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Existence and Uniqueness of solutions of Volterra integrodifferential equations in Banach Space

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Abstract

In this paper, we intend to study the existence and uniqueness of solution of Volterra integrodifferential equation of the type

$$u'(t) = f(t, u(t)) + \int_0^t a(t,s)g(s,u(s))ds$$

$$u(0) = x$$

under some suitable conditions on the functions, f, g and Kernel, $a(t,s)$ as mentioned later. The method employed in our analysis is based on the ideas of Hussain [2].

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1 Introduction and results

In 1970, M. G. Crandall [1] has studied the Cauchy problem

$$u' = g(u,t), u(0) = x.$$

In our paper, we consider the volterra integrodifferential equation of the form,

$$u'(t) = f(t,u(t)) + \int_0^t a(t,s)g(s,u(s))ds, u(0) = x \tag{1.1}$$

where $g, f \in C[R \times E, E]$ and the function $a : R_+ \times R_+ \to R$, is a Hölder continuous.
E is Banach space and x,y $\in$ E. We denote S = S (x, r); U = U (x, r) the closed and open sphere of the centre x and radius r, respectively. We define,

$$< x,y >= \frac{1}{2} \lim_{h \to 0} \frac{1}{h}(||x+hy|| - ||x-hy||),\ for\ x,y\ \in\ E$$

(see [3] and [4]) The functions f,g $\in$ C[I $\times$ S, E] are said to be demicontinuous, if it is continuous from $I \times S$ in strong topology to the weak topology of E, (see[[7]]).

We make the use of the following hypothesis in our subsequent discussion;

Hypothesis 1. *(H_1) Let f,g $\in$ C[I $\times$ S, E] be demicontinuous such that for sufficiently small positive constant σ and $\nu_i \in$ S(x, r) and $u_i \in U(v_i, \sigma) \cap$ S(x, r). $(i = 1, 2)$)*

(i)

$$\begin{aligned} < u_1 - u_2, f(t,v_1) + \int_0^t a(t,s)g(s,v_1(s))ds - f(t,v_2) \\ - \int_0^t a(t,s)g(s,v_2(s))ds > \\ \leq \alpha(||u_1 - v_1||, ||u_2 - v_2||) + \beta(t)\int_0^t |a(t,s)|||g(s,v_1(s)) \\ - g(s,v_2)(s)||ds + \beta(t)||u_1 - u_2||. \end{aligned}$$

(ii)

$$||g(t,u_1) - g(t,v_1)|| \leq L(t)||u_1 - v_1||;$$

where α is a real valued continuous function and β and L(t) are nonnegative continuous functions on I.

Hypothesis 2. *Let $a : [0,T] \times [0,T] \to R$ is continuous and satisfies Hölder continuity condition in the first and second place with exponent ρ i.e., there exists a positive constant $b_0 > 0$ such that $|a(t_1, s_1) - a(t_2, s_2)| \le b_0(|t_1 - t_2|^\rho + |s_1 - s_2|^\rho)$.*

For all $t_1, t_2, s_1, s_2 \in [O,\ T]$ we need the following Lemma in the proofs of our main results

We need the following Lemmas in our subsequent discussion;

Lemma 1.1. *(Martin [5]) Let u be the E-valued function on real interval J such that*

$$\frac{d}{dt}u(t) \text{ and } \frac{d}{dt}||u(t)||, \text{ exist for every } t \in J.$$

Then

$$\frac{d}{dt}||u(t)|| =< u(t), \frac{d}{dt}u(t) >, \text{ for almost every } t \in J.$$

Lemma 1.2. *((Pachpatte [6]) Let u(t), f(t) and g(t) be real valued nonnegative continuous functions defined on $I = [0, \infty)$ and h(t) be a positive and nondecreasing contninuous function defined on I, for which the inequality*

$$u(t) \le h(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right))ds$$

holds for all t ∈ I. Then

$$u(t) \le h(t)[1 + \int_0^t f(s)\exp\left(\int_0^s [f(\tau) + g(\tau)]d\tau\right)ds],$$

for all t ∈ I.

