

Proof. For each $u_0 \in \mathcal{X}$. Let $u_{2n+1} = \mathcal{R}_1 u_{2n}$ and $u_{2n+2} = \mathcal{R}_2 u_{2n+1}$ for $n = 0, 1, 2, \dots$, we prove that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$. It follows from (2.1) for $y = u_{2n}$ and $z = u_{2n-1}$ that

$$\begin{aligned} p(u_{2n+1}, u_{2n}) &= p(\mathcal{R}_1 u_{2n}, \mathcal{R}_2 u_{2n-1}) \\ &\leq \alpha_1 \Lambda_1^p(u_{2n}, u_{2n-1}) + \alpha_2 \Lambda_2^p(u_{2n}, u_{2n-1}), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Lambda_1^p(u_{2n}, u_{2n-1}) &= \psi\Big(p(u_{2n}, \mathcal{R}_1 u_{2n}) \frac{1 + p(u_{2n-1}, \mathcal{R}_2 u_{2n-1})}{1 + p(u_{2n}, u_{2n-1})}\Big) \\ &= \psi\Big(p(u_{2n}, u_{2n+1}) \frac{1 + p(u_{2n-1}, u_{2n})}{1 + p(u_{2n}, u_{2n-1})}\Big) \\ &= \psi\Big(p(u_{2n+1}, u_{2n}) \frac{1 + p(u_{2n-1}, u_{2n})}{1 + p(u_{2n-1}, u_{2n})}\Big) \text{ (by (P3))} \\ &= \psi\Big(p(u_{2n+1}, u_{2n})\Big), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \Lambda_2^p(u_{2n}, u_{2n-1}) &= \max\Big\{\psi(p(u_{2n}, u_{2n-1})), \psi(p(u_{2n}, \mathcal{R}_1 u_{2n})), \\ &\qquad \psi\Big(\frac{1}{2}[p(u_{2n-1}, \mathcal{R}_1 u_{2n}) + p(u_{2n}, \mathcal{R}_2 u_{2n-1})]\Big), \\ &\qquad \psi\Big(\frac{p(u_{2n}, \mathcal{R}_1 u_{2n})[1 + p(u_{2n-1}, \mathcal{R}_2 u_{2n-1})]}{1 + p(u_{2n}, u_{2n-1})}\Big)\Big\} \\ &= \max\Big\{\psi(p(u_{2n}, u_{2n-1})), \psi(p(u_{2n}, u_{2n+1})), \end{aligned}$$

(2.6)

$$\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n+1})+p(u_{2n},u_{2n})]\Big),$$

$$\psi\Big(\frac{p(u_{2n},u_{2n+1})[1+p(u_{2n-1},u_{2n})]}{1+p(u_{2n},u_{2n-1})}\Big)\Big\} \tag{2.7}$$

$$\begin{aligned}&\leq \max\Big\{\psi(p(u_{2n-1},u_{2n})),\psi(p(u_{2n+1},u_{2n})),\\ &\quad\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n})+p(u_{2n+1},u_{2n})]\Big),\\ &\quad\psi\Big(\frac{p(u_{2n+1},u_{2n})[1+p(u_{2n-1},u_{2n})]}{1+p(u_{2n-1},u_{2n})}\Big)\Big\}\text{ (by (P3) and (P4))}\end{aligned} \tag{2.8}$$

$$\begin{aligned}&=\max\Big\{\psi(p(u_{2n-1},u_{2n})),\psi(p(u_{2n+1},u_{2n})),\\ &\quad\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n})+p(u_{2n+1},u_{2n})]\Big)\Big\}.\end{aligned} \tag{2.9}$$

The following cases arise.

Case (i) If $\Lambda_2^p(u_{2n},u_{2n-1})=\psi(p(u_{2n+1},u_{2n}))$, then from (2.4), (2.5), (2.9) and using the property of ψ that

$$\begin{aligned}p(u_{2n+1},u_{2n})&\leq \alpha_1\,\psi\Big(p(u_{2n+1},u_{2n})\Big)+\alpha_2\,\psi\Big(p(u_{2n+1},u_{2n})\Big)\\ &=(\alpha_1+\alpha_2)\psi\Big(p(u_{2n+1},u_{2n})\Big)\\ &<(\alpha_1+\alpha_2)p(u_{2n+1},u_{2n})\\ &<p(u_{2n+1},u_{2n}),\text{ (since, }(\alpha_1+\alpha_2)<1)\end{aligned} \tag{2.10}$$

a contradiction.

