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Fixed point theorems for Suzuki-generalized Ćirić type nonlinear contractions on a metric space endowed with a locally T-transitive binary relation

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Abstract

In this paper, we extend a class of Suzuki-generalized nonlinear contractions due to Pant (Appl. Gen. Topol. 19 (1) (2018), 163-172) to a class of Suzuki-generalized Ćirić type nonlinear contractions employing a locally T-transitive binary relation and utilized the same to prove some fixed point results. Our newly proved results unify and generalize several fixed point theorems of the existing literature essentially due to Alam and Imdad, Agarwal $et\ al.$, and others.

Keywords and phrases: $\mathcal{R}$-complete metric spaces, locally T-transitive binary relations, completeness of a binary relation $\mathcal{R}$.

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1 Introduction

Banach contraction principle [1] (or, in short BCP) plays an important role in nonlinear analysis. Several noted generalizations of BCP via ϕ-contraction are available in the existing literature $e.g$ [2, 3, 4, 5]. As usual a control function is a mapping $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) < t$ for each $t > 0$. A self-mapping T defined on a metric space (X,d) is said to be a nonlinear contraction with respect to control function ϕ (or, in short, ϕ-contraction) if $d(Tx,Ty) \leq \phi(d(x,y))$ (for all $x,y \in X$). This principle has been generalized and extended in the several ways. Browder [2] extended Banach contraction principle to a class of nonlinear contractions which was later improved by Boyd and Wong [3], Mukherjea [4] and Jotić [5]. The class of control functions due to Boyd and Wong [3] can be described as follows:

$$\Omega = \Big\{\phi : [0,\infty) \to [0,\infty) : \phi(t) < t \text{ for each } t > 0 \\ \text{and} \limsup_{r\to t^+} \phi(r) < t \text{ for each } t > 0\Big\}.$$

The following fixed point result employing a control function is due to Pant [6]:

Theorem 1.1. *[6] Let (X, d) be a complete metric space. Assume that a mapping $T : X \to X$ satisfies the following contractive condition:*

(I) *if there exists strictly increasing right continuous control function ϕ such that*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Longrightarrow \quad d(Tx,Ty) \leq \phi(\mathcal{N}(x,y)) \forall x,y \in X,$$

(where $\mathcal{N}(x,y) := max\big\{d(x,y), d(x,Tx), d(y,Ty)\big\}$). Then T has a unique fixed point in X.

On the other hand, in the course of last several years, the BCP has been extended and generalized to ordered metric spaces by numerous researchers namely: Ran and Reurings [7] and Nieto and Rodríguez-Lopéz [8], O'Regan and Petruşel [9] and Agarwal $et\ al.$ [10]. For the results of this kind one can be referred to [12, 11, 13] and references cited therein.
The aim of this paper is to extend for Suzuki-generalized Ćirić type of nonlinear contractions and utilized the same to prove fixed point results employing a locally

T-transitive binary relation. Our newly proved results generalize several fixed point theorems of the existing literature specially due to Pant [6], Alam and Imdad [22], Agarwal $et\ al.$ [10] and others. Finally, an example is given for the genuineness of our newly proved result.

2 Preliminaries

Throughout this manuscript, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of natural numbers and whole numbers respectively ($i.e.$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). Given a non empty set X, a subset $\mathcal{R}$ of X^2 is called a binary relation on X. For simplicity, we sometimes write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$.

Out of various kind of binary relations, the following are relevant to our present discussion:
A binary relation $\mathcal{R}$ defined on a nonempty set X is called

- "amorphous" if it has no specific property at all,
- "universal" if $\mathcal{R} = X^2$,
- "empty" if $\mathcal{R} = \emptyset$,
- "reflexive" if $(x, x) \in \mathcal{R}\ \ \forall\, x \in X$,
- "symmetric" if $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,
- "antisymmetric" if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ imply $x = y$,
- "transitive" if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ imply $(x, z) \in \mathcal{R}$,
- "complete" if $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}\ \ \forall\, x, y \in X$,
- "partial order" if $\mathcal{R}$ is reflexive, antisymmetric and transitive.

