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Galerkin and perturbed collocation methods for solving a class of linear fractional integro-differential equations

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Abstract

In this study, we consider a class of linear fractional integro-differential equations which consist of the same order of fractional differentiation and fractional integration. The approximate solutions of such type of the equations are presented using the Galerkin and perturbed collocation analysis method. In the Caputo sense, fractional differentiation and fractional integration are used. Finally, some illustrative examples are presented to demonstrate the effectiveness of the methods. From the computational viewpoint,

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the Galerkin method is more efficient and easier than the perturbed collocation method.

1 Introduction

In the last three decades, the theory of fractional calculus has been the focus of many studies due to it's wide range of applications such as in biological sciences, chemical sciences, chemical physics, optics, signal processing etc. Most often, the equations arising from the mathematical modelling of fractional order integro differential equations are difficult to solve analytically. This is because many of them do not have solutions in closed form, and thus seeking an approximate solution by numerical techniques becomes useful.

To achieve this, a lot of numerical methods have been used by many researchers. [1] employed multiple perturbed collocation Tau method to solve higher order linear and nonlinear boundary value problems by employing Chebyshev basis functions as the approximate solution. The authours concluded that as the degree of approximant N increases the results obtained by the proposed method converges rapidly to the exact solution. (Oyedepo et. al., 2019) [2], used the Legendre Galerkin Method to solve fractional order Fredholm integro-differential equations successfully and it was seen that the results obtained by the method employed converge to the exact solution at lower values of N. (Ghasemi et. al., 2007) [3], applied Wavelet- Galerkin method(WGM) with homotopy perturbation method(HPM) to solve nonlinear fractional integro-differential equations. Conclusion was drawn that HPM converge faster to exact solution than the Wavelet- Galerkin method. (Rawashdeh, 2006) [4], presents the collocation method which need to be added to solve the fractional integro-differential equation. The wavelet-Galerkin method (WGM) which is used to solve the integro-differential equation can be found in [5]. A comparison between the wavelet-Galerkin method and the Adomain decomposition method to solve the integro-differential equation is given by [6]. (Fakhar-Izadi and Dehghan, 2012) [7] used an efficient pseudo-spectral Legendre Galerkin method for solving a nonlinear partial integro-differential equation arising in population dynamics. A discrete Galerkin method for fractional integro-differential equations is employed by [8]. (Olayiwola et. al., 2020) [9], successfully employed the Legendre polynomial in solving Volterra integro-differential equations using the collocation analysis method. The findings of the study show that, the solution obtained by the means of the basis function used yielded the desired accuracy when compared with the exact solution. (Samah, 2010) [10], applied the collocation method combined with the least square method and the Adomian decomposition method to solve both linear and nonlinear fractional integro-differential equations. Other methods used by many researchers are perturbation collocation method [11], mixed interpolation collocation method [12], least square method and shifted polynomials [13], He's Homotopy perturbation method [14, 15]. Other studies which apply novel methods for solving fractional integro-differential equation and other forms of equation are found in (see for example, [16-25]). In this paper we considered the use of Galerkin method and perturbed collocation method to solve a class of linear fractional integro-differential equations and comparisons are made between the two methods. The class of the fractional integro-differential equation is of the form:

$$D^p u(x) = g(x) + J^p u(x) \ , \qquad a \le x \le b. \tag{1.1}$$

With the initial condition: $u(0) = u_0$, where $g(x)$ is a continuous function on (x, u) for $u \in \mathbb{R}$, $a > 0$ and $0 < x < a$, u_0 is a real positive constant, D^p denotes the Caputo fractional derivatives and J^p denotes the Caputo fractional integral operator.