Case (ii) If $\Lambda_2^p(u_{2n},u_{2n-1})=\psi(p(u_{2n-1},u_{2n}))$, then from (2.4), (2.5), (2.9) and using the property of ψ that

$$\begin{aligned}p(u_{2n+1},u_{2n})&\leq \alpha_1\,\psi\Big(p(u_{2n+1},u_{2n})\Big)+\alpha_2\,\psi\Big(p(u_{2n-1},u_{2n})\Big)\\ &\leq \alpha_1\,p(u_{2n+1},u_{2n})+\alpha_2\,p(u_{2n-1},u_{2n}),\end{aligned}$$

or

$$(1-\alpha_1)p(u_{2n+1},u_{2n}) \leq \alpha_2\, p(u_{2n-1},u_{2n}),$$

or

$$p(u_{2n+1},u_{2n}) \leq \Big(\frac{\alpha_2}{1-\alpha_1}\Big)p(u_{2n-1},u_{2n}). \tag{2.11}$$

Case (iii) If $\Lambda_2^p(u_{2n},u_{2n-1}) = \psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n+1},u_{2n})]\Big)$, then from (2.4), (2.5), (2.9) and using the property of ψ that

$$\begin{aligned} p(u_{2n+1},u_{2n}) &\leq \alpha_1\,\psi\Big(p(u_{2n+1},u_{2n})\Big) + \alpha_2\,\psi\Big(\frac{1}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n+1},u_{2n})]\Big) \\ &\leq \alpha_1\, p(u_{2n+1},u_{2n}) + \frac{\alpha_2}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n+1},u_{2n})], \end{aligned}$$

or

$$\begin{aligned} 2p(u_{2n+1},u_{2n}) &\leq 2\alpha_1\, p(u_{2n+1},u_{2n}) + \alpha_2\, p(u_{2n-1},u_{2n}) + \alpha_2\, p(u_{2n+1},u_{2n}) \\ &= (2\alpha_1+\alpha_2)\, p(u_{2n+1},u_{2n}) + \alpha_2\, p(u_{2n-1},u_{2n}), \end{aligned}$$

or

$$(2-2\alpha_1-\alpha_2)p(u_{2n+1},u_{2n}) \leq \alpha_2\, p(u_{2n-1},u_{2n}),$$

or

$$p(u_{2n+1},u_{2n}) \leq \Big(\frac{\alpha_2}{2-2\alpha_1-\alpha_2}\Big)p(u_{2n-1},u_{2n}). \tag{2.12}$$

Put $\theta = \max\Big\{\frac{\alpha_2}{1-\alpha_1}, \frac{\alpha_2}{2-2\alpha_1-\alpha_2}\Big\} < 1$, since $(\alpha_1+\alpha_2) < 1$. Then from (2.12), we obtain

$$p(u_{2n+1},u_{2n}) \leq \theta\, p(u_{2n-1},u_{2n}), \tag{2.13}$$

which implies

$$p(u_{n+1},u_n) \leq \theta\, p(u_n,u_{n-1}). \tag{2.14}$$

Let $\mathcal{D}_n = p(u_{n+1}, u_n)$ and $\mathcal{D}_{n-1} = p(u_n, u_{n-1})$. Then from (2.14), it can be concluded that

$$\mathcal{D}_n \leq \theta\, \mathcal{D}_{n-1} \leq \theta^2\, \mathcal{D}_{n-2} \leq \cdots \leq \theta^n\, \mathcal{D}_0. \tag{2.15}$$

Therefore, since $0 \leq \theta < 1$, taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} p(u_{n+1}, u_n) = 0. \tag{2.16}$$

Now, we shall show that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$.

Thus for any $n, m \in \mathbb{N}$ with $m > n$, then we have

$$\begin{aligned}
p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{n+m-1}, u_m) \\
&\quad - p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \cdots - p(u_{n+m-1}, u_{n+m-1}) \\
&\leq \theta^n p(u_0, u_1) + \theta^{n+1} p(u_0, u_1) + \cdots + \theta^{n+m-1} p(u_0, u_1) \\
&= \theta^n [p(u_0, u_1) + \theta p(u_0, u_1) + \cdots + \theta^{m-1} p(u_0, u_1)] \\
&= \theta^n [1 + \theta + \cdots + \theta^{m-1}] \mathcal{D}_0 \\
&\leq \theta^n \Big(\frac{1 - \theta^{m-1}}{1 - \theta} \Big) \mathcal{D}_0.
\end{aligned}$$