Definition 2.1. *[15, 16, 14] Let X be a nonempty set equipped with partial order $\preceq$, A self-mapping T defined on X is called increasing (or isotone or order-preserving) if for any $x, y \in X$,*

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

The following notion is formulated by using a suitable property with a view to relax the continuity requirement of the underlying mapping especially in the hypotheses of a fixed point theorem due to Nieto and Rodríguez-López [8].

Definition 2.2. *[17] Let (X,d) be a metric space equipped with a partial order $\preceq$. We say that the triplet $(X, d, \preceq)$ has "ICU (increasing-convergence-upper bound) property" if every increasing convergent sequence in X is bounded above by its limit (as an upper bound).*

Inspired by Roldán-López-de-Hierro $et\ al.$ [18], Alam and Imdad intoduced the following: ($i.e.$, a notion originated from T-transitive subset of X^2 is essentially due to [18]).

Definition 2.3. *[19] Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ defined on X is called "T-transitive" if for any $x, y, z \in X$,*

$$(Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}.$$

Inspired by Turinici [21, 20], Alam and Imdad [19] introduced the following notions by localizing the transitivity condition.

Definition 2.4. *[19] Let X be a nonempty set. A binary relation $\mathcal{R}$ defined on X is called "locally transitive" if for each (effectively) $\mathcal{R}$-preserving sequence $\{x_n\} \subset X$ (with range $E = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_E$ is transitive.*

Clearly, the notions of "T-transitivity" and "locally transitivity" both are relatively weaker than the notions of transitivity, but they are independent of each other. In order to make them compatible, Alam and Imdad [19] introduced the following notion of transitivity.

Definition 2.5. *[19] Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ defined on X is called locally T-transitive if for each (effectively) $\mathcal{R}$-preserving sequence $\{x_n\} \subset T(X)$ (with range $E = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_E$ is transitive.*

Proposition 2.1. *[19] Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Then* *(i)* $\mathcal{R}$ *is* T*-transitive* $\Leftrightarrow \mathcal{R}|_{T(X)}$ *is transitive,*

(ii) $\mathcal{R}$ *is locally* T*-transitive* $\Leftrightarrow \mathcal{R}|_{T(X)}$ *is locally transitive,*

(iii) $\mathcal{R}$ *is transitive* $\Rightarrow \mathcal{R}$ *is locally transitive* $\Rightarrow \mathcal{R}$ *is locally* T*-transitive,*

(iv) $\mathcal{R}$ *is transitive* $\Rightarrow \mathcal{R}$ *is* T*-transitive* $\Rightarrow \mathcal{R}$ *is locally* T*-transitive.*

3 Relevant Notions and Auxiliary Results

In this section, for the sake of completeness, we summarize some relevant definitions and basic results for our subsequent discussion:

Definition 3.1. *[22] Let* $\mathcal{R}$ *be a binary relation on a nonempty set* X *and* $x, y \in X$*. We say that* x *and* y *are* $\mathcal{R}$*-comparative if either* $(x, y) \in \mathcal{R}$ *or* $(y, x) \in \mathcal{R}$*. We denote it by* $[x, y] \in \mathcal{R}$*.*

Definition 3.2. *[23] Let* X *be a nonempty set and* $\mathcal{R}$ *a binary relation on* X*.*

(i) *The inverse or transpose or dual relation of* $\mathcal{R}$*, denoted by* $\mathcal{R}^{-1}$*, is defined by* $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$*.*

(ii) *The symmetric closure of* $\mathcal{R}$ *(denoted by* $\mathcal{R}^s$*) is defined to be the set* $\mathcal{R} \cup \mathcal{R}^{-1}$ *(*$i.e.$*,* $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$*). Indeed,* $\mathcal{R}^s$ *is the smallest symmetric relation on* X *containing* $\mathcal{R}$*.*