Some properties of the operator D^p and J^p may be found in [podulbny, 1999], and we mention the following:

$$\mathrm{D}^p x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-p)} x^{n-p} \tag{1.2}$$

$$\mathrm{J}^p x^n = \frac{\Gamma(n+1)}{\Gamma(n+1+p)} x^{n+p} \quad \text{for } x > 0,\ p \ge 0,\ n > -1\,. \tag{1.3}$$

2 Review of Legendre and shifted Legendre polynomials

The Legendre polynomials $L_k(x)\,; k = 0, 1, 2, 3, \ldots$ are the eigenfunctions of the singular Sturm-Liouville problem.

$$\left(\left(1 - t^2\right) L'_k(t)\right)' k(k+1) L_k(t) = 0 \qquad t \in [-1, 1]\,. \tag{2.1}$$

The Legendre polynomials satisfy the recursive relation

$$L_{k+1}(t) = \frac{2k+1}{k+1} t L_k(t) - \frac{k}{k+1} L_{k-1}(t) \ , \quad k = 1, 2, 3, \ldots. \tag{2.2}$$

Where $L_0(t) = 1$ and $L_1(t) = x$ which are thus generated by the Legendre relation

$$L_k(t) = \frac{1}{2^k k!}\frac{d^k}{dx^k}\left(t^2-1\right)^k; \qquad k = 0, 1, 2, \ldots. \tag{2.3}$$

In other to use these polynomials on the interval [0, 1], we define the so called shifted Legendre polynomial by introducing the change of variable t, $t = 2x - 1$; let the shifted Legendre polynomial $L_k(2x-1)$ be denoted by ${L^*}_k(x)$. Then ${L^*}_k(x)$ can be obtained as follows.

$${L^*}_{k+1}(x) = \frac{(2k+1)(2x-1)}{k+1}{L^*}_k(x) - \frac{k}{(k+1)}{L^*}_{k-1}(x) \tag{2.4}$$

Where:

$$\begin{aligned}
{L^*}_0(x) &= 1\\
{L^*}_1(x) &= 2x-1\\
{L^*}_2(x) &= 6x^2-6x+1\\
{L^*}_3(x) &= 20x^3-30x^2+12x-1\\
{L^*}_4(x) &= 70x^4-140x^3+90x^2-20x+1\\
{L^*}_5(x) &= 252x^5-630x^4+560x^3-210x^2+30x-1.
\end{aligned}$$

2.1 Chebyshev and shifted Chebyshev polynomials

Chebyshev polynomials are sequence of orthogonal polynomials related to de-Moivre's formula and can be defined recursively. One usually distinguishes between chebyshev polynomials of first kind which are denoted by T_n and chebyshev polynomials of second kind which are denoted by U_n.

2.2 Chebyshev polynomials of first kind

Chebyshev polynomials of first kind $T_r(x)$ is defined as:

$$T_n(x) = \cos\left(\mathrm{n}\cos^{-1}x\right), \qquad -1 \le x \le 1. \tag{2.5}$$

Or equivalently

$$\mathrm{T}_n(x) = \cos n\theta, \quad \text{where} \quad \theta = \cos^{-1}\mathrm{x} \tag{2.6}$$

$$\mathrm{T}_n(x) = \cos n\theta, \quad \text{where} \quad \theta = \cos^{-1}x. \tag{2.7}$$

The few chebyshev polynomials of the first kind are;

$$\begin{aligned}
&T_n(x) \\
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{aligned}$$

2.3 The Shifted Chebyshev polynomials

For convenience and for the sake of problems that exist in intervals other than $-1 \leq x \leq 1$, $T_n(x)$ is in this subsection normalized to a general finite range $a \leq x \leq b$ as follows:

$$T^*{}_N(x) = \cos\left(\mathrm{N}\cos^{-1}x\right); \quad -1 \leq x \leq 1\,. \tag{2.8}$$

And the recurrence relation is given by

$$T^*{}_{N+1}(x) = 2xT^*{}_N(x) - T^*{}_{N-1}(x)\,, \qquad N \geq 1. \tag{2.9}$$

Where N is the degree of the polynomial.
In general, Chebyshev polynomial valid in $a \leq x \leq b$ is given as

$$T^*{}_N(x) = \cos\left[N\cos^{-1}\left(\frac{2x-b-a}{b-a}\right)\right]; \qquad -1 \leq x \leq 1. \tag{2.10}$$