Taking the limit as $n, m \to \infty$ in the above inequality, we get $p(u_n, u_m) \to 0$, since $0 < \theta < 1$, hence $\{u_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$. Hence, by Lemma 1.1, this sequence will also Cauchy in $(\mathcal{X}, d_p)$. In addition, since $(\mathcal{X}, p)$ is complete, $(\mathcal{X}, d_p)$ is also complete. Thus there exists $v \in \mathcal{X}$ such that $u_n \to v$ as $n \to \infty$. Moreover, by Lemma 1.1,

$$p(v, v) = \lim_{n\to\infty} p(v, u_n) = \lim_{n,m\to\infty} p(u_n, u_m) = 0, \tag{2.17}$$

implies

$$\lim_{n\to\infty} d_p(v, u_n) = 0. \tag{2.18}$$

Now, we shall show that v is a common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$. Notice that due to (2.17), we have $p(v, v) = 0$. By (2.1) with $y = u_{2n}$ and $z = v$ and using (2.17), we have

$$\begin{aligned} p(u_{2n+1}, \mathcal{R}_2 v) &= p(\mathcal{R}_1 u_{2n}, \mathcal{R}_2 v) \\ &\leq \alpha_1\, \Lambda_1^p(u_{2n}, v) + \alpha_2\, \Lambda_2^p(u_{2n}, v), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \Lambda_1^p(u_{2n}, v) &= \psi\Big(p(u_{2n}, \mathcal{R}_1 u_{2n}) \frac{1 + p(v, \mathcal{R}_2 v)}{1 + p(u_{2n}, v)}\Big) \\ &= \psi\Big(p(u_{2n}, u_{2n+1}) \frac{1 + p(v, \mathcal{R}_2 v)}{1 + p(u_{2n}, v)}\Big). \end{aligned} \tag{2.20}$$

Passing to limit as $n \to \infty$ and using the properties of ψ and (2.17), we obtain

$$\Lambda_1^p(u_{2n}, v) \to 0, \tag{2.21}$$

and

$$\begin{aligned} \Lambda_2^p(u_{2n}, v) = \max\Big\{&\psi(p(u_{2n}, v)), \psi(p(u_{2n}, \mathcal{R}_1 u_{2n})), \\ &\psi\Big(\frac{1}{2}[p(z, \mathcal{R}_1 u_{2n}) + p(u_{2n}, \mathcal{R}_2 v)]\Big), \\ &\psi\Big(\frac{p(u_{2n}, \mathcal{R}_1 u_{2n})[1 + p(v, \mathcal{R}_2 v)]}{1 + p(u_{2n}, v)}\Big)\Big\} \\ = \max\Big\{&\psi(p(u_{2n}, v)), \psi(p(u_{2n}, u_{2n+1})), \\ &\psi\Big(\frac{1}{2}[p(v, u_{2n+1}) + p(u_{2n}, \mathcal{R}_2 v)]\Big), \\ &\psi\Big(\frac{p(u_{2n}, u_{2n+1})[1 + p(v, \mathcal{R}_2 v)]}{1 + p(u_{2n}, v)}\Big)\Big\}. \end{aligned} \tag{2.22}$$

Passing to limit as $n \to \infty$ and using the properties of ψ and (2.17) in equation

(2.22), we obtain

$$\begin{aligned}\Lambda_2^p(u_{2n}, v) &= \max\Big\{0, 0, \psi\Big(\frac{p(v, \mathcal{R}_2 v)}{2}\Big), 0\Big\} \\ &= \psi\Big(\frac{p(v, \mathcal{R}_2 v)}{2}\Big) < \frac{p(v, \mathcal{R}_2 v)}{2} < p(v, \mathcal{R}_2 v). \end{aligned} \tag{2.23}$$

Now from equations (2.19), (2.21) and (2.23), we obtain

$$\begin{aligned}p(u_{2n+1}, \mathcal{R}_2 v) &\leq \alpha_1 \,.0 + \alpha_2\, p(v, \mathcal{R}_2 v) \\ &= \alpha_2\, p(v, \mathcal{R}_2 v). \end{aligned} \tag{2.24}$$

Passing to limit as $n \to \infty$, we obtain

$$\begin{aligned}p(v, \mathcal{R}_2 v) &\leq \alpha_2\, p(v, \mathcal{R}_2 v) \\ &< p(v, \mathcal{R}_2 v), \text{ since } \alpha_2 < 1, \end{aligned} \tag{2.25}$$

which is a contradiction. Hence $p(v, \mathcal{R}_2 v) = 0$, that is, $v = \mathcal{R}_2 v$. This shows that v is a fixed point of $\mathcal{R}_2$. By similar fashion, we can show that $v = \mathcal{R}_1 v$. Thus v is a common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Now, we shall show the uniqueness of common fixed point. Assume that v' is another common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R}_1 v' = v' = \mathcal{R}_2 v'$ with $v \neq v'$. Using (2.1) for $y = v$, $z = v'$ and using the properties of ψ and (2.17), we have