Proposition 3.1. *[22] For a binary relation* $\mathcal{R}$ *defined on a nonempty set* X*,*

$$(x, y) \in \mathcal{R}^s \Longleftrightarrow [x, y] \in \mathcal{R}.$$

Definition 3.3. *[22] Let* $\mathcal{R}$ *be a binary relation defined on a nonempty set* X*. A sequence* $\{x_n\} \subset X$ *is called "*$\mathcal{R}$*-preserving" if*

$$(x_n, x_{n+1}) \in \mathcal{R} \ \ \forall \, n \in \mathbb{N}_0.$$

Definition 3.4. *[22] Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ defined on X is called T-closed if for any $x, y \in X$,*

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}.$$

Proposition 3.2. *[24] Let X be a nonempty set endowed with a binary relation $\mathcal{R}$ and T be a self-mapping on X such that $\mathcal{R}$ is T-closed, then $\mathcal{R}^s$ is also T-closed.*

Proposition 3.3. *[19] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X and T be a self-mapping on X. If $\mathcal{R}$ is T-closed, then for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also T^n-closed, where T^n denotes n^{th} iterate of T.*

Definition 3.5. *[24] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

Notice that every complete metric space is $\mathcal{R}$-complete. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with usual completeness.

Definition 3.6. *[24] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X with $x \in X$. A mapping $T : X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.*

Clearly, every continuous mapping is $\mathcal{R}$-continuous, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with usual continuity.
The following notion is a generalization of d-self-closedness of a partial order relation $(\preceq)$ contained in Turinici [25, 26]:

Definition 3.7. *[22] A binary relation $\mathcal{R}$ defined on a metric space (X, d) is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \;\; \forall\, k \in \mathbb{N}_0$.*

Given a binary relation $\mathcal{R}$ and a self-mapping T on a nonempty set X, we use the following notations:

(i) $F(T)$:=the set of all fixed points of T,

(ii) $X(T,\mathcal{R}) := \{x \in X : (x,Tx) \in \mathcal{R}\}$,

(iii) $\mathcal{N}(x,y) := max\big\{d(x,y), d(x,Tx), d(y,Ty)\big\}$,

(iv) $\mathcal{M}(x,y) := max\big\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\{d(x,Ty)+d(y,Tx)\}\big\}$.

Remark 3.1. *Observe that* $\mathcal{N}(x,y) \le \mathcal{M}(x,y)$ $(\forall x,y \in X)$.

Proposition 3.4. *[20] Let* (X,d) *be a metric space and* $\{x_n\}$ *a sequence in* X*. If* $\{x_n\}$ *is not a Cauchy, then there exist* $\epsilon > 0$ *and two subsequences* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *of* $\{x_n\}$ *such that*

(i) $k \le m_k < n_k \ \ \forall\, k \in \mathbb{N}$,

(ii) $d(x_{m_k}, x_{n_k}) > \epsilon \ \ \forall\, k \in \mathbb{N}$,

(iii) $d(x_{m_k}, x_{n_{k-1}}) \le \epsilon \ \ \forall\, k \in \mathbb{N}$.

Moreover, suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*, then*

(iv) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,

(v) $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon$.

Proposition 3.5. *[17] Let* $\phi \in \Omega$*. If* $\{t_n\} \subset (0,\infty)$ *is a sequence such that* $t_{n+1} \le \phi(t_n)\ \forall\, n \in \mathbb{N}_0$*, then* $\lim_{n\to\infty} t_n = 0$.

Proposition 3.6. *If* (X,d) *is a metric space,* $\mathcal{R}$ *is a binary relation on* X*,* T *is a self-mapping on* X *and* $\phi \in \Omega$*, then the following contractivity conditions are equivalent:*

(I) $\frac{1}{2}d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \phi(\mathcal{M}(x,y)) \ \ \forall\, x,y \in X \ with (x,y) \in \mathcal{R}$,

(II) $\frac{1}{2}d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \phi(\mathcal{M}(x,y)) \ \ \forall\, x,y \in X \ with [x,y] \in \mathcal{R}$.