And the recurrence relation is given as

$$T^*{}_{N+1}(x) = 2\left(\frac{2x-b-a}{b-a}\right)T^*{}_N(x) - T^*{}_{N-1}(x)\,.$$

Few terms of the shifted chebyshev polynomials valid in the interval $[0,1]$ are given below:

$$T^*{}_0\left(x\right)=1$$
$$T^*{}_1\left(x\right)=2x-1$$
$$T^*{}_2\left(x\right)=8x^2-8x+1$$
$$T^*{}_3\left(x\right)=32x^3-48x^2+18x-1$$
$$T^*{}_4\left(x\right)=128x^4-256x^3+100x^2-32x+1$$
$$T^*{}_5\left(x\right)=512x^5-128x^4+1120x^3-400x^2+50x-1.$$

3 Construction of the methods

This section, we discussed the Numerical application of Legendre Polynomial Basis Function on the Solution of a class of linear fractional integro-differential Equation using the Galerkin Method and Perturbed Collocation method.

3.1 Galerkin Method (GM)

Here, this method will be used to solve linear fractional integro-differential equations of the form:

$$D^p u\left(x\right)=g\left(x\right)+J^p u\left(x\right)\ ,\quad u_i\left(0\right)=\varnothing_i(0)\qquad a\leq x\leq b. \tag{3.1}$$

Where $g(x)$ is a smooth known function and $u(x)$ is a function to be determined.

We assumed a trial solution of the form:

$$u\left(x\right)=u_N\left(x\right)=\sum_{k=0}^{N}a_k\ L^*{}_k\left(x\right). \tag{3.2}$$

Where $L^*{}_k\left(x\right)$ is the shifted Legendre polynomial and $a_k\ ,\ k=0(1)N$ are unknown constants to be determined.

Substitute the assumed solution of equation (3.2) in to equation (3.1) to obtain:

$$D^p\left(\sum_{k=0}^{N}a_k\ L^*{}_k\left(x\right)\right)=g\left(x\right)+J^p\left(\sum_{k=0}^{N}a_k\ L^*{}_k\left(x\right)\right). \tag{3.3}$$

Applying (1.2) and (1.3) on (3.3) to obtain:

$$\sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1-p)} x^{k-p} - \sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1+p)} x^{k+p} = g(x). \tag{3.4}$$

And therefore, the residual $R(u,x)$ will be:

$$R(u,x) = \sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1-p)} x^{k-p} - \sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1+p)} x^{k+p} - g(x) = 0. \tag{3.5}$$

To determine the constant coefficients, a_k , $k = 0, 1, 2, 3, \ldots$ we find the inner product of (3.5) with the basis function $L^*{}_k(x)$, $k = 0, 1, 2, 3, \ldots$ to get:

$$\int_a^b (R(u,x))\,(L^*{}_k(x))\,dx = 0\,, \qquad a \le x \le b\,, \quad k = 0, 1, 2, \ldots N. \tag{3.6}$$

Equation (3.6) is further simplified to give rise to $N+1$ linear algebraic system of equations with $N+1$ number of constants for $L^*{}_k(x)$, $k = 0, 1, 2, \ldots N$.

The system of equations obtained from (3.6) is solved to get values for the unknown constants. The values are now substituted in to the assumed solution given in equation (3.2) to get the approximate solution. It is significant to mention that when the problem contains some initial conditions, we first apply those conditions before implementing the Galerkin procedure to obtain the remaining number of required equations.

3.2 Perturbed collocation method (PCM)

Here, this method will be used to solve fractional integro-differential equations of the form:

$$D^p y(x) = f(x) + J^p y(x)\ , \quad y_i(0) = \varnothing_i(0) \qquad r \le x \le s. \tag{3.7}$$

Where $f(x)$ is a known smooth function and $y(x)$ is a function to be determined.

$$y(x) = y_N(x) = \sum_{k=0}^{N} a_k L^*{}_k(x)\,. \tag{3.8}$$

Where $L^*{}_k(x)$ is shifted Legendre polynomial basis function and N is the degree of the assumed approximant. Then equation (3.7) is slightly perturbed to obtain:

$$D^p y(x) = f(x) + J^p y(x) + G_n(x) \tag{3.9}$$

$$G_n(x) = \sum_{v=1}^{n} \tau_v T^*{}_{(n-v+1)}(x) \tag{3.10}$$

is called the perturbation term, $T^*(x)$ is shifted Chebyshev polynomial basis function.