$$\begin{aligned}p(v, v') &= p(\mathcal{R}_1 v, \mathcal{R}_2 v') \\ &\leq \alpha_1\, \Lambda_1^p(v, v') + \alpha_2\, \Lambda_2^p(v, v'), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned}\Lambda_1^p(v, v') &= \psi\Big(p(v, \mathcal{R}_1 v)\frac{1 + p(v', \mathcal{R}_2 v')}{1 + p(v, v')}\Big) \\ &= \psi\Big(p(v, v)\frac{1 + p(v', v')}{1 + p(v, v')}\Big) = 0, \end{aligned} \tag{2.27}$$

and

$$\Lambda_2^p(v, v') = \max\Big\{\psi(p(v, v')), \psi(p(v, \mathcal{R}_1 v)), \psi\Big(\frac{1}{2}[p(v', \mathcal{R}_1 v) + p(v, \mathcal{R}_2 v')]\Big), \psi\Big(\frac{p(v, \mathcal{R}_1 v)[1 + p(v', \mathcal{R}_2 v')]}{1 + p(v, v')}\Big)\Big\}$$

$$= \max\Big\{\psi(p(v, v')), \psi(p(v, v)), \psi\Big(\frac{1}{2}[p(v', v) + p(v, v')]\Big), \psi\Big(\frac{p(v, v)[1 + p(v', v')]}{1 + p(v, v')}\Big)\Big\}$$
$$= \max\Big\{\psi(p(v, v')), 0, \psi(p(v, v')), 0\Big\} = \psi(p(v, v')). \tag{2.28}$$

From equations (2.26), (2.27), (2.28) and using the property of ψ, we obtain

$$p(v, v') \leq \alpha_1 .0 + \alpha_2\, \psi(p(v, v')) = \alpha_2\, \psi(p(v, v')) < \alpha_2\, p(v, v') < p(v, v'), \text{ since } \alpha_2 < 1,$$

which is a contradiction. Hence, $p(v, v') = 0$, that is, $v = v'$. Thus the common fixed point of $\mathcal{R}_1$ and $\mathcal{R}_2$ is unique. This completes the proof. □

Theorem 2.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous self-maps on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{F}_1^m y, \mathcal{F}_2^n z) \leq L_1\, \mathcal{H}_1^p(y, z) + L_2\, \mathcal{H}_2^p(y, z), \tag{2.29}$$

for all $y, z \in \mathcal{X}$, where m and n are some positive integers,

$$\mathcal{H}_1^p(y, z) = \psi\Big(p(y, \mathcal{F}_1^m y)\frac{1 + p(z, \mathcal{F}_2^n z)}{1 + p(y, z)}\Big), \tag{2.30}$$

and

$$\begin{aligned}\mathcal{H}_2^p(y,z) = \max\Big\{&\psi(p(y,z)),\\ &\psi(p(y,\mathcal{F}_1^m y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{F}_1^m y) + p(y,\mathcal{F}_2^n z)]\Big),\\ &\psi\Big(\frac{p(y,\mathcal{F}_1^m y)[1+p(z,\mathcal{F}_2^n z)]}{1+p(y,z)}\Big)\Big\},\end{aligned} \tag{2.31}$$

for all $\psi \in \Psi$, and $L_1, L_2 \in [0,1)$ with $L_1 + L_2 < 1$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{X}$.

Proof. Since $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ satisfy the conditions of the Theorem 2.1. So $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ have a unique common fixed point. Let w be the common fixed point. Then we have

$$\begin{aligned}\mathcal{F}_1^m w = w &\Rightarrow \mathcal{F}_1(\mathcal{F}_1^m w) = \mathcal{F}_1 w\\ &\Rightarrow \mathcal{F}_1^m(\mathcal{F}_1 w) = \mathcal{F}_1 w.\end{aligned}$$

If $\mathcal{F}_1 w = w_0$, then $\mathcal{F}_1^m w_0 = w_0$. So $\mathcal{F}_1 w$ is a fixed point of $\mathcal{F}_1^m$. Similarly, $\mathcal{F}_2^n(\mathcal{F}_2 w) = \mathcal{F}_2 w$, that is, $\mathcal{F}_2 w$ is a fixed point of $\mathcal{F}_2^n$.
Now, using equations (2.29) and (2.17) and using the properties of ψ, we have

$$\begin{aligned}p(w,\mathcal{F}_1 w) &= p(\mathcal{F}_1^m w, \mathcal{F}_1^m(\mathcal{F}_1 w))\\ &\leq L_1\, \mathcal{H}_1^p(w,\mathcal{F}_1 w) + L_2\, \mathcal{H}_2^p(w,\mathcal{F}_1 w),\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}\mathcal{H}_1^p(w,\mathcal{F}_1 w) &= \psi\Big(p(w,\mathcal{F}_1^m w)\frac{1+p(\mathcal{F}_1 w, \mathcal{F}_2^n(\mathcal{F}_1 w))}{1+p(w,\mathcal{F}_1 w)}\Big)\\ &= \psi\Big(p(w,w)\frac{1+p(\mathcal{F}_1 w,\mathcal{F}_1 w)}{1+p(w,\mathcal{F}_1 w)}\Big)\\ &= \psi(0) = 0,\end{aligned} \tag{2.33}$$