4 Main Results

We begin this section with our main results as follows:

Theorem 4.1. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ and T be a self-mapping on X. Suppose that following conditions hold:*

(a) *(X, d) is $\mathcal{R}$-complete,*

(b) *$X(T, \mathcal{R})$ is non-empty,*

(c) *$\mathcal{R}$ is T-closed and locally T-transitive,*

(d) *if there exists $\phi \in \Omega$ along with ϕ enjoys the increasing property such that*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq \phi(\mathcal{M}(x, y)) \ \forall\, x, y \in X \ with (x, y) \in \mathcal{R},$$

(e) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed.*

Then T has a fixed point.

Proof. In view of (b), let $x_0 \in X(T, \mathcal{R})$, then $(x_0, Tx_0) \in \mathcal{R}$. As $\mathcal{R}$ is T-closed and using Proposition 3.2 ,we have

$$(x_n, x_{n+1}) \in \mathcal{R} \ \ \forall n \in \mathbb{N}_0 \tag{4.1}$$

where $x_n = T^n x_0 = Tx_{n-1}$.

Therefore the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. For, if $d(x_{n_0+1}, x_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then, we have $T(x_{n_0}) = x_{n_0}$ so that x_{n_0} is a fixed point of T and hence we are through.

On the other hand, if $d(x_{n+1}, x_n) > 0 \ \forall \ n \in \mathbb{N}_0$, implies $\frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, then applying the contractivity condition (d) to (4.1), using triangle inequality of d and increasingness of ϕ, we deduce, for all $n \in \mathbb{N}_0$,

$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \phi(\mathcal{M}(x_n, x_{n+1}))$
$= \phi(max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}(d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})\})$
$= \phi(max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}d(x_n, x_{n+2})\})$
$\leq \phi(max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}))\}\})$
$\leq \phi\Big(max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\Big)$
so that

$$d(x_{n+1}, x_{n+2}) \quad \leq \quad \phi(max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{4.2}$$

In case if $max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$ and using the property of ϕ, then by (4.2), we obtain $d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2})$, which is contradiction and hence (4.2) reduces to

$$d(x_{n+1}, x_{n+2}) \quad \leq \quad \phi(d(x_n, x_{n+1})). \tag{4.3}$$

In view of (4.3) and Proposition 3.5, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.4}$$

Now, we claim that $\{x_n\}$ is a Couchy sequence sequence. To do this, suppose that $\{x_n\}$ is not a Couchy. Owing to Proposition 3.4, there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $k \leq m_k < n_k$, $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k - 1}) < \epsilon$. Further, in view of (4.4) and Proposition 3.4, we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{4.5}$$

For a fixed $\epsilon > 0$ and in view of (4.4), there exists some positive integer $k \in \mathbb{N}$ such that $\frac{1}{2}d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{m_k}, x_{n_k})$ for $k \leq m_k < n_k$. In view of (4.1) and locally T-transitivity of $\mathcal{R}$, we have $(x_{n_k}, x_{m_k}) \in \mathcal{R}$ (for all $k \in \mathbb{N}$). Now, utilizing the contractive condition (d), we obtain

$$d(Tx_{n_k}, Tx_{m_k}) \leq \phi(\mathcal{M}(x_{n_k}, x_{m_k})).$$

Denote $\delta_k := d(x_{n_k}, x_{m_k})$ and $\eta_{n_k} := d(x_{n_k}, x_{n_{k+1}})$, which on again utilizing the triangular inequality, contractive condition (d) and increasing property of ϕ, we have