Now, substituting (3.8) and (3.10) into (3.9) we obtain;

$$D^p\left(\sum_{k=0}^{N} a_k L^*{}_k(x)\right) - J^p\left(\sum_{k=0}^{N} a_k L^*{}_k(x)\right) = f(x) + \sum_{v=1}^{n} \tau_v T^*{}_{(n-v+1)}(x). \tag{3.11}$$

Where $\tau_v (v = 1(1)n)$ are the free tau parameters to be determined and $a_k,\ k = 0, 1, 2, \dots N$ are the unknown constants also to be determined.

Applying (1.2) and (1.3) on (3.11) to obtain:

$$\sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1-p)} x^{k-p} - \sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1+p)} x^{k+p}$$

$$= f(x) + \sum_{v=1}^{n} \tau_v T^*{}_{(n-v+1)}(x). \tag{3.12}$$

Equation (3.12) is further simplified and then collocated at equally spaced interior points, $x = x_i$ on $[r, s]$; $x_i = r + \frac{(s-r)i}{N}$; $\quad i = 1, 2, 3, \dots\dots N$ to obtain a system of linear algebraic equations, including those obtained from the use of initial conditions.

The values are then substituted into the assumed solution given in equation (3.8) to give the required approximate solution.

4 Numerical Applications

We solved the below examples for $N = 4$ as defined in equations (3.2) and (3.8).

Example 4.1. *Consider the linear fractional order integro-differential equation:*

$$D^p u(x) = g(x) - J^p u(x)\ , \quad u(0) = 0, \quad x \in [0, 1]. \tag{4.1}$$

Where

$$g(x)=\frac{\Gamma(4)}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}}+\frac{\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}x^{\frac{7}{2}}-\frac{\Gamma(3)}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}}-\frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}}\ ,\quad p=\frac{1}{2}\,. \tag{4.2}$$

The exact solution is $u(x)=x^3-x^2$.

TABLE 1: Results for example 4.1

x	*Exact*	*Galerkin Method (GM)*	*Error (GM)*	*Perturbed collocation method (PCM)*	*Error (PCM)*
0.0	0.00000000000000	0.00000000000238	2.377e-12	-0.00000000000315	3.149e-12
0.1	-0.00900000000000	-0.00899999997700	2.309e-11	-0.00900113112700	1.131e-06
0.2	-0.03200000000000	-0.03199999996000	4.424e-11	-0.03200188411000	1.884e-06
0.3	-0.06300000000000	-0.06299999994000	6.463e-11	-0.06300193559000	1.936e-06
0.4	-0.09600000000000	-0.09599999992000	8.344e-11	-0.09600117452000	1.175e-06
0.5	-0.12500000000000	-0.12499999990000	1.003e-10	-0.12499970230000	2.977e-07
0.6	-0.14400000000000	-0.14400000000000	1.151e-10	-0.14399783240000	2.168e-06
0.7	-0.14700000000000	-0.14700000000000	1.284e-10	-0.14699609110000	3.909e-06
0.8	-0.12800000000000	-0.12799999980000	1.409e-10	-0.12799521660000	4.783e-06
0.9	-0.08100000000000	-0.08099999979000	1.539e-10	-0.08099615948000	3.841e-06
1.0	0.00000000000000	0.00000000016641	1.688e-10	-0.00000008272043	8.292e-08

Example 4.2. *Consider the fractional integro-differential*

$$D^{p}u(x)=\frac{1}{\sqrt{\pi}}\left(\frac{256}{315}x^{\frac{9}{2}}+\frac{16}{3}x^{\frac{3}{2}}+\frac{96}{35}x^{\frac{7}{2}}-\frac{16}{15}x^{\frac{5}{2}}\right)-J^{P}u(x)\,,$$

$$p=\frac{1}{2}\ ,\ \ u(0)=0. \tag{4.3}$$

Where the exact solution $u(x)=x^4-x^3+2x^2$.