and

$$\mathcal{H}_2^p(w, \mathcal{F}_1 w) = \max\Big\{\psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1^m(\mathcal{F}_1 w))),$$

$$\psi\Big(\frac{1}{2}[p(\mathcal{F}_1 w, \mathcal{F}_1^m w) + p(w, \mathcal{F}_2^n(\mathcal{F}_1 w))]\Big),$$

$$\begin{aligned}
&\psi\Big(\frac{p(w, \mathcal{F}_1^m w)[1 + p(\mathcal{F}_1 w, \mathcal{F}_2^n(\mathcal{F}_1 w))]}{1 + p(w, \mathcal{F}_1 w)}\Big)\Big\} \\
&= \max\Big\{\psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1 w)), \\
&\quad \psi\Big(\frac{1}{2}[p(\mathcal{F}_1 w, w) + p(w, \mathcal{F}_1 w)]\Big), \\
&\quad \psi\Big(\frac{p(w, w)[1 + p(\mathcal{F}_1 w, \mathcal{F}_1 w)]}{1 + p(w, \mathcal{F}_1 w)}\Big)\Big\} \\
&= \max\Big\{\psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1 w)), \psi(p(w, \mathcal{F}_1 w), 0\Big\} \\
&= \psi(p(w, \mathcal{F}_1 w)). \qquad (2.34)
\end{aligned}$$

From equations (2.32)-(2.34) and using the property of ψ, we obtain

$$\begin{aligned}
p(w, \mathcal{F}_1 w) &\le L_1 .0 + L_2\, \psi(p(w, \mathcal{F}_1 w)) = L_2\, \psi(p(w, \mathcal{F}_1 w)) \\
&< L_2\, p(w, \mathcal{F}_1 w) < p(w, \mathcal{F}_1 w), \text{ since } L_2 < 1,
\end{aligned}$$

which is a contradiction. Hence, we deduce that $p(w, \mathcal{F}_1 w) = 0$, that is, $w = \mathcal{F}_1 w$ for all $w \in \mathcal{X}$. Similarly, we can show that $w = \mathcal{F}_2 w$. This shows that w is a common fixed point of $\mathcal{F}_1$ and $\mathcal{F}_2$. For the uniqueness of w, let $w' \neq w$ be another common fixed point of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then clearly w' is also a common fixed point of $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ which implies $w = w'$. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{X}$. This completes the proof. □

If we take $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{S}$ in Theorem 2.1, then we have the following result as corollary.

Corollary 2.1. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$*

satisfying the condition:

$$p(\mathcal{S}y, \mathcal{S}z) \leq \beta_1\, \mathcal{Q}_1^p(y, z) + \beta_2\, \mathcal{Q}_2^p(y, z), \tag{2.35}$$

for all $y, z \in \mathcal{X}$*, where*

$$\mathcal{Q}_1^p(y, z) = \psi\Big(p(y, \mathcal{S}y)\frac{1 + p(z, \mathcal{S}z)}{1 + p(y, z)}\Big),$$

and

$$\mathcal{Q}_2^p(y, z) = \max\Big\{\psi(p(y, z)), \psi(p(y, \mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z, \mathcal{S}y) + p(y, \mathcal{S}z)]\Big), \psi\Big(\frac{p(y, \mathcal{S}y)[1 + p(z, \mathcal{S}z)]}{1 + p(y, z)}\Big)\Big\},$$

for all $\psi \in \Psi$*, where* $\beta_1, \beta_2 \in [0, 1)$ *with* $\beta_1 + \beta_2 < 1$*. Then* $\mathcal{S}$ *has a unique fixed point in* $\mathcal{X}$*.*

If we take $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{G}$ in Theorem 2.2, then we have the following result as corollary.

Corollary 2.2. *Let* $\mathcal{G}$ *be a self-map on a complete partial metric space* $(\mathcal{X}, p)$ *satisfying the inequality for some positive integer* n*:*

$$p(\mathcal{G}^n y, \mathcal{G}^n z) \leq s_1\, \mathcal{M}_1^p(y, z) + s_2\, \mathcal{M}_2^p(y, z), \tag{2.36}$$

for all $y, z \in \mathcal{X}$*, where*

$$\mathcal{M}_1^p(y, z) = \psi\Big(p(y, \mathcal{G}^n y)\frac{1 + p(z, \mathcal{G}^n z)}{1 + p(y, z)}\Big),$$

and

$$\mathcal{M}_2^p(y, z) = \max\Big\{\psi(p(y, z)), \psi(p(y, \mathcal{G}^n y)), \psi\Big(\frac{1}{2}[p(z, \mathcal{G}^n y) + p(y, \mathcal{G}^n z)]\Big), \psi\Big(\frac{p(y, \mathcal{G}^n y)[1 + p(z, \mathcal{G}^n z)]}{1 + p(y, z)}\Big)\Big\},$$

for all $\psi \in \Psi$*, and* $s_1, s_2 \in [0, 1)$ *with* $s_1 + s_2 < 1$*. Then* $\mathcal{G}$ *has a unique fixed point in* $\mathcal{X}$*.*