$$\begin{aligned}
&d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})\\
&= d(x_{n_k}, x_{n_k+1}) + d(Tx_{n_k}, Tx_{m_k}) + d(x_{m_k+1}, x_{m_k})\\
&\leq \eta_{n_k} + \phi(\mathcal{M}(x_{n_k}, x_{m_k})) + \eta_{m_k}\\
&= \eta_{n_k} + \phi(max\{d(x_{n_k}, x_{m_k}), \eta_{n_k}, \eta_{m_k}, \tfrac{1}{2}\{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})\}\})\\
&\qquad + \eta_{m_k}\\
&\leq \eta_{n_k} + \phi(max\{d(x_{n_k}, x_{m_k}), \eta_{n_k}, \eta_{m_k}, \tfrac{1}{2}\{d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1})\\
&\qquad + d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1})\}\}) + \eta_{m_k}\\
&\leq \eta_{n_k} + \phi(max\{d(x_{n_k}, x_{m_k}), \eta_{n_k}, \eta_{m_k}, \delta_k + \eta_{m_k}, \delta_{k+1} + \eta_{m_k}\}) + \eta_{m_k}\\
&\leq \eta_{n_k} + \phi(max\{\delta_k + \eta_{m_k}, \delta_{k+1} + \eta_{m_k}\}) + \eta_{m_k}
\end{aligned}$$

so that

$$d(x_{n_k}, x_{m_k}) \leq \eta_{n_k} + \phi(max\{\delta_k + \eta_{m_k}, \delta_{k+1} + \eta_{m_k}\}) + \eta_{m_k}. \tag{4.6}$$

Utilizing the fact that $\delta_k + \eta_{m_k} \to \epsilon$ (in view of (4.4) and (4.5)) in the real line and definition of Ω, yeilds that

$$\limsup_{k\to\infty} \phi(\delta_k + \eta_{m_k}) = \limsup_{\delta_k+\eta_{m_k}\to\epsilon^+} \phi(\delta_k + \eta_{m_k}) < \epsilon. \tag{4.7}$$

Letting $k \to \infty$ in (4.6) and using (4.7), reduces to

$$\epsilon = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \limsup_{k\to\infty} \phi(\delta_k + \eta_{m_k}) = \limsup_{\delta_k+\eta_{m_k}\to\epsilon^+} \phi(\delta_k + \eta_{m_k}) < \epsilon, \tag{4.8}$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Similarly, in the other case one can have that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is $\mathcal{R}$-complete, there exists $z \in X$ such that $x_n \xrightarrow{d} z$ as $n \to \infty$.

Finally, we prove that z is a fixed point of T. To accomplish this, suppose that T is $\mathcal{R}$-continuous. As $\{x_n\}$ is $\mathcal{R}$-preserving with $x_n \xrightarrow{d} z$, $\mathcal{R}$-continuity of T implies that $x_{n+1} = T(x_n) \xrightarrow{d} T(z)$. Using the uniqueness of limit, we obtain $T(z) = z$, $i.e.$, z is a fixed point of T.

Alternately, assume that $\mathcal{R}$ is d-self-closed. As $\{x_n\}$ is $\mathcal{R}$-preserving such that $x_n \xrightarrow{d} z$, the d-self-closedness of $\mathcal{R}$ guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, z] \in \mathcal{R}$ $(\forall\, k \in \mathbb{N}_0)$.

Now, we claim that (for all $k \in \mathbb{N}_0$)

$$\frac{1}{2}d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, z) \text{ or } \frac{1}{2}d(x_{n_K+1}, x_{n_k+2}) \leq d(x_{n_k+1}, z). \tag{4.9}$$

Arguing by contradiction, we assume that (for some $k_o \in \mathbb{N}_0$)

$$\frac{1}{2}d(x_{n_{k_o}}, x_{n_{k_o}+1}) > d(x_{n_{k_o}}, z) \text{ and } \frac{1}{2}d(x_{n_{k_o}+1}, x_{n_0+2}) > d(x_{n_{k_o}+1}, z)$$

Applying the triangle inequality, we obtain

$$\begin{aligned} d(x_{n_{k_o}}, x_{n_{k_o}+1}) &\leq d(x_{n_{k_o}}, z) + d(x_{n_{k_o}+1}, z) \\ &< \frac{1}{2}d(x_{n_{k_o}}, x_{n_{k_o}+1}) + \frac{1}{2}d(x_{n_{k_o}+1}, x_{n_{k_o}+2}) \\ &< \frac{1}{2}\{d(x_{n_{k_o}}, x_{n_{k_o}+1}) + d(x_{n_{k_o}}, x_{n_{k_{o+1}}})\} = d(x_{n_{k_o}}, x_{n_{k_o}+1}), \end{aligned}$$

a contradiction. Therefore, ((4.9) for all $k \in \mathbb{N}_0$) holds immediately.