Now we state our main theorem as follows;

Theorem 1.1. *Assume that hypothesis $(H_1) - (H_2)$ hold. Then there exists one and only one strongly continuous, once weakly continuously differentiable function on some interval [0, ρ] of I which satisfies equation (1.1).*

Proof. Since $f(t, u)$ and $g(s, u)$ are demicontinuous on I $times$ S, Where S is a closed sphere, $I = [0, T]$ and $a(t, s)$ is Hölder continuous. Then there exist constants $0 < r_0 < r$ and $0 < T_0 < T$, $b_0 > 0$ and $M_i > 0$, $i = 1, 2$, such that

$$|a(t_1, s_1) - a(t_2, s_2)| \le b_0(|t_1 - t_2|^\rho + |s_1 - s_2|^\rho)$$

$$||f(t,u)|| \leq M_1, ||g(t,u)|| \leq M_2 \ \ for \ all \ \ (t,u) \ \in \ [0,T_0] \times S(x,r)$$

and

$$\begin{aligned} |a(t,s)| &\leq |a(t,s) - a(0,0)| + |a(0,0)| \\ &\leq b_0[|t|^\rho + |s|^\rho] + |a(0,0)| \\ &= b_0[T^\rho + T^\rho] + |a(0,0)| \\ &= 2b_0 T^\rho + |a(0,0)| \\ &= r, \end{aligned}$$

where $r = 2b_0T^\rho + |a(0,0)|$.

Let $\rho = \min\{\frac{r_0}{M_1+\gamma M_2 T}, T_0\}$ and define the approximate solutions $u_n(\ . \ )$ $(n = 1, 2, ...)$ by $u_n = x$ for $t \leq 0$

$$u_n(t) = x + \int_0^t f(s, u_n(s - \epsilon_n))ds + \int_0^t \int_0^s a(s,\tau) g(\tau, u_n(\tau - \epsilon_n)) d\tau ds \qquad for \ 0 \leq t \leq \rho \quad (1.2)$$

where $\epsilon_n = \frac{\rho}{n}$.

Then u_n is well defined and strongly continuous, once weakly continuously differentiable on [0, ρ] for $f(s, u_n(s-\epsilon_n))$ and $g(\tau, u_n(\tau - \epsilon_n))$ are bounded, weakly continuous functions of s and τ on [0, ρ] respectively. Since $f(s, u_n(s-\epsilon_n))$ and $g(\tau, u_n(\tau-\epsilon_n))$ are Bochner integrable functions of s and τ respectively on [0, ρ], the strong derivative $u_n'(t)$ of $u(t)$ exists for all t $\in$ [0, ρ] and

$$\begin{aligned}
||u_n(t)-u_n(\tau)|| &\le |\int_\tau^t \left\{||f(s,u_n(s-\epsilon_n))||+\int_0^s |a(s,\tau)|||g(\tau,u_n(\tau-\epsilon_n))||d\tau\right\} ds| \\
&\le |\int_\tau^t \left\{M_1+\int_0^s rM_2 d\tau\right\} ds| \\
&= |\int_\tau^t \{M_1+\gamma M_2 s\}\, ds| \\
&\le M_1|t-\tau|+\gamma M_2\frac{|t^2-\tau^2|}{2} \\
&= M_1|t-\tau|+\gamma M_2\frac{|(t-\tau)(t+\tau)|}{2} \\
&= |t-\tau|[M_1+\frac{\gamma M_2|t+\tau|}{2}] \\
&\le |t-\tau|[M_1+\frac{\gamma M_2|t+\tau|}{2}] \\
&= [M_1+\gamma M_2 T]|t-\tau| \\
&= M|t-\tau|
\end{aligned}$$

where $M = M_1 + \gamma M_2 T$.

Thus $\frac{d}{dt}||u_n - u_m(t)||$ exist for a.e t $\in$ [0, ρ] and

$$\begin{aligned}
\frac{d}{dt}||u_n(t)-u_m(t)|| &=< u_n(t)-u_m(t), u_n'(t)-u_m'(t) > \\
&=< u_n(t)-u_m(t), f(t,u_n(t-\epsilon_n)) \\
&+\int_0^t a(t,s)g(s,u_n(s-\epsilon_n))ds - f(t,u_m(t-\epsilon_m)) \\
&-\int_0^t a(t,s)g(s,u_m(s-\epsilon_m))ds
\end{aligned}$$

for all t $\in$ [0, ρ] Let n_0 be a natural number such that $\epsilon_n \le \min\{\sigma, \frac{\sigma}{M}\}$ for $n \ge n_0$ where $M = M_1 + \gamma M_2 T$. Then

$$\frac{d}{dt}||u_n(t) - u_m(t)|| \leq \alpha(||u_n(t) - u_n(t - \epsilon_n)||, ||u_m(t) - u_m(t - \epsilon_m)||)$$
$$+ \beta(t) \int_0^t |a(t, s)|||g(s, u_n(s - \epsilon_n)) - g(s, u_m(s - \epsilon_m))||ds$$
$$+ \beta(t)||u_n(t) - u_m(t)||$$
$$= \alpha(||u_n(s) - u_n(s - \epsilon_n)||, ||u_m(s) - u_m(s - \epsilon_m)||)$$
$$+ \beta(t) \int_0^t \gamma L(s)||u_n(s - \epsilon_n) - u_m(s - \epsilon_m)||ds$$
$$+ \beta(t)||u_n(t) - u_m(t)||.$$