Proof. Let $\mathcal{U} = \mathcal{G}^n$, then from (2.36), we have

$$p(\mathcal{U}y, \mathcal{U}z) \leq s_1\, \mathcal{M}_1^p(y,z) + s_2\, \mathcal{M}_2^p(y,z),$$

for all $y, z \in \mathcal{X}$, where

$$\mathcal{M}_1^p(y,z) = \psi\Big(p(y,\mathcal{U}y)\frac{1+p(z,\mathcal{U}z)}{1+p(y,z)}\Big),$$

and

$$\mathcal{M}_2^p(y,z) = \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{U}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{U}y)+p(y,\mathcal{U}z)]\Big), \psi\Big(\frac{p(y,\mathcal{U}y)[1+p(z,\mathcal{U}z)]}{1+p(y,z)}\Big)\Big\},$$

So by Corollary 2.1, $\mathcal{U}$, that is, $\mathcal{G}^n$ has a unique fixed point u_0. But $\mathcal{G}^n(\mathcal{G}u_0) = \mathcal{G}(\mathcal{G}^n u_0) = \mathcal{G}u_0$. So $\mathcal{G}u_0$ is also a fixed point of $\mathcal{G}^n$. Hence $\mathcal{G}u_0 = u_0$, i.e., u_0 is a fixed point of $\mathcal{G}$. Since the fixed point of $\mathcal{G}$ is also a fixed point of $\mathcal{G}^n$, so the fixed point of $\mathcal{G}$ is unique. This completes the proof. □

Corollary 2.3. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$. Suppose that there exists a nondecreasing function $\psi \in \Psi$ satisfying the condition:*

$$\begin{aligned} p(\mathcal{S}y, \mathcal{S}z) \leq\ & \beta_1\, \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big) \\ & + \beta_2\, \psi\Big(\max\Big\{p(y,z), \\ & p(y,\mathcal{S}y), \frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)], \\ & \frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big\}\Big), \end{aligned} \tag{2.37}$$

for all $y, z \in \mathcal{X}$, where $\beta_1, \beta_2 \in [0,1)$ with $\beta_1 + \beta_2 < 1$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Proof. It follows from Corollary 2.1 by taking that if $\psi \in \Psi$ is a nondecreasing function, we have

$$\mathcal{Q}_2^p(y,z) = \psi\Big(\max\Big\{p(y,z), p(y,\mathcal{S}y), \frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)], \frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big\}\Big).$$

□

Remark 2.1. *It is clear that the conclusions of the Corollary 2.3 remain valid if in condition (2.37), the second term of the right-hand side is replaced by one of the following terms:*

$$\beta_2\,\psi(p(y,z)); \quad \beta_2\,\psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big);$$

$$\beta_2\,\max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y))\Big\};$$

$$\text{or}\quad \beta_2\,\max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big)\Big\}.$$

Corollary 2.4. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X},p)$. Suppose that there exist five positive constants a_j, $j=1,2,3,4,5$ with $\sum_{j=1}^{5} a_j < 1$ satisfying the inequality:*

$$\begin{aligned} p(\mathcal{S}y,\mathcal{S}z) \le{}& a_1\left(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\right) + a_2\,p(y,z) \\ &+ a_3\,p(y,\mathcal{S}y) + a_4\,\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)] \\ &+ a_5\,\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}, \end{aligned} \tag{2.38}$$

for all $y,z\in\mathcal{X}$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Proof. It follows from Corollary 2.1 with $\psi(t) = (a_1+a_2+a_3+a_4+a_5)t$. □

As a special case, we obtain partial metric space versions of Banach ([7]) and Chatterjae ([13]) fixed point results from Corollary 2.4.