On using assumption (d) (in view of (4.9)), Proposition 3.6 and $[x_{n_k}, z] \in \mathcal{R}$ $(\forall\, k \in \mathbb{N}_0)$, we have

$$\begin{aligned} &d(x_{n_k+1}, Tz) = d(Tx_{n_k}, Tz) \leq \phi(\mathcal{M}(x_{n_k}, z)) \\ &\quad = \phi\Big(max\Big\{d(x_{n_k}, z), d(z, Tz), d(x_{n_k}, x_{n_k+1}), \frac{1}{2}\Big\{d(x_{n_k}, Tz) + d(x_{n_k+1}, z)\Big\}\Big\}\Big) \end{aligned} \tag{4.10}$$

We need to discus four cases:

Case (i): If $max\Big\{d(x_{n_k}, z), d(z, Tz), d(x_{n_k}, x_{n_k+1}), \frac{1}{2}\Big\{d(x_{n_k}, Tz) + d(x_{n_k+1}, z)\Big\}\Big\} =$

$d(z,Tz)$. Taking $k\to\infty$ in (4.10) and definition of Ω ($i.e.$, $\phi(t)<t$ for all $t>0$ and $\lim\limits_{s\to t^+}\sup\phi(s)<t$ for all $t>0$), we obtain

$$d(z,Tz)=\lim_{k\to\infty}d(x_{n_k+1},Tz)\le\limsup_{d(x_{n_k+1},Tz)\to d(z,Tz)^+}\phi(d(x_{n_k+1},Tz))<d(z,Tz),$$

which is a contradiction unless $T(z)=z$.

Case (ii): If $max\Big\{d(x_{n_k},z),d(z,Tz),d(x_{n_k},x_{n_k+1}),\frac{1}{2}\big\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\big\}\Big\}$ $=d(x_{n_k},x_{n_k+1})$. Taking $k\to\infty$ in (4.10), definition of Ω ($i.e.$, $\phi(t)<t$ for all $t>0$ and $\lim\limits_{s\to t^+}\sup\phi(s)<t$ for all $t>0$) and fact that ($d(x_{n+1},x_n)>0$ $\forall\, n\in\mathbb{N}_0$), we obtain

$$d(z,Tz)=\lim_{k\to\infty}d(x_{n_k+1},Tz)\le\lim_{k\to\infty}d(x_{n_k},x_{n_k+1})=0,$$

which is again a contradiction unless $T(z)=z$.

Case (iii): If $max\Big\{d(x_{n_k},z),d(z,Tz),d(x_{n_k},x_{n_k+1}),\frac{1}{2}\big\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\big\}\Big\}$ $=\frac{1}{2}\big\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\big\}$. Set $t_k:=\frac{1}{2}\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\}\to\frac{1}{2}d(z,Tz)^+$, since $t_k\to\frac{1}{2}d(z,Tz)$, as $k\to\infty$.

Taking $k\to\infty$ in (4.10) and definition of Ω ($i.e.$, $\phi(t)<t$ for all $t>0$ and $\lim\limits_{s\to t^+}\sup\phi(s)<t$ for all $t>0$), we obtain

$$\begin{aligned}d(z,Tz)=\lim_{k\to\infty}d(x_{n_k+1},Tz)&\le\lim_{k\to\infty}\phi(\frac{1}{2}\big\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\big\})\\&\le\limsup_{k\to\infty}\phi(\frac{1}{2}\big\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\big\})\\&=\limsup_{t_k\to\frac{1}{2}d(z,Tz)^+}\phi(t_k)<\frac{1}{2}d(z,Tz),\end{aligned}$$

which is a contradiction unless $T(z)=z$.