TABLE 2: Results for example 4.2

x	*Exact*	*Galerkin Method (GM)*	*Error (GM)*	*Perturbed collocation method (PCM)*	*Error (PCM)*
0.0	*0.00000000000000*	*0.00000000005000*	*5.000e-11*	*-0.00000000005000*	*5.000e-11*
0.1	*0.01910000000000*	*0.019099999930000*	*6.804e-11*	*0.01909934378000*	*6.562e-07*
0.2	*0.07360000000000*	*0.0735999998900000000*	*1.146e-10*	*0.07359903895000*	*9.610e-07*
0.3	*0.16110000000000*	*0.1610999998000000000*	*1.792e-10*	*0.16109929210000*	*7.080e-07*
0.4	*0.28160000000000*	*0.2815999997000000000*	*2.522e-10*	*0.28160014100000*	*1.410e-07*
0.5	*0.43750000000000*	*0.4374999996000000000*	*3.250e-10*	*0.43750145580000*	*1.456e-06*
0.6	*0.63360000000000*	*0.6335999995000000000*	*3.898e-10*	*0.63360293770000*	*2.938e-06*
0.7	*0.87710000000000*	*0.8770999996000000000*	*4.400e-10*	*0.87710411970000*	*4.120e-06*
0.8	*1.17760000000000*	*1.1776000000000000000*	*4.698e-10*	*1.17760436700000*	*4.366e-06*
0.9	*1.54710000000000*	*1.5470999990000000000*	*4.744e-10*	*1.54710287400000*	*2.874e-06*
1.0	*2.00000000000000*	*2.0000000000000000000*	*4.500e-10*	*1.99999867100000*	*1.330e-06*

Example 4.3. *Consider the fractional integro-differential equation:*

$$D^{p}u\left(x\right)=\frac{1}{\sqrt{\pi}}\left(\frac{16}{5}x^{\frac{5}{2}}-\frac{32}{35}x^{\frac{7}{2}}\right)-J^{P}u\left(x\right)\ ,\quad p=\frac{1}{2}\ ,\ \ u\left(0\right)=0\quad 0\leq x\leq 1. \tag{4.4}$$

Where the exact solution $u\left(x\right)=x^{3}$.

TABLE 3: Results for example 3

x	Exact	Galerkin Method (GM)	Error (GM)	Perturbed collocation method (PCM)	Error (PCM)
0.0	0.00000000000000	-0.00000000009511	9.511e-11	0.0000000000129	1.292e-11
0.1	0.00100000000000	0.00099999999590	4.141e-12	0.00099902635130	9.736e-07
0.2	0.00800000000000	0.00800000007200	7.232e-11	0.00799766408300	2.336e-06
0.3	0.02700000000000	0.02700000014000	1.390e-10	0.02699607997000	3.920e-06
0.4	0.06400000000000	0.06400000019000	1.997e-10	0.06399446133000	5.539e-06
0.5	0.12500000000000	0.12500000030000	2.576e-10	0.12499301600000	6.984e-06
0.6	0.21600000000000	0.21600000030000	3.146e-10	0.21599197220000	8.028e-06
0.7	0.34300000000000	0.34300000030000	3.721e-10	0.34299157910000	8.421e-06
0.8	0.51200000000000	0.51200000050000	4.305e-10	0.51199210590000	7.894e-06
0.9	0.72900000000000	0.72900000050000	4.894e-10	0.72899384250000	6.157e-06
1.0	1.00000000000000	1.00000000100000	5.475e-10	0.99999709950000	2.900e-06

5 conclusion

In this paper, we applied the Galerkin method combined with the perturbed collocation method to solve a class of linear fractional integro-differential equation consisting of the same order fractional differentiation and fractional integration with the aid of shifted Legendre basis function. This study showed that Galerkin method is more efficient and gives better results than those from perturbed collocation method.

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