Corollary 2.5. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$. Suppose that there exists $\mu \in [0,1)$ such that one of the following conditions hold:*

$$p(\mathcal{S}y, \mathcal{S}z) \leq \mu\, p(y,z),$$

$$p(\mathcal{S}y, \mathcal{S}z) \leq \frac{\mu}{2}\, [p(z, \mathcal{S}y) + p(y, \mathcal{S}z)],$$

for all $y, z \in \mathcal{X}$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Proof. It follows from Corollary 2.4 by taking (1) $a_2 = \mu$ and $a_1 = a_3 = a_4 = a_5 = 0$ and (2) $a_4 = \mu$ and $a_1 = a_2 = a_3 = a_5 = 0$. □

If we take $\beta_1 = 0$, $\beta_2 = 1$ and

$$\max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y) + p(y,\mathcal{S}z)]\Big),$$
$$\psi\Big(\frac{p(y,\mathcal{S}y)[1 + p(z,\mathcal{S}z)]}{1 + p(y,z)}\Big)\Big\} = \psi(p(y,z)),$$

in Corollary 2.1, then we obtain the following result.

Corollary 2.6. *Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{S}y, \mathcal{S}z) \leq \psi(p(y,z)),$$

for all $y, z \in \mathcal{X}$ and $\psi \in \Psi$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

If we take $\psi(t) = k\,t$, where $0 < k < 1$ is a constant in Corollary 2.6, then we obtain the following result.

Corollary 2.7. *(see [23]) Let $\mathcal{S}$ be a self-map on a complete partial metric space $(\mathcal{X}, p)$ satisfying the condition:*

$$p(\mathcal{S}y, \mathcal{S}z) \leq k\, p(y,z),$$

for all $y, z \in \mathcal{X}$ and $k \in [0,1)$ is a constant. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

Remark 2.2. *Corollary 2.7 generalizes Banach contraction mapping principle ([7]) from complete metric space to the setting of complete partial metric space.*

Now, we give some examples in support of the result.

Example 2.1. *Let* $\mathcal{X} = \{1, 2, 3, 4\}$ *and* $p\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *be defined by*

$$p(y,z) = \begin{cases} |y-z| + \max\{y,z\}, & \text{if } y \neq z, \\ y, & \text{if } y = z \neq 1, \\ 0, & \text{if } y = z = 1, \end{cases}$$

for all $y, z \in \mathcal{X}$. *Then* $(\mathcal{X}, p)$ *is a complete partial metric space.*

Define the mapping $\mathcal{S}\colon \mathcal{X} \to \mathcal{X}$ *by*

$$\mathcal{S}(1) = 1,\ \mathcal{S}(2) = 1,\ \mathcal{S}(3) = 2,\ \mathcal{S}(4) = 2.$$

Now, we have

$$p(\mathcal{S}(1), \mathcal{S}(2)) = p(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p(1,2),$$

$$p(\mathcal{S}(1), \mathcal{S}(3)) = p(1,2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4}p(1,3),$$

$$p(\mathcal{S}(1), \mathcal{S}(4)) = p(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p(1,4),$$

$$p(\mathcal{S}(2), \mathcal{S}(3)) = p(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p(2,3),$$

$$p(\mathcal{S}(2), \mathcal{S}(4)) = p(1,2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4}p(2,4),$$

$$p(\mathcal{S}(3), \mathcal{S}(4)) = p(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p(3,4).$$

Thus, $\mathcal{S}$ *satisfies all the conditions of Corollary 2.7 with* $k = \frac{3}{4} < 1$. *Now by applying Corollary 2.7,* $\mathcal{S}$ *has a unique fixed point. Indeed* 1 *is the required unique fixed point in this case.*

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Example 2.2. *Let* $\mathcal{X} = [0, \infty)$ *and* $p\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *be defined by* $p(y,z) = \max\{y,z\}$ *for all* $y, z \in \mathcal{X}$. *Then* $(\mathcal{X}, p)$ *is a complete partial metric space. Con-*

sider the mappings $\mathcal{S}\colon \mathcal{X} \to \mathcal{X}$ *defined by*

$$\mathcal{S}(y) = \begin{cases} 0, & \text{if } 0 \leq y < 1, \\ \frac{y^2}{1+y}, & \text{if } y \geq 1, \end{cases}$$

and $\psi\colon [0,\infty) \to [0,\infty)$ *is defined by* $\psi(t) = \frac{3t}{4}$.

We have the following cases:

Case (i) *If* $y, z \in [0,1)$ *and assume that* $y \geq z$*, then we have*

$$p(\mathcal{S}(y), \mathcal{S}(z)) = 0,$$

$$\begin{aligned} \mathcal{Q}_1^p(y,z) &= \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big) \\ &= \psi\Big(\frac{y(1+z)}{(1+y)}\Big) = \frac{3y(1+z)}{4(1+y)}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_2^p(y,z) &= \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big), \\ &\qquad \psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\} \\ &= \max\Big\{\psi(y), \psi(y), \psi(\frac{y+z}{2}), \psi\Big(\frac{y(1+z)}{1+y}\Big)\Big\} = \psi(y) = \frac{3y}{4}. \end{aligned}$$

Hence from above inequalities, we see that

$$p(\mathcal{S}(y), \mathcal{S}(z)) = 0 \leq \beta_1\, \mathcal{Q}_1^p(y,z) + \beta_2\, \mathcal{Q}_2^p(y,z).$$

Thus the inequality holds.