Case (iv): If $max\Big\{d(x_{n_k},z),d(z,Tz),d(x_{n_k},x_{n_k+1}),\frac{1}{2}\big\{d(x_{n_k},Tz)+d(x_{n_k+1},z)\big\}\Big\}$ $=d(x_{n_k},z)$.

We assert that

$$d(x_{n_k+1},Tz)\le d(x_{n_k},z)\ \ \forall\, k\in\mathbb{N}. \tag{4.11}$$

On account of two different possibilities occurring here, we consider a partition of $\mathbb{N}$ *i.e.*, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

(1) $d(x_{n_k}, z) = 0 \ \forall\, k \in \mathbb{N}^0$,

(2) $d(x_{n_k}, z) > 0 \ \forall\, k \in \mathbb{N}^+$.

In case (1), we have $d(Tx_{n_k}, Tz) = 0 \leq d(x_{n_k}, z) \ \forall\, k \in \mathbb{N}^0$. In case (2), and definition of Ω, we have $d(x_{n_k+1}, Tz) \leq \phi(d(x_{n_k}, z)) < d(x_{n_k}, z)$ for all $k \in \mathbb{N}^+$. Hence in both the cases, we get $d(x_{n_k+1}, Tz) \leq d(x_{n_k}, z) \ \ \forall\, k \in \mathbb{N}$, which by using the fact that $x_{n_k} \xrightarrow{d} z$ as $k \to \infty$, yields that $x_{n_k+1} \xrightarrow{d} T(z)$. Again, owing to the uniqueness of limit, we obtain $T(z) = z$ so that z is a fixed point of T. □

Now, we deduce some corollaries, which are sharpened versions (in the context of contractive conditions and binary relations) of Theorem 1.1 due to Pant [6].

Corollary 4.1. *If in the hypotheses of Theorem 4.1, control function (utilized in the assumption (d)) is replaced by strictly inceasing right continuous control function besides retaining the rest of the hyotheses, then the conclusion of Theorem 4.1 remains true.*

In view of Proposition 2.1, one can have the following:

Corollary 4.2. *Theorem 4.1 remains true if locally T-transitivity of $\mathcal{R}$ (utilized in assumption (c)) is replaced by any one of the following conditions besides retaining rest of the hypotheses:*

(i) *$\mathcal{R}$ is transitive,*

(ii) *$\mathcal{R}$ is T-transitive,*

(iii) *$\mathcal{R}$ is locally transitive.*

Theorem 4.2. *If in the hypotheses of Theorem 4.1, the completeness of $\mathcal{R}$ is added, then T has a unique fixed point.*

Proof. As $F(T) \neq \emptyset$ (in view of Theorem 4.1), choose $x, y \in F(T)$ $(i.e., x = T(x) \text{ and } y = T(y))$. We are required to show that $x = y$. Suppose that $x \neq y$. Since $\mathcal{R}$ is complete $[x, y] \in \mathcal{R}$ $(\forall x, y \in X)$ and $x = T(x)$, $y = T(y)$, we have $0 = \frac{1}{2}d(x, Tx) < d(x, y)$. Now, applying the contractive condition (d) to this fact, $\frac{1}{2}d(x, Tx) < d(x, y)$, $([x, y] \in \mathcal{R} \;\; \forall x, y \in X)$,

$$\begin{aligned} d(x, y) = d(Tx, Ty) &\leq \phi(\mathcal{M}(x, y)) \\ &\leq \phi(\max\{d(x, y), d(x, Tx), d(y, Ty), \\ &\quad \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}) \\ &< d(x, y), \end{aligned}$$

a contradiction unless $x = y$. Hence T has a unique fixed point. □

5 An Illustrative Example

In this section, we construct an example to demonstrate the utility of our newly proved result over Theorem 1.1 due to Pant [6].