Integrating inequality from 0 to t, we have

$$||u_n(t) - u_m(t)|| \leq \int_0^t \alpha(||u_n(s) - u_n(s - \epsilon_n)||, ||u_m(s) - u_m(s - \epsilon_m)||)$$
$$+ \int_0^t \beta(s) \int_0^s \gamma L(\tau)||u_n(\tau - \epsilon_n) - u_m(\tau - \epsilon_m)||d\tau ds$$
$$+ \int_0^t \beta(s)||u_n(s) - u_m(s))||ds.$$

Taking limits on both the sides as m, n $\to \infty$, we have

$$\lim_{m,n\to\infty} ||u_n(t) - u_m(t)|| \leq \int_0^t \alpha(0, 0)ds + \lim_{m,n\to\infty} \int_0^t \beta(s) \int_0^s \gamma L(\tau)||u_n(\tau - \epsilon_n)$$
$$- u_m(\tau - \epsilon_m)||d\tau ds + \lim_{m,n\to\infty} \int_0^t \beta(s)||u_n(s) - u_m(s)||ds$$

i.e.,

$$\lim_{m,n\to\infty} ||u_n(t) - u_m(t)|| \leq 0 + \int_0^t \beta(s)\{\lim_{m,n\to\infty} ||u_n(s) - u_m(s)||$$
$$+ \int_0^s \lim_{m,n\to\infty} \gamma L(\tau)||u_n(\tau - \epsilon_n) - u_m(\tau - \epsilon_n)||d\tau\}ds.$$

$$\leq 0 + \int_0^t \beta(s) \lim_{m,n\to\infty} ||u_n(s) - u_m(s)||$$
$$+ \int_0^t \beta(s) \int_0^s \gamma L(\tau) \lim_{m,n\to\infty} ||u_n(\tau) - u_m(\tau)||d\tau ds.$$

Now an application of Pachpatte's inequality established in [6].

We have,

$$\lim_{m,n\to\infty} ||u_n(t) - u_m(t)|| \le 0[1 + \int_0^t \beta(s)\exp\left(\int_0^t [\beta(\tau) + \gamma L(\tau)]d\tau\right) ds]$$
$$= 0.$$

So that $\lim_{m,n\to\infty} ||u_n(t) - u_m(t)|| = 0$ uniformly on [0, ρ].
Consequently $\lim_{m,n\to\infty} u_n(t) = u(t)$ exist uniformly on [0, ρ] and u satisfies the equation (1.1).

Uniqueness : Let v(t) be another strongly continuous, once weekly continuously differentiable solution of (1.2), Then

$$\begin{aligned}\frac{d}{dt}||u(t) - v(t)|| &=< u(t) - v(t), f(t, u(t)) + \int_0^t a(t,s)g(s,u(s))ds - f(t, v(t)) \\ &\quad - \int_0^t a(t,s)g(s,v(s))ds > \\ &\le \alpha(||u(t) - u(t)||, ||v(t) - v(t)||) \\ &\quad + \beta(t)\int_0^t |a(t,s)|||g(s,v(s)) - g(s,v(s))||ds \\ &\quad + \beta(t)||u(t) - v(t)||\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}||u(t) - v(t)|| \le \alpha(0,0) \\ &+ \beta(t)\int_0^t \gamma L(s)||u(s) - v(s)||ds + \beta(t)||u(t) - v(t)||\end{aligned}$$

Integrating from 0 to t

$$\begin{aligned}||u(t) - v(t)|| &\le \int_0^t \beta(s)\int_0^s \gamma L(\tau)||u(\tau - v(\tau))||d\tau ds + \int_0^t \beta(s)||u(s) - v(s)||ds \\ &= 0 + \int_0^t \beta(s)||u(s) - v(s)||ds + \int_0^t \beta(s)\int_0^s \gamma L(\tau)||u(\tau) - v(\tau)||d\tau ds\end{aligned}$$

Again by Pachpatte's Inequality, we have

$$
\begin{aligned}
||u(t) - v(t)|| &\leq 0[1 + \int_0^t \beta(s) \exp(\int_0^s [\beta(\tau) + \gamma L(\tau) d\tau]) ds] \\
&= 0.
\end{aligned}
$$

This implies that $u(t) = v(t)$ for t ϵ [0, ρ], and hence the uniqueness is proved. □

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