Case (ii) *If* $z \in [0,1)$, $y \geq 1$ *and assume that* $y \geq z$*, then we have*

$$p(\mathcal{S}(y), \mathcal{S}(z)) = \max\Big\{\frac{y^2}{1+y}, 0\Big\} = \frac{y^2}{1+y},$$

and

$$\begin{aligned}\mathcal{Q}_1^p(y,z) &= \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big)\\ &= \psi\Big(\frac{y(1+z)}{(1+y)}\Big) = \frac{3y(1+z)}{4(1+y)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}_2^p(y,z) &= \max\Big\{\psi(p(y,z)),\psi(p(y,\mathcal{S}y)),\psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big),\\ &\qquad \psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\}\\ &= \max\Big\{\psi(y),\psi(y),\psi\Big(\frac{1}{2}\Big[\frac{y^2}{1+y}+y\Big]\Big),\psi\Big(\frac{y(1+z)}{1+y}\Big)\Big\} = \psi(y) = \frac{3y}{4}.\end{aligned}$$

Using contractive condition (2.35), we have

$$\frac{y^2}{1+y} \le \beta_1\left(\frac{3y(1+z)}{4(1+y)}\right) + \beta_2\left(\frac{3y}{4}\right).$$

If we take $y = 1$ and $z = 0$, then the above inequality reduces to

$$\frac{1}{2} \le \Big(\frac{3\beta_1}{8}\Big) + \Big(\frac{3\beta_2}{4}\Big),$$

or

$$4 \le 3\beta_1 + 6\beta_2.$$

The above inequality is satisfied for (i) $\beta_1 = \frac{1}{5}$ and $\beta_2 = \frac{3}{5}$, (ii) $\beta_1 = \frac{1}{5}$ and $\beta_2 = \frac{2}{3}$ with $\beta_1 + \beta_2 < 1$.

Case (iii) *If $y \ge z \ge 1$ and assume that $y \ge z$, then we have*

$$p(\mathcal{S}(y),\mathcal{S}(z)) = \max\Big\{\frac{y^2}{1+y},\frac{z^2}{1+z}\Big\} = \frac{y^2}{1+y},$$

and

$$\mathcal{Q}_1^p(y,z) = \psi\Big(p(y,\mathcal{S}y)\frac{1+p(z,\mathcal{S}z)}{1+p(y,z)}\Big)$$
$$= \psi\Big(\frac{y(1+z)}{(1+y)}\Big) = \frac{3y(1+z)}{4(1+y)},$$

and

$$\mathcal{Q}_2^p(y,z) = \max\Big\{\psi(p(y,z)), \psi(p(y,\mathcal{S}y)), \psi\Big(\frac{1}{2}[p(z,\mathcal{S}y)+p(y,\mathcal{S}z)]\Big),$$
$$\psi\Big(\frac{p(y,\mathcal{S}y)[1+p(z,\mathcal{S}z)]}{1+p(y,z)}\Big)\Big\}$$
$$= \max\Big\{\psi(y), \psi(y), \psi\Big(\frac{1}{2}[z+y]\Big), \psi\Big(\frac{y(1+z)}{1+y}\Big)\Big\} = \psi(y) = \frac{3y}{4}.$$

Using contractive condition (2.35), we have

$$\frac{y^2}{1+y} \leq \beta_1\Big(\frac{3y(1+z)}{4(1+y)}\Big) + \beta_2\Big(\frac{3y}{4}\Big).$$

If we take $y = z = 1$, then the above inequality reduces to

$$\frac{1}{2} \leq \frac{3\beta_1}{4} + \frac{3\beta_2}{4},$$

or

$$2 \leq 3\beta_1 + 3\beta_2.$$

The above inequality is satisfied for (i) $\beta_1 = \frac{1}{5}$ and $\beta_2 = \frac{1}{2}$, (ii) $\beta_1 = \frac{1}{3}$ and $\beta_2 = \frac{2}{5}$ and (iii) $\beta_1 = \frac{1}{4}$ and $\beta_2 = \frac{4}{7}$ with $\beta_1 + \beta_2 < 1$. Thus, in all the above cases $\mathcal{S}$ satisfies all the conditions of Corollary 2.1. Hence, $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$, indeed, $y = 0$ is the required point.

Conclusion

In this paper, we establish some unique common fixed point theorems in the framework of complete partial metric spaces involving auxiliary function and give some consequences of the established results as corollaries. We also give some examples in support of the results. The results of findings in this paper extend and generalize several results from the existing literature regarding partial metric spaces.

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