Example 5.1. *Let $X = [0, 4)$ equipped with usual metric d and a binary relation $\mathcal{R} = \{(0, 0), (0, 1), (1, 0), (1, 1), (3, 0)\}$. Notice that, $\mathcal{R}$ is not transitive but it is T-transitive. Hence in view of Proposition 2.1, it is locally T-transitive. Define a self-mapping T on X by*

$$T(x) = \begin{cases} 0, & x \in [0, 1], \\ 1, & x \in (1, 4), \end{cases}$$

Clearly, $\mathcal{R}$ is T-closed and (X, d) is $\mathcal{R}$-complete. Define a function $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{2}t$, then ϕ is increasing and also a member of Ω. It can be easily *seen that contractive condition* (d) *of Theorem 4.1 are satisfied for all* $(x, y) \in \mathcal{R}$ *except at* $(x, y) = (3, 0)$. *This means we need to varify condition* (d) *at* $(3, 0) \in \mathcal{R}$. *If we take* $x = 3$, *then* $1 = \frac{1}{2}d(3, T3) < d(3, 0) = 3$ *implies that*

$$1 = d(T3, T0) \leq \phi(max\{d(3, 0), d(3, T3), d(0, T0), \frac{1}{2}\{d(3, T0) + d(0, T3)\}\}) = \phi(3) = \frac{3}{2} \ \ for \ \ (3, 0) \in \mathcal{R}.$$

Secondly, if we choose $x = 0$, *then*
$0 = \frac{1}{2}d(0, T0) \leq d(0, 0) = 0 \implies 0 = d(T0, T0) \leq \phi(\mathcal{M}(0, 0)) = 0$, *for* $(0, 0) \in \mathcal{R}$ *and*
$0 = \frac{1}{2}d(0, T0) < d(0, 1) = 1 \implies 0 = d(T0, T1) \leq \phi(\mathcal{M}(0, 1)) = \frac{1}{2}$, *for* $(0, 1) \in \mathcal{R}$.
Taking any $\mathcal{R}$*-preserving sequence* $\{x_n\}$ *such that* $x_n \xrightarrow{d} x$. *As* $(x_n, x_{n+1}) \in \mathcal{R}$, *for all* $n \in \mathbb{N}$, *there exists* $N \in \mathbb{N}$ *such that* $x_n = x \in \{0, 1\}$, for all $n \geq N$. *Hence* $\mathcal{R}$ *is* d*-self closed. Thus, all the conditions of Theorem 4.1 are satisfied and hence* T *has a fixed point (namely* $x = 0$*). As* (X, d) *is not complete and* $\mathcal{R}$ *is not partial order, therefore present example can not be covered by Theorem1.1 (due to Pant [6]) and due to Agarwal* $et\ al$. *[Theorem 2.3, [10]] respectively, which shows that utility our newly proved result.*

Now, we deduce some special cases, which are sharpened versions of several known fixed point theorems of the existing literature.

(1) Under the universal relation ($i.e.$, $\mathcal{R} = X^2$), Theorem 4.2 deduces to generalized form of Theorem 1.1 due to Pant [6].

(2) On choosing $\mathcal{M}(x, y)$ to be $\mathcal{N}(x, y)$, $\mathcal{R}$ to be a partial order relation $\preceq$ and $\phi \in \Omega$ to be a continuous control function (not necessarily increasing) in Theorem 4.2, we obtain sharpened version of Theorem 2.3 due to Agarwal $et\ al$. [10]. Observe that the class of continuous control mappings is properly contained in the class of control mappings ($i.e.$, Ω) due to Boyd and Wong [3].

(3) Setting $\phi(t) = \alpha t$ (where $\alpha \in [0, 1)$) in Theorem 4.2, we obtain a sharpened version of Theorem 3.1 due to Alam and Imdad [22]. Observe that under this setting, the additional condition of locally T-transitivity on the involved binary relation is not necessary, which is uniformity with Alam and Imdad [22